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Research Paper

ON THE ZEROS OF POLAR DERIVATIVES OF POLYNOMIALS

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ABSTRACT:-*We extend some existing results on the zeros ofpolar derivatives of polynomials by considering more general coefficient conditions. As special cases the extended results yield much simpler expressions for the upper bounds of zeros of those existing results.*
Mathematics Subject Classification: *30C10, 30C15.*

Keywords:*- Zeros of polynomial,*Eneström-*Kakeya theorem, Polar derivatives.*

I. INTRODUCTION

Let $\mathrm{D}_\alpha\mathrm{P}(\mathrm{z}) = \mathrm{nP}(\mathrm{z}) + (\alpha - \mathrm{z})\mathrm{P}'(\mathrm{z})$denote the polar derivative of a polynomial P(z) of degree nwith respect to real number α. The polynomial $\mathrm{D}_\alpha\mathrm{P}(\mathrm{z})$is of degree at most n-1 and it generalizes the ordinary derivative in the sense that $\lim_{\alpha\to\infty}\frac{\mathrm{D}_\alpha\mathrm{P}(\mathrm{z})}{\alpha} = \mathrm{P}'(\mathrm{z})$. Many results on the location of zeros of polynomials are available in the literature. In literature [3-5] attempts have been made to extend and generalize the Eneström-Kakeya theorem. Existing results in the literature also show that there is a need to find bounds for special polynomials, for example, for those having restrictions on the coefficient, there is always need for refinement of results in this subject to find location of zeros of polar derivatives of polynomials. Among them the Eneström-Kakeya theorem [1-2] given below is well known in the theory of zero distribution of polynomials.

Theorem A. (Eneström-Kakeyatheorem): LetP(z)=$\sum_{i=0}^{n} a_i z^i$*be a polynomial of degree n*such that $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_n$ then allthe zeros of P(z) lie in |z|≤1.

The following theorems B and C due to P.Ramulu and G.L. Reddy [6]

Theorem B. Let P(z)= $\sum_{i=0}^{n} a_i z^i$*be a polynomial of degree n and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α such that
$na_0 \le (n-1)a_1 \le (n-2)a_2 \le \cdots \le 3a_{n-3} \le 2a_{n-2} \le a_{n-1}$ $if\ \alpha = 0$ then all the zeros of polar derivative $D_0P(z)$lie in$|z| \le \frac{1}{|a_{n-1}|}[a_{n-1} - na_0 + |na_0|]$.

Theorem C. Let P(z)= $\sum_{i=0}^{n} a_i z^i$*be a polynomial of degree n and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α such that
$na_0 \ge (n-1)a_1 \ge (n-2)a_2 \ge \cdots \ge 3a_{n-3} \ge 2a_{n-2} \ge a_{n-1}$ $if\ \alpha = 0$ then all the zeros of polar derivative $D_0P(z)$ lie in$|z| \le \frac{1}{|a_{n-1}|}[|na_0| + na_0 - a_{n-1}]$.

Here we establish the following results.

Theorem 1. Let P(z)= $\sum_{i=0}^{n} a_i z^i$*be a polynomial of degree n and*$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α*such that*

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$$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i \; for \; i = 0,1,\ldots,n-2.$$

then all the zeros of polar derivative $D_\alpha P(z)$ *lie in*

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|].$$

Corollary 1. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α *such that*

$$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i > 0 \; for \; i = 0,1,\ldots,n-2.$$

then all the zeros of polar derivative $D_\alpha P(z)$ *lie in* $|z| \leq 1$.

Remark 1. By taking $a_i > 0 \; for \; i = 0,1,2,\ldots,n-1$, in theorem 1, then it reduces to Corollary 2.
Remark 2. By taking $\alpha = 0$ in theorem 1, then it reduces to Theorem B.

Theorem 2. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α *such that*

$$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i \; for \; i = 0,1,\ldots,n-2.$$

then all the zeros of polar derivative $D_\alpha P(z)$ *lie in*

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}].$$

Corollary 2. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α such that

$$0 < [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i \; for \; i = 0,1,\ldots,n-2.$$

then all the zeros of polar derivative $D_\alpha P(z)$ *lie in*

$$|z| \leq \frac{1}{n\alpha a_n + a_{n-1}}[2(\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}].$$

Remark 3. By taking $a_i > 0 \; for \; i = 0,1,2,\ldots,n-1$, in theorem2, then itreduces to Corollary 2.
Remark 4. By taking $\alpha = 0$ in theorem 2, then it reduces to Theorem C.

II. PROOFS OF THE THEOREMS

Proof of the Theorem 1.

LetP(z) $= a_n z^n + a_{n-1}z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$ be a polynomial of degree n

Let $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ be polar derivative of P(z) with respect to real number α of degree n-1.

$$\Rightarrow D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + \cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]$$

Letus consider the polynomial Q(z) $= (1 - z)\, D_\alpha P(z)$ so that

$$Q(z) = (1-z)([n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-2} + \cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0])$$

$$= -[n\alpha a_n + a_{n-1}]z^n + \{[n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} + [(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}]z^{n-2} + \cdots + [3\alpha a_3 + (n-2-2\alpha)a_2(n-1)a_1]z^2 + [2\alpha a_2 + (n-1-\alpha)a_1 - na_0]z + [\alpha a_1 + na_0]\}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1 \; for i = 0,1,2,\ldots,n-2$.

Now

$$|Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1} - \{|n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}||z|^{n-1} + |(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}||z|^{n-2} + \cdots + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1||z|^2 + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0||z| + |\alpha a_1 + na_0|\}$$

$$\geq \ |n\alpha a_n + a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{|\ n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}| + \frac{|(n-1)\alpha a_{n-1} + (2{+}2\alpha - n\alpha)a_{n-2} - 3a_{n-3}|}{|z|} + \cdots + \frac{|3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1|}{|z|^{n-3}} + \frac{2\alpha a_2 + (n-1-\alpha)a_1 - na_0|}{|z|^{n-2}} + \frac{|\alpha a_1 + na_0|}{|z|^{n-1}}\}\ ]$$

$$\geq |n\alpha a_n + a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\ \{\ |\ n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}\ | + |(n-1)\alpha a_{n-1} + (2{+}2\alpha - n\alpha)a_{n-2} - 3a_{n-3}\ | + \cdots + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1| + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0| + |\alpha a_1 + na_0|\ \}]$$

$$\geq |n\alpha a_n + a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{\ [n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}] + [\ (n-1)\alpha a_{n-1} + (2{+}2\alpha - n\alpha)a_{n-2} - 3a_{n-3}\ ] + \cdots + [3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1] + [2\alpha a_2 + (n-1-\alpha)a_1 - na_0] + |\alpha a_1 + na_0|\}]$$

$$\geq |n\alpha a_n + a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\ \}]$$

$$> 0\ if\ |z| > \frac{1}{|n\alpha a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|].$$

This shows that ifQ(z)> 0
provided$|z| > \frac{1}{|n\alpha\, a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|]$.

Hence all the zeros of Q(z) with |z|$>$ 1 lie in

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|].$$

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros ofpolar derivative $D_\alpha P(z)$ are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.

Proof of the Theorem 2.

LetP(z) $= a_n z^n + a_{n-1}z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$be a polynomial of degree n

Let $D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z)$ be polar derivative of P(z) with respect to real number$\alpha$of degree n-1.

$$\Rightarrow D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n{-}1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n{-}2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + \cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]$$

Letus consider the polynomial Q(z) $=$ ($1 -$ z) $D_\alpha P(z)$ so that

$$Q(z) = (1-z)([n\alpha a_n + a_{n-1}]z^{n-1} + [(n{-}1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n{-}2)\alpha a_{n-2} + 3a_{n-3}]z^{n-2} + \cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0])$$

$$= -[n\alpha a_n + a_{n-1}]z^n + \{[\ n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} + [\ (n-1)\alpha a_{n-1} + (2{+}2\alpha - n\alpha)a_{n-2} - 3a_{n-3}]z^{n-2} + \cdots + [3\alpha a_3 + (n-2-2\alpha)a_2(n-1)a_1]z^2 + [2\alpha a_2 + (n-1-\alpha)a_1 - na_0]z + [\alpha a_1 + na_0]\ \}$$

Also if $|z| > 1$ then$\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < 1\ for i = 0,1,2,\ldots,n-2$.

Now

$$|Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1} - \{|\ n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}||z|^{n-1}$$

$$+\left|(n-1)\alpha a_{n-1}+(2+2\alpha-n\alpha)a_{n-2}-3a_{n-3}\right||z|^{n-2}+\cdots+|3\alpha a_3+(n-2-2\alpha)a_2-(n-1)a_1||z|^2$$
$$+|2\alpha a_2+(n-1-\alpha)a_1-na_0||z|+|\alpha a_1+na_0|\,\}$$

$$\geq\ |n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha a_n+a_{n-1}|}\{|\,n\alpha a_n+(1+\alpha-n\alpha)a_{n-1}-2a_{n-2}|$$
$$+\frac{|(n-1)\alpha a_{n-1}+(2+2\alpha-n\alpha)a_{n-2}-3a_{n-3}|}{|z|}$$
$$+\cdots+\frac{|3\alpha a_3+(n-2-2\alpha)a_2-(n-1)a_1|}{|z|^{n-3}}+\frac{2\alpha a_2+(n-1-\alpha)a_1-na_0|}{|z|^{n-2}}+\frac{|\alpha a_1+na_0|}{|z|^{n-1}}\}\,]$$

$$\geq|n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha a_n+a_{n-1}|}\ \{\,|\,n\alpha a_n+(1+\alpha-n\alpha)a_{n-1}-2a_{n-2}\,|$$
$$+|(n-1)\alpha a_{n-1}+(2+2\alpha-n\alpha)a_{n-2}-3a_{n-3}\,|$$
$$+\cdots+|3\alpha a_3+(n-2-2\alpha)a_2-(n-1)a_1|$$
$$+|2\alpha a_2+(n-1-\alpha)a_1-na_0|+|\alpha a_1+na_0|\ \}]$$

$$\geq|n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha a_n+a_{n-1}|}\{\,[2a_{n-2}-(1+\alpha-n\alpha)a_{n-1}-n\alpha a_n]$$
$$+[3a_{n-3}\ -(2+2\alpha-n\alpha)a_{n-2}-(n-1)\alpha a_{n-1}]$$
$$+\cdots+[(n-1)a_1-(n-2-2\alpha)a_2-3\alpha a_3]$$
$$+[na_0-(n-1-\alpha)a_1-2\alpha a_2]+|\alpha a_1+na_0|\}]$$

$$\geq|n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha a_n+a_{n-1}|}\{|\alpha a_1+na_0|+(\alpha a_1+na_0)-n\alpha a_n-a_{n-1}\,\}]$$

> 0 if $|z|>\frac{1}{|n\alpha\, a_n+a_{n-1}|}[|\alpha a_1+na_0|+(\alpha a_1+na_0)-n\alpha a_n-a_{n-1}]$.

This shows that if Q(z)>0
provided$|z|>\frac{1}{|n\alpha\, a_n+a_{n-1}|}[|\alpha a_1+na_0|+(\alpha a_1+na_0)-n\alpha a_n-a_{n-1}\ ]$.

Hence all the zeros of Q(z) with |z|$>$ 1 lie in

$$|z|\leq\frac{1}{|n\alpha a_n+a_{n-1}|}[|\alpha a_1+na_0|+(\alpha a_1+na_0)-n\alpha a_n-a_{n-1}\ ].$$

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of polar derivative$D_\alpha P(z)$ are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 2.

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