Quest Journals
Journal of Research in Humanities and Social Science
Volume 13 ~ Issue 4 (April 2025) pp: 122-130
ISSN(Online):2321-9467
www.questjournals.org

Research Paper

Numerical Solution of Neutral Fractional Integro-Differential Equations via Fourth-Degree Hat Functions and Euler-Maruyama Approximation for Weakly Singular Kernels

حم عذدي نهمعادلات انتكامهيت-انتفاضهيت انكسريت انمحايذة باستخذاو دوال انقبعت من انذرجت انرابعت وطريقت أويهر- مارويايا مع نواة متفرقت بشكم ضعيف

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Abstract:
This paper presents a novel methodology for solving fractional integro- differential equations (FIDEs) using fourth-degree hat functions (FDHFs). The fundamental properties of fractional calculus, including Riemann-Liouville integrals and Caputo derivatives, are explored and employed to formulate accurate numerical solutions. The study incorporates mathematical assumptions to ensure the existence and uniqueness of solutions, and the Euler-Maruyama method is applied for numerical approximation. Illustrative examples using the double Laplace transform demonstrate the efficacy of the proposed approach, yielding precise solutions that align with theoretical results. The paper highlights the significance of fractional calculus in modeling systems with long- term memory and chaotic behavior, offering a robust tool for solving complex equations in fields such as engineering and physics.

Keywords:
Fractional integro-differential equations • Fourth-degree hat functions • Euler-Maruyama method • Weakly singular kernels • Caputo derivative • Riemann-Liouville integral • Double Laplace transform • Numerical approximation • Stochastic differential equations • Memory effects.

***R**eceived 13 Apr., 2025; **R**evised 24 Apr., 2025; **A**ccepted 26 Apr., 2025*

المستخلص:

تهجف هحه الرقة إلى تقجي مشهجية ججيجة لحل السعادلات التكاملية-التفاضمية الكدخية باستخجام دوال

القبعة م الجرجة الخابعة .(FDHFs) ت استكذاف الخرائص الأساسية لحداب الكدر، بسا في

ذلغ تكاملات ريسان-ليفيمي ومذتقات كابت، واستخجامها لرياغة حمل عجدية دقيقة. تزسشت الجراسة افتخاضات رياضية لزسان وجد الحمل وتفخدها، كسا ت تطبيق طخيقة أويمخ-
ماروياما لتقخيب الحمل العجدية. ت تقجي أمثمة تضيحية باستخجام تحيل لابلاس السددوج لإظهار فعالية الطخيقة
السقتخحة، حيث ت الحرل عمى حمل دقيقة تتافق مع الشتائج الشعخية. تبخز هحه الرقة أهسية
حداب الكدر في نسحجة الأنعسة ذات الحاكخة الطيمة والدمك الفضي، وتقجم أداة قية لحل السعادلات السعقجة في مجالات مثل الهشجسة والفيدياء.

I. Introduction

In fractional calculus, the notion of derivatives and integrals is generalized to any real and even complex order. The concept of fractional computation arose in 1695 when G.W Leibniz suggested that there was a possibility of fractional differentiation of the order. Many standard properties are broken by fractional differential and integral operators, including the standard product (Leibniz) rule, the standard chain rule, the semi-group property for orders of derivatives, and the semi-group property for dynamic maps. The violation of the Leibniz rule's standard form is a characteristic property of non-integer order derivatives. On the other hand, long-term memory and non-local dynamics are two of the most important applications of fractional derivatives and integrals of non-integer order. Fractional calculus has a long and illustrious history that goes back more than 300 years. Nevertheless, for a long time, it was regarded as a pure mathematical field that lacked real-world applications. In the last few decades, the subject of fractional-order calculus has gotten a lot of attention because it allows you to represent a system more precisely without (or with minimal) approximation. Furthermore, this approach is a good tool for analyzing fractional dimension systems with long-term "memory" and chaotic behavior, and it is advantageous to model the behavior of a process in fractional-order because the response will include many values that would otherwise be ignored by integer-order due to approximations. As a result, fractional calculus has piqued the interest of scientists and engineers alike. For instance, fractional calculus models have been found to be a useful tool for describing the mechanics of viscoelastic materials and anomalous particle transport in groundwater. Signal processing, control of dynamic systems, wave propagation, medicine, economics, and finance are some of the other applications of fractional calculus models.[5]

2. Preliminary

We will use the following conventions unless stated otherwise. The expectation corresponding to a probability distribution P will be denoted by E. If A is a vector or matrix, its transpose will be represented by A^T. The notation |. | will be used to denote both the Euclidean norm on Rd and the trace norm on $R^{d\times r}$. In other words, if x ∈ Rd, |x| will refer to the Euclidean norm, and if A is a matrix, |A| will represent the trace norm. The indicator function of a set S will be denoted by 1_S, where $1_S(x) = 1$ if x ∈ S and 0 otherwise. For two real numbers a and b, we will use the notation a ∨ b := max(a, b) and a ∧ b := min(a, b). Additionally, the uppercase letter C (with or without subscripts) will be used to represent a positive constant whose value may vary depending on its context, but it will always be independent of the step size h. Finally, we will introduce four mild assumptions that will be used later for the nonlinear functions f_i (i = 1, . . . , n) and g_j (j = 0, 1, 2).[3]

3. Assumptions

i) ∃ $L_1 > 0$ such that $\forall t_1, t_2, s \in [0, T]$ and $\forall z \in R^d$, g_1 and g_2 satisfy the condition:[8]

$$|g_j(t_1, s, z) - g_j(t_2, s, z)| \le L_1(1 + |z|)|t^1 - t_2|,\ j = 1, 2.$$

ii) ∃ $L_2 > 0$ such that $\forall t, s_1, s_2 \in [0, T]$ and $\forall z \in R^d$, g_0, g_1, g_2 and fi for i = 1, . . . , n satisfy the condition:

$$\begin{cases} |g_0(s_1,z)-g_0(s_2,z)| \vee |g_j(t,s_1,z)-g_j(t,s_2,z)| \le L_2(1+|z|)|s_1-s_2|, & j=1,2, \\ |f_i(s_1,z)-f_i(s_2,z)| \le L_2(1+|z|)|s_1-s_2|, & i=1,2,\dots,n \end{cases}$$

iii) ∀ integer m ≥ 1, $\exists K_m \ge 0$ depending only on m, such that ∀t, s ∈ [0, T] and $\forall z_1, z_2 \in R^d$ with $|z_1| \vee |z_2| \le m$, g_0, g_1, g_2 and, fi for i = 1,…, n hold the local Lipsichitz condition:

$$\begin{cases} |g_0(s,z_1)-g_0(s,z_2)| \vee |g_j(t,s,z_1)-g_j(t,s,z_2)| \le K_m|z_1-z_2|, & j=1,2, \\ |f_i(s,z_1)-f_i(s,z_2)| \le K_m|z_1-z_2|, & i=1,2,\dots,n \end{cases}$$

iv) ∃ L > 0 such that ∀t, s ∈ [0, T] and $\forall z \in R^d$, g_0, g_1, g_2, and fi for i = 1,…, n satisfy the linear growth condition:

$$\begin{cases} |g_0(s,z)| \vee |g_i(t,s,z)| \le L(1+|z|), & j=1,2, \\ |f_i(s,z)| \le L(1+|z|), \quad \text{for} & i=1,2,\dots,n \end{cases}$$

Remark(3.1)[8] We emphasize that the local Lipschitz condition mentioned above (i.e., Assumption 3) is weaker than the next global Lipschitz condition in order to represent the generality of our conclusions: ∃ K > such that ∀t, s ∈ [0, T] and $\forall z_1, z_2 \in R^d$, g_0, g_1, g_2 and f_i(i = 1,…,n) satisfy the inequalities

$$\begin{cases}|g_0(s,z_1)-g_0(s,z_2)|\vee|g_j(t,s,z_1)-g_j(t,s,z_2)|\le K|z_1-z_2|, & j=1,2,\\ |f_i(s,z_1)-f_i(s,z_2)|\le K|z_1-z_2|, & i=1,2,\dots,n\end{cases} \quad (1)$$

4. Well-posedness of stochastic fractional neutral integro-differential equation

We will examine the existence, uniqueness, and continuous dependence on the initial value of the exact solution to the stochastic fractional integro-differential equation (1) using the preparation from the previous part.[3]

5. Existence and uniqueness of solution of stochastic fractional neutral integro-differential equation

We first propose the EM approximation to facilitate in the proof of the existence result. For each integer $N \geq 1$, EM-approximation can be shown as[5]

$$z^N(t)=z_0+\sum_{i=1}^{n}\int_0^t F_i\left(t,s,\hat{z}^N(s)\right)ds+\int_0^t G_0\left(t,s,\hat{z}^N(s)\right)ds$$
$$+\int_0^t G_1\left(t,s,\hat{z}^N(s)\right)ds+\int_0^t G_2\left(t,s,\hat{z}^N(s)\right)dW(s), \quad (2)$$

where the simplest step process $\hat{z}^N(s)=\sum_{n=0}^{N}(t_n)1_{[t_n,t_{n+1}]}(t)$ and the mesh points $t_n = nh$ ($n = 0, \ldots, N$) with $h = \frac{T}{n}$.

Lemma(5.1)[9] If assumption (4) is true, then there is a positive constant C that does not depend on the value of N. This constant satisfies the following inequalities for any integer $p \geq 2$, and for all values of t in the interval $[0, T]$:

$$E[|z^N(t)|p]\le C \quad \text{and}, \ [|\hat{z}^N(t)|p]\le C,\ \forall t\in[0,T].$$

Proof. First, we prove the case where $p > 2$. Let $k \geq 1$ be an integer. We define the stopping time

$$\rho_m^N = T\wedge \inf\{t\in[0,T] : |z^N(t)|\ge m\},$$

where ρ_m^N m $\uparrow$ T almost surely as $m \to \infty$. For convenience, we set $z_m^N(t) = z^N(t \wedge \rho_m^N)$ and $\hat{z}^N(t) = \hat{z}^N(t \wedge \rho_m^N)$ for all $t \in [0, T]$. Using Holder inequality, Burkholder-Davis- Gundy inequality, assumption (4), and the fact that $E[|z_0|^p]< +\infty$, we can deduce from (2) that there exists a positive constant $q \in (0, p\alpha)$, which only depends on $p > 2$ and $0 < \alpha \leq \alpha_i \leq 1$, so that

$$E\left[\left|z_m^N(t)\right|^p\right]\le\frac{(n+4)^p}{\Gamma^p(\alpha)}\left\{\Gamma^p(\alpha)E\left[|z_0(t)|^p\right]\right.$$
$$+\sum_{i=0}^{n}\frac{\Gamma^p(\alpha)}{\Gamma^p(\alpha_i)}E\left(\left|\int_0^{t\wedge\rho_m^p}\left(t\wedge\rho_m^p-s\right)^{\alpha_i-1}f_i\left(s,\hat{z}_m^N(s)\right)ds\right|^p\right)$$
$$+E\left(\left|\int_0^{t\wedge\rho_m^p}\left(t\wedge\rho_m^p-s\right)^{\alpha-1}f_i\left(s,\hat{z}_m^N(s)\right)ds\right|^p\right)$$
$$+B^p(\alpha,1-\beta_1)E\left(\left|\int_0^{t\wedge\rho_m^p}\left(t\wedge\rho_m^p-s\right)^{\alpha_i-1}\sup_{s\le v\le t\wedge\rho_m^p}\left|g_1\left(v,s,\hat{z}_m^N(s)\right)\right|ds\right|^p\right)$$
$$\left.+B^p(\alpha,1-\beta_2)E\left(\left|\int_0^{t\wedge\rho_m^p}\left(t\wedge\rho_m^p-s\right)^{2(\alpha-\beta_2)}\sup_{s\le v\le t\wedge\rho_m^p}\left|g_2\left(v,s,\hat{z}_m^N(s)\right)\right|^2ds\right|^{\frac{p}{2}}\right)\right\}$$
$$\le C_1\left\{1+\sum_{i=1}^{n}\left(\int_0^{t\wedge\rho_m^N}\left(t\wedge\rho_m^N-s\right)^{\frac{-\alpha_1-q}{p-1}}ds\right)^{p-1}\right.$$
$$\int_0^{t\wedge\rho_m^N}\left(t\wedge\rho_m^N-s\right)\left(1+E\left[\left|\hat{z}_m^N(s)\right|^p\right]\right)ds$$

$$+\left[\int_0^{t\wedge\rho_m^N}\left(t\wedge\rho_m^N-s\right)^{\frac{2(p\alpha-q)}{p-2}}ds\right]^{\frac{p-2}{2}}\int_0^{t\wedge\rho_m^N}\left(t\wedge\rho_m^N-s\right)^{q-1}\left(1+E\left[\left|\hat{z}_m^N(s)\right|^p\right]\right)ds\Bigg\}$$

$$\leq C_2\left(1+\int_0^{t\wedge\rho_m^N}\left(t\wedge\rho_m^N-s\right)^{q-1}\left(1+E\left[\left|\hat{z}_m^N(s)\right|^p\right]\right)ds\right)$$

By taking the supremum on both sides of the equation, we find that the positive constants C_1 and C_2 are independent of m and N.

$$\sup_{0\leq\lambda\leq t}E\left[\left|z_m^N(\lambda)\right|^p\right]\leq C_2\left\{1+\sup_{0\leq\lambda\leq t}\int_0^{\lambda\wedge\rho_m^N}\left(\lambda\wedge\rho_m^N-s\right)^{q-1}\sup_{0\leq\eta\leq t}E\left[\left|z_m^N(s)\right|^p\right]ds\right\}$$

If we replace $v=\frac{s}{\lambda\wedge\rho_m^N}$, we will get

$$\sup_{0\leq\lambda\leq t}E\left[\left|z_m^N(\lambda)\right|^p\right]$$

$$\leq C_2\left\{1+\sup_{0\leq\lambda\leq t}\left(\lambda\wedge\rho_m^N-s\right)^q\int_0^1(1-v)^{q-1}\sup_{0\leq\eta\leq\left(\lambda\wedge\rho_m^N\right)v}E\left[\left|z_m^N(\eta)\right|^p\right]dv\right\}$$

$$\leq C_2\left\{1+t^q\int_0^1(1-v)^{q-1}\sup_{0\leq\eta\leq tv}E\left[\left|z_m^N(\eta)\right|^p\right]dv\right\}$$

When we alternate with s = tv, then we have

$$\sup_{0\leq\lambda\leq t}E\left[\left|z_m^N(\lambda)\right|^p\right]\leq C_2\left\{1+\int_0^t(t-s)^{q-1}\sup_{0\leq\eta\leq tv}E\left[\left|z_m^N(\eta)\right|^p\right]dv\right\}$$

which with the application of weakly singular Gronwall's inequality yields

$$E\left[\left|z_m^N(t)\right|^p\right]\leq C,\quad\forall t\in[0,T].$$

Letting $m\to+\infty$ and using Fatou's lemma to indicate

$$E\left[\left|z^N(t)\right|^p\right]\leq C,\quad\forall t\in[0,T].$$

Additionally, by using the same logic and approach as in the previous proof for the scenario where $p\geq 2$, we can also derive the same results when p = 2. However, instead of utilizing Holder's inequality, we will substitute it with Cauchy-Schwarz's inequality.

6. Fractional Calculus

Fractional calculus is a branch of mathematics that studies the properties of integrals and derivatives with non-integer orders of integration and differentiation (called fractional integrals and derivatives). The Riemann-Liouville and Caputo definitions are the most widely used for fractional integrals and derivatives. This article is based on the Caputo definition of fractional derivative because it is the only one that has the same form as integer-order differential equations in initial conditions.[10]

Now, the fractional integral of Riemann-Liouville and the Caputo derivative are defined as follows:

Definition(6.1)[10]. Let y(t) be a continuous function with $t>0$. The Riemann-Liouville fractional integral operator of order α, $\alpha\geq 0$ of the function y(t) is defined as follows:

$$J_t^{\alpha}y(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds, & \alpha>0,\\ y(t), & \alpha=0.\end{cases}\qquad(3)$$

where $\Gamma(.)$ is the fractional-order gamma function.

Definition(6.2)[11] Let y(t) be a continuous function with $t>0$. The Caputo fractional derivative of order $\alpha>0$ of the function y(t) is defined as follows:

$$ {}^{C}D^{\alpha}y(t)=\frac{1}{\Gamma(q-\alpha)}\int_{0}^{t}(t,s)^{q-\alpha-1}y^{(q)}(s)ds, \quad t\in[0,T], \qquad (4)$$

where $y^{(q)}(s)=\frac{d^{q}y(s)}{ds^{q}}$, $q\in N$, and $q-1<\alpha\leq q$.

The following formula establishes the relationship between the Caputo fractional derivative and the Riemann-Liouville fractional integral:

$$({}^{c}D^{\alpha}J^{\alpha}_{t}y)(t)=y(t), \qquad (5)$$

$$J_{t}^{\alpha}\,{}^{C}D^{\alpha}y(t)=y(t)-\sum_{r=0}^{q-1}y^{(r)}(0)\frac{t^{r}}{r!},\ q-1<\alpha\leq q. \qquad (6)$$

7. Fourth-Degree Hat Functions and Their Properties

In order to construct the FDHFs, assume that the interval $\Omega=[0,T]$ is divided into n equidistant subintervals, and then each of these subintervals must be divided again into four equidistant subintervals with a length equal to h, where $h=\frac{T}{4n}$ and $n\in N$. The FDHFs form a set of (4n+1) linearly independent functions in $L^{2}[0,T]$. These functions are defined as follows: [1]

$$\xi_{0}(t)=\begin{cases}\frac{(t-h)(t-3h)(t-4h)}{24h^{4}}, & 0\leq t\leq 4h\\ 0, & \text{otherwise.}\end{cases} \qquad (7)$$

if $k=1,2,\ldots,n-1$,

$$\xi_{4k}(t)=\begin{cases}\frac{(t-(4k+1)h)(t-(4k+2)h)(t-(4k+3)h)(t-(4k+3)h)}{24h^{4}}, & 4(k-1)h\leq t\leq 4(k+1)h\\ \frac{(t-(4k+1)h)(t-(4k+2)h)(t-(4k+3)h)(t-(4k+4)h)}{24h^{4}}, & 4kh\leq t\leq 4(k+1)h\\ 0, & \text{otherwise.}\end{cases} \qquad (8)$$

if $k=1,2,\ldots n$,

$$\xi_{4k-1}(t)=\begin{cases}\frac{-(t-4kh)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{6h^{4}}, & 4(k-4)h\leq t\leq(4k)h,\\ 0, & \text{otherwise.}\end{cases} \qquad (9)$$

$$\xi_{4k-2}(t)=\begin{cases}\frac{(t-4kh)(t-(4k-2)h)(t-(4k-3)h)(t-(4k-4)h)}{4h^{4}}, & (4k-4)h\leq t\leq 4kh,\\ 0, & \text{otherwise.}\end{cases} \qquad (10)$$

$$\xi_{4k-3}(t)=\begin{cases}\frac{(t-(4k-2)h)(t-(4k-1)h)(t-(4k-4)h)(t-(4k-4)h)}{6h^{4}}, & (4k-4)h\leq t\leq 4kh,\\ 0, & \text{otherwise.}\end{cases} \qquad (11)$$

and

$$\xi_{4n}(t)=\begin{cases}\frac{(t-(T-h))(t-(T-2h))(t-(T-3h))(t-(T-4h))}{6h^{4}}, & T-4h\leq t\leq T,\\ 0, & \text{otherwise}\end{cases} \qquad (12)$$

According to the definition (3), FDHFs have the following properties:

(i) According to the definition of FDHFs, there is a significant relation:

$$\xi_{i}(jh)=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases} \qquad i,j=0,1,2,\ldots,4n. \qquad (13)$$

(ii) The total sum of FDHFs is one, implying:

$$\sum_{i=0}^{4n}\xi_{i}(t)=1. \qquad (14)$$

(iii) The functions $\xi_{0}(t), \xi_{1}(t),\ldots,\xi_{4n}(t)$ are linearly independent for all $t\in[0,T]$.

(iv) Any function $y(t)\in L^{2}[0,T]$ can be approximated in terms of FDHFs as:

$$y(t)\approx y_{4n}(t)=\sum_{i=0}^{4n}y_{k}\xi_{i}(t)=Y^{T}\Xi(t)=\Xi^{T}(t)Y, \qquad (15)$$

where $\Xi(t)=[\xi_{0}(t), \xi_{1}(t), \xi_{2}(t), \ldots, \xi_{4n}(t)]^{T}$, and $Y=[y_{0}, y_{1}, \ldots, y_{4n}]^{T}$.

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The use of FDHFs to approximate a function y(t) is significant because the coefficients y_κ in Eq.(5.1.15) are given by:

$$y_\kappa = y(\kappa h),\ \kappa = 0,1,..,4n. \tag{16}$$

(v) Any function $K(t, s) \in L^2([0,T]\times[0,T])$ can be approximated in terms of FDHFs as:

$$K(t,s) \approx K_{4n}(t,s) = \sum_{r=0}^{4n}\sum_{\kappa=0}^{4n} K_{\kappa r}\xi(t)\xi_r(t) = \Xi^T D\Xi(t) = \Xi^T(s)D^T\Xi(t), \tag{17}$$

where, $K_{\kappa r}(t, s) = L(\kappa h, rh)$, $\forall \kappa$, $r = 0,1,2, ...,4n$.

8. Methodology

The double Laplace transform method for obtaining the general solution v(x, t) of Equation (5.3.1) is developed in this segment. Transforming Equation (5.3.1) by double Laplace transform, we obtain[10]

$$s^\alpha \overline{v}(p,s) - \sum_{j=0}^{m-1} s^{\alpha-1-j} L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\} + L_x L_t\left\{\int_0^t (t-\tau)^{-b}\frac{\partial^4 v(x,0)}{\partial t^4}d\tau\right\} = \overline{K}(p,s), \tag{18}$$

where $\overline{K}(p,s) = L_x L_t[K(p,s)]$.

Convolution Theorem, If DLT of f(x,t) and single Laplace transform of g (t) are given by $L_x L_t\ [f(x, t)] = \overline{f}(p, s)$ and $L_t\ \{g(t)\} = \overline{g}(s)$ then

$$L_x L_t\{g(t) * f(x,t)\} = L_x L_t\left\{\int_0^t g(t-y)f(x,t)dy\right\} = \overline{g}(s)\overline{f}(x,t), \tag{19}$$

where $g(t) * f(x,t) = \int_0^t g(t-y)f(x,t)dy$.

Using convolution theorem Equation (19), we get

$$\begin{aligned} &s^\alpha \overline{v}(p,s) - \sum_{j=0}^{m-1} s^{\alpha-1-j} L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\} \\ &+ L_t\left[t^{-b}\right] L_x L_t\left\{\int_0^t (t-\tau)^{-b}\frac{\partial^4 v(x,0)}{\partial t^4}d\tau\right\} = \overline{K}(p,s). \end{aligned} \tag{20}$$

The fourth order partial derivatives double Laplace transform formula is

$$L_x L_t\left\{\frac{\partial^4 v(x,0)}{\partial t^4}\right\} = p^4\overline{f}(p,s) - \sum_{j=0}^{3} p^{3-j} L_t\left\{\frac{\partial^j v(0,t)}{\partial t^j}\right\}, \tag{21}$$

Using Equation (5.3.7) in Equation (5.3.7), we obtain

$$\begin{aligned} &s^\alpha \overline{v}(p,s) - \sum_{j=0}^{m-1} s^{\alpha-1-j} L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\} \\ &+ \frac{\Gamma(1-b)}{s^{1-b}}\left[s^\alpha \overline{v}(p,s) - \sum_{k=0}^{3} p^{3-k} L_t\left\{\frac{\partial^k u(0,t)}{\partial t^k}\right\}\right] = \overline{K}(p,s). \end{aligned} \tag{22}$$

Further, transforming Equations (17) and (18) by single Laplace transform, we get

$$L_x\left\{\frac{\partial^j v(x,0)}{\partial t^j}\right\} = \overline{f}_j(p),\ L_t\left\{\frac{\partial^k u(0,t)}{\partial t^k}\right\} = \overline{g}_k(s), \quad j = 0,1,2,\dots,m-1, \text{ and } k = 1,2,3. \tag{23}$$

By putting Equation (5.3.9) in Equation (5.3.8) and we obtain by simplifying

$$\overline{v}(p,s) = \frac{1}{\left[s^\alpha + \frac{\Gamma(1-b)}{s^{1-b}}p^4\right]}\left[\overline{K}(p,s) + \sum_{j=0}^{m-1} s^{\alpha-1-j}\overline{f}_j(p)\frac{\Gamma(1-b)}{s^{1-b}}\sum_{j=0}^{3} p^{3-j}\overline{g}_k(s)\right]. \tag{24}$$

We obtain the solution of Equation (15) by using the inverse DLT to Equation (24).

$$v(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{\left[s^\alpha + \frac{\Gamma(1-b)}{s^{1-b}}p^4\right]}\left[\overline{K}(p,s) + \sum_{j=0}^{m-1} s^{\alpha-1-j}\overline{f}_j(p)\frac{\Gamma(1-b)}{s^{1-b}}\sum_{j=0}^{3} p^{3-j}\overline{g}_k(s)\right]\right] \tag{25}$$

In this case, we assume that the inverse DLT of Equation (25) exists.

3 Results and discussion

In this part, we provide examples to exhibit the applicability of the previous technique.

Example (8.1)[11]: By changing $b = \frac{1}{2}$, $m = 1$ and $K(x,t) = x^4 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 32t^{\frac{3}{2}} + 48\sqrt{t}$ in Equation (15),

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \int_0^t (t-\tau)^{-\frac{1}{2}} \frac{\partial^4 v(x,0)}{\partial t^4} d\tau = x^4 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 32t^{\frac{3}{2}} + 48\sqrt{t}, \quad x, t \geq 0, \; 0 < \alpha \leq 1, \tag{26}$$

subject to

$$v(x,0) = f_0(x) = x^4, \; v(0,t) = g_0(x) = 0, \; \frac{\partial v(0,t)}{\partial x} = g_1(t) = 0, \quad \frac{\partial^2 v(0,t)}{\partial x^2} = g_2(t) = 0, \frac{\partial^3 v(0,t)}{\partial x^3} = g_3(t) = 0. \tag{27}$$

Transforming Equation (5.3.13) by single Laplace transform, we get

$$\bar{f}_0(p) = \frac{24}{p^5}, \quad \bar{g}_k(s) = 0, \quad k = 0,1,2,3, \tag{28}$$

Transforming K (x, t) by double Laplace transform, we get

$$\overline{K}(p,s) = \frac{24}{p^5}\frac{1}{s^{2-\alpha}} + 32\frac{1}{p}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}} + 48\frac{1}{p}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}}. \tag{29}$$

Substituting above in Equation(25), we get solution of Equation(26):

$$v(x,t) = L_x^{-1} L_t^{-1}\left[\frac{1}{\left[s^{\alpha} + \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} p^4\right]}\left[\frac{24}{p^5}\frac{1}{s^{2-\alpha}} + 32\frac{1}{p}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + 48\frac{1}{p}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + s^{\alpha-1}\frac{24}{p^5}\right]\right] \tag{30}$$

$$v(x,t) = L_x^{-1} L_t^{-1}\left[\frac{1}{\left[s^{\alpha} + \sqrt{\frac{\pi}{s}} p^4\right]}\left[\left(\frac{1}{s^2} + \frac{1}{s}\right)\frac{24}{p^5}\left[s^{\alpha} + \sqrt{\frac{\pi}{s}} p^4\right]\right]\right] = L_x^{-1} L_t^{-1}\left[\left(\frac{1}{s^2} + \frac{1}{s}\right)\frac{24}{p^5}\right] = (t+1)x^4. \tag{31}$$

Figure 1. shows the exact solution $v(x,t) = (t+1)x^4$ utilizing a variety of values of $0 \leq x \leq 1$ and $0 \leq t \leq 15$.

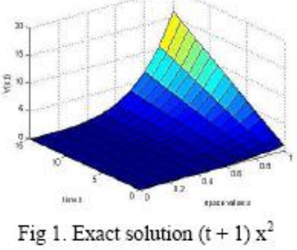

Fig 1. Exact solution $(t + 1) x^2$

Example(8.2)[11]. By changing $b = \frac{1}{2}$, $m = 2$ and

$$K(x,t) = 2x^4 \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{128}{5} t^{\frac{5}{2}},$$

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} + \int_0^t (t-\tau)^{-\frac{1}{2}} \frac{\partial^4 v(x,0)}{\partial t^4} d\tau = 2x^4 \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{128}{5} t^{\frac{5}{2}}, \ x,t \geq 0, \ 0 < \alpha \leq 2, \tag{32}$$

subject to

$$v(x,0) = f_0(x) = 0, \ \frac{\partial v(x,0)}{\partial x} = f_1(t) = 0, \ v(0,t) = g_0(x) = 0,$$
$$\frac{\partial v(0,t)}{\partial x} = g_1(t) = 0, \frac{\partial^2 v(0,t)}{\partial x^2} = g_2(t) = 0, \frac{\partial^3 v(0,t)}{\partial x^3} = g_3(t) = 0. \tag{33}$$

Transforming Equation (5.3.19) by single Laplace transform, we get

$$\bar{f}_0(p) = \bar{f}_1(p) = 0, \ \ \bar{g}_k(s) = 0, \ \ k = 0,1,2,3. \tag{34}$$

Transforming K (x, t) by double Laplace transform, we get

$$\overline{K}(p,s) = 2\frac{24}{p^5}\frac{1}{s^{3-\alpha}} + \frac{128}{5}\frac{1}{p}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}}. \tag{35}$$

Substituting above in Equation(32), we get solution of Equation(31):

$$v(x,t) = L_x^{-1} L_t^{-1}\left[\frac{1}{\left[s^{\alpha} + \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} p^4\right]}\left[2\frac{24}{p^5}\frac{1}{s^{3-\alpha}} + \frac{128}{5}\frac{1}{p}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + 48\frac{1}{p}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}}\right]\right] \tag{36}$$

Computing, we get desired solution:

$$v(x,t) = L_x^{-1} L_t^{-1}\left[\frac{\Gamma(5)}{p^5}\frac{\Gamma(3)}{s^3}\right] = x^4 t^2. \tag{37}$$

Figure 2. shows the exact solution v (x, t) = $x^4 t^2$ utilizing a variety of values of $0 \leq x \leq 1$ and $0 \leq t \leq 15$.

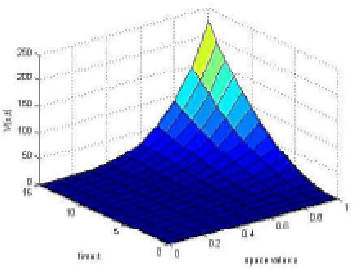

Fig 2. Exact solution $x^4 t^2$.

Conclusion:

This study presents a numerical solution for neutral fractional integro-differential equations with weakly singular kernels using fourth-degree hat functions and Euler-Maruyama approximation. The equation is transformed into a linear algebraic system via numerical quadrature, while the stochastic Euler-Maruyama scheme models system noise. Numerical results demonstrate the efficiency and accuracy of the proposed method compared to known analytical solutions.

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