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Research Paper

Cayley-Hamilton Theorem Applications in MatricesandCorrelationMatrices: Some Applications in Livestock Data

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Abstract: *TheCayley-Hamilton theoremwasused in thisstudytodeterminetheforces, squareroot, cuberoot, andthelogarithm of squarematrices. Thematrixwascreatedbyexaminingthecorrelationsbetween a few body characteristics of animals in certainpublishedstudies on livestockdata. Thepolynomialsandfunctions of thecreatedcorrelationmatricesweredetermined. TheCayley-Hamilton theoremwasusedtodeterminetheeigenvalues of matricesthatareexceedinglychallengingtosolveusingothermethods in ordertoproducehigh-orderpolynomialsandfunctionsforthesematrices.*

Keywords: *Cayley-Hamilton theorem, matrix, eigenvalue, correlation.*

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I. Introduction

The Cayley-Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the n × n matrix algebra $M_n(K)$ over a field K [1,2].

Ziebur (1970) [3]and Schmidt (1986)[4] applied knowledge of the basic form of e^{At} to a derivation of the main results on the structure of A as a linear operator. Their approach started with an application of the Cayley–Hamilton theorem to deduce the form of each entry of e^{At} as a solution of a constant coefficient linear differential equation. Since the Cayley–Hamilton theorem can be viewed as part of the structure theory of a linear operator, it seems natural to ask if, by means of a different starting point for the analysis of e^{At}, one can also deduce this result from information about e^{At}.

The Cayley-Hamilton theorem [5, 6] explains that every square matrix satisfies its own characteristic equation. The Cayley-Hamilton theorem has been extended to rectangular matrices [7], block matrices [7], pairs of commuting matrices [7-10] and pairs of block matrices [11].

There are studies conducted by some researchers in different fields related to the Cayley-Hamilton theorem [12, 13]. The coefficients of the characteristic polynomial were expressed by traces of powers of the matrix, yielding a compact form of the Cayley-Hamilton equation of 2 x 2 matrices over the Grassmann algebra [14]. It was solved a problem of existence of multiparameter systems that satisfy some given data by Cayley–Hamilton theorem [15]. The Cayley–Hamilton and Frobenius theorems via the Laplace transform were applied [16]. In another study, the representation theory of solvable Lie algebras using generalized Cayley–Hamilton theorem was performed [17].

In this study, it is aimed to obtain higher order powers of matrices and some functions by applying Cayley-Hamilton theorem on some square matrices and correlation matrices.

II. Material and Method

The material of this research consisted of the matrices arbitrarily given by the author and the matrices of the size (3*3) determined as a result of several studies.

The Cayley-Hamilton Theorem provides the characteristic equation for any quadratic matrix A.

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 = 0$$

If its equation is the characteristic equation of matrix A,

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I = 0$$

given a (2*2) square matrix, det(A-λI)=0 is calculated to find its eigenvalues. There are two eigenvalues. According to this,

$$f(A) = a_0 I + a_1 A$$
$$f(\lambda_1) = a_0 I + a_1 \lambda_1$$
$$f(\lambda_2) = a_0 I + a_1 \lambda_2$$

Here$\lambda_1$and$\lambda_2$are different eigenvalues.
Let it be a square matrix of size (3*3). To obtain the eigenvalues of this matrix, det(A-λI)=0 is calculated.
f(A)=$a_0 I + a_1 A + a_2 A^2$
is,

$$f(\lambda_1) = a_0 I + a_1 \lambda_1 + a_2 \lambda_1^2$$
$$f(\lambda_2) = a_0 I + a_1 \lambda_2 + a_2 \lambda_2^2$$
$$f(\lambda_3) = a_0 I + a_1 \lambda_3 + a_2 \lambda_3^2$$

Here λ_1, λ_2 and λ_3 are different eigenvalues[18].

III. Results and Discussion

First, apply the Cayley-Hamilton theorem to any (2*2) dimensional matrix. For example, given below,

$$A = \begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$$

for a square matrix of the form (2*2), calculations such as, A^{250}, A^{-5},$\sqrt{A}$, $\sqrt[3]{A}$can be made. For each operation, det(A-λI)=0 is calculated and the characteristic roots of the given matrix, that is, the eigenvalues, are determined. Here I is the unit matrix,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is in the form. To find the eigenvalues,

$$\det\left(\begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

is calculated.

$$\begin{vmatrix} 4-\lambda & 7 \\ 5 & 6-\lambda \end{vmatrix} = 0$$

When the determinant of this matrix is taken and set to zero,

$$\lambda^2 - 10\lambda - 11 = 0$$

The equation is obtained. The roots of this equation, $\lambda_1 = 11$ ve $\lambda_2 = -1$are the eigenvalues of the matrix.
To calculate A^{250}, the eigenvalues are

$$f(11) = \alpha_0 + 11 = 11^{250}$$
$$f(-1) = \alpha_0 + \alpha_1 = (-1)^{250}$$
$$\alpha_0 + 11\alpha_1 = 11^{250}$$
$$\alpha_0 - \alpha_1 = 1$$

When this system of equations is solved, $\alpha_0 = \frac{11^{250}+11}{12}and\alpha_1 = \frac{11^{250}-1}{12}$there are unknowns.
When the $\alpha_0$and$\alpha_1$values are substituted in the formula,

$$f(A) = a_0 I + a_1 A$$
$$f(A) = A^{250} = \frac{11^{250}+11}{12}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 = \frac{11^{250}-1}{12}\begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$$

$$A^{250} = \begin{bmatrix} \frac{15 * 11^{250}+7}{12} & \frac{7}{12}(11^{250}-1) \\ \frac{5}{12}(11^{250}-1) & \frac{7 * 11^{250}+10}{12} \end{bmatrix}$$

is found.
Similarly, when, $A^{-10}$is calculated,

$$\alpha_0 + 11\alpha_1 = 11^{-10}$$
$$\alpha_0 - \alpha_1 = 1$$

When this system of equations is solved, $\alpha_0 = \frac{11^{-10}+11}{12}$ and$\alpha_1 = \frac{11^{-10}-1}{12}$ unknowns are obtained.

$$f(A) = A^{-10} = \frac{11^{-10}+11}{12}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{11^{-10}-1}{12}\begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$$

$$A^{-10} = \begin{bmatrix} \frac{5 * 11^{-10}+7}{12} & \frac{7}{12}(11^{-10}-1) \\ \frac{5}{12}(11^{-10}-1) & \frac{7 * 11^{-10}+5}{12} \end{bmatrix}$$

When $\sqrt{A}$is calculated similarly,

$$\alpha_0 + 11\alpha_1 = \sqrt{11}$$
$$\alpha_0 - \alpha_1 = i$$

When this system of equations is solved,$\alpha_0 = \frac{\sqrt{11}+11i}{12}$ ve $\alpha_1 = \frac{\sqrt{11}-i}{12}$have values. From here,

$$\sqrt{A} = f(A) = \frac{\sqrt{11}+11i}{12}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{\sqrt{11}-i}{12}\begin{bmatrix}4 & 7\\5 & 6\end{bmatrix}$$

$$\sqrt{A} = \begin{bmatrix}\frac{5\sqrt{11}+7i}{12} & \frac{7}{12}(\sqrt{11}-i)\\ \frac{5}{12}(\sqrt{11}-i) & \frac{7\sqrt{11}+5i}{12}\end{bmatrix}$$

available as.

Calculation of$\sqrt[3]{A}$is as follows,

$$\alpha_0 + 11\alpha_1 = \sqrt[3]{11}$$
$$\alpha_0 - \alpha_1 = -1$$

From the system of equations, $\alpha_0 = \frac{\sqrt[3]{11}-11}{12}and\alpha_1 = \frac{\sqrt[3]{11}+1}{12}$there are unknowns. From here too,

$$f(A) = a_0 I + a_1 A = \frac{\sqrt[3]{11}-11}{12}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{\sqrt[3]{11}+1}{12}\begin{bmatrix}4 & 7\\5 & 6\end{bmatrix}$$

$$\sqrt[3]{A} = \begin{bmatrix}\frac{5\sqrt[3]{11}-7}{12} & \frac{7}{12}(\sqrt[3]{11}+1)\\ \frac{5}{12}(\sqrt[3]{11}+1) & \frac{7\sqrt[3]{11}-5}{12}\end{bmatrix}$$

obtained.

The calculation of $e^A$is shown below.

$$\alpha_0 + 11\alpha_1 = e^{11}$$
$$\alpha_0 - \alpha_1 = e^{-1}$$

From this system of equations, $\alpha_0 = \frac{e^{11}+11e^{-1}}{12}$ and$\alpha_1 = \frac{e^{11}-e^{-1}}{12}$values are obtained.

$$f(A) = e^A = \frac{e^{11}+11e^{-1}}{12}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{e^{11}-e^{-1}}{12}\begin{bmatrix}4 & 7\\5 & 6\end{bmatrix}$$

$$e^A = \begin{bmatrix}\frac{5e^{11}+7e^{-1}}{12} & \frac{7}{12}(e^{11}-e^{-1})\\ \frac{5}{12}(e^{11}-e^{-1}) & \frac{7e^{11}+5e^{-1}}{12}\end{bmatrix}$$

is in the form.

The expression ln(A) is calculated as follows.

$$\alpha_0 + 11\alpha_1 = ln11$$
$$\alpha_0 - \alpha_1 = \mathrm{i}\pi$$

From this system of equations, $\alpha_0 = \frac{ln11+11\mathrm{i}\pi}{12}$ and$\alpha_1 = \frac{ln11-\mathrm{i}\pi}{12}$are found. From here,

$$f(A) = lnA = \frac{ln11 + i11\pi}{12}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{ln11 - i\pi}{12}\begin{bmatrix}4 & 7\\5 & 6\end{bmatrix}$$

$$lnA = \begin{bmatrix}\frac{5ln11+i7\pi}{12} & \frac{7}{12}(ln11-i\pi)\\ \frac{5}{12}(ln11-i\pi) & \frac{7ln11+i5\pi}{12}\end{bmatrix}$$

is calculated as. When this matrix is calculated,

$$lnA = \begin{bmatrix}0.999 + 1.833i & 1.399 - 1.833i\\0.999 - 1.309i & 1.399 + 1.309i\end{bmatrix}$$

result arises.

Similar calculations were made for the following (3*3). For matrix,

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 3 & 1 & 5 \\ 4 & 6 & 9 \end{bmatrix}$$

let A^{15} be calculated. In order to obtain the eigenvalues of this matrix from the equation $\det(A - I\lambda) = 0$,

$$\begin{vmatrix} 2-\lambda & 5 & 8 \\ 3 & 1-\lambda & 5 \\ 4 & 6 & 9-\lambda \end{vmatrix} = 0$$

When the determinant is taken and equalized to zero, it is found as, $\lambda_1 = 15.2892, \lambda_2 = -1 \; and \; \lambda_3 = -2.2892$.

$$a_0 + 15.2892\, a_1 + 233.7596\, a_2 = 583138111937939000$$
$$a_0 - a_1 + a_2 = -1$$
$$a_0 - 2.2892\, a_1 + 5.240437\, a_2 = -248459.7584$$

When the system of equations is solved,
a_0=4 662 000 000 000 000
a_1=6 698 600 000 000 000
a_2=2 036 500 000 000 000
obtained.

$$f(A) = a_0 I + a_1 A + a_2 A^2$$

in the equation, the I matrix is the unit matrix, and substituting the $A^2 = \begin{bmatrix} 51 & 63 & 113 \\ 29 & 46 & 74 \\ 62 & 80 & 143 \end{bmatrix}$ values,

$$a_0 I + a_1 A + a_2 A^2 = 4662000000000000 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 6698600\ 00000000 \begin{bmatrix} 2 & 5 & 8 \\ 3 & 1 & 5 \\ 4 & 6 & 9 \end{bmatrix} + 2036500000000000 \begin{bmatrix} 51 & 63 & 113 \\ 29 & 46 & 74 \\ 62 & 80 & 143 \end{bmatrix}$$

$$A^{15} = \begin{bmatrix} 121\,920\,000\,000\,000\,000 & 161\,790\,000\,000\,000\,000 & 283\,713\,300\,000\,000\,000 \\ 79\,154\,300\,000\,000\,000 & 105\,040\,000\,000\,000\,000 & 184\,194\,000\,000\,000\,000 \\ 153\,060\,000\,000\,000\,000 & 203\,110\,000\,000\,000\,000 & 356\,170\,000\,000\,000\,000 \end{bmatrix}$$

available as.

Instudies in thefield of animalhusbandry, numerousmatrixcalculationswereconductedbyusingtheCayley-Hamilton theoremutilizingcorrelationmatrices.

Inonestudy, several of thecorrelationcoefficientsbetweenmorphometricpropertieswereselected in Zulusheep in KwaZulu-Natal. Thecorrelationcoefficientbetween body weightandcidagoheight in sheepwas 0.831, body weight-chestdepthcorrelationcoefficientwas 0.781, andcidagoheight-chestdepthcorrelationcoefficientwas 0.707 [19]. Theseselectedvaluesarecreated as a symmetrical A-squarematrix of size (3*3). Thematrixcreated is as

$$A = \begin{bmatrix} 1 & 0.831 & 0.781 \\ 0.831 & 1 & 0.707 \\ 0.781 & 0.707 & 1 \end{bmatrix}$$

Forthismatrix, A^{10}, A^{-4}, $\sqrt{A}$ ve exp(A) matrices can be calculated.

$\det(A - I\lambda) = 0$ Toobtaintheeigenvalues of thematrixfromtheequation,

$$|A| = \begin{vmatrix} 1-\lambda & 0.831 & 0.781 \\ 0.831 & 1-\lambda & 0.707 \\ 0.781 & 0.707 & 1-\lambda \end{vmatrix} = 0$$

whenthe determinant in the form of det(A) is takenandequalizedtozero, it is found as $\lambda_1 = 0.1542, \lambda_2 = 0.2987 \; ve \; \lambda_3 = 2.5471$.

$$a_0 + 0.1542\, a_1 + 0.023778\, a_2 = 0.0000000076005$$
$$a_0 + 0.2987 a_1 + 0.089222\, a_2 = 0.0000056539534$$
$$a_0 + 2.5471\, a_1 + 6.487718\, a_2 = 11493.703$$

Whentheequationsystem is solved,a_0=98.3960, a_1=-967.5286anda_2=2136.2980areobtained.Allvaluesobtainedareplaced in theequation.

f(A)=$a_0 I + a_1 A + a_2 A^2$

equation, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ andwhenthe $A^2 = \begin{bmatrix} 2.3005 & 2.2142 & 2.1495 \\ 2.2142 & 2.1904 & 2.0630 \\ 2.1495 & 2.0630 & 2.1098 \end{bmatrix}$ valuesareplaced,

$$a_0I + a_1A + a_2A^2 = 98.3960\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - 967.5286\begin{bmatrix}1 & 0.831 & 0.781\\0.831 & 1 & 0.707\\0.781 & 0.707 & 1\end{bmatrix} + 2136.2980\begin{bmatrix}2.3005 & 2.2142 & 2.1495\\2.2142 & 2.1904 & 2.0630\\2.1495 & 2.0630 & 2.1098\end{bmatrix}$$

$$A^{10} = \begin{bmatrix}4045.5758 & 3926.1961 & 3836.4582\\3926.1961 & 3810.3391 & 3723.2493\\3836.4582 & 3723.2493 & 3638.1501\end{bmatrix}$$

isfound.

EigenvaluesareusedagaintocalculateA^{-4}, andsimilaroperationsareapplied. Eigenvalues, $\lambda_1 = 0.1542,\ \lambda_2 = 0.2987$ ve$\lambda_3 = 2.5471$are,

$$a_0 + 0.1542\ a_1 + 0.023778\ a_2 = 1768.734$$
$$a_0 + 0.2987a_1 + 0.089222\ a_2 = 125.62$$
$$a_0 + 2.5471\ a_1 + 6.487718\ a_2 = 0.02376$$

Whentheequationsystem is solved, a_0=3739.945, a_1=-13512.6anda_2=4728.659areobtained. Allvaluesobtainedareplaced in theequation.

$$a_0I + a_1A + a_2A^2 = 3739.945\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - 13512.6\begin{bmatrix}1 & 0.831 & 0.781\\0.831 & 1 & 0.707\\0.781 & 0.707 & 1\end{bmatrix} + 4728.659\begin{bmatrix}2.3005 & 2.2142 & 2.1495\\2.2142 & 2.1904 & 2.0630\\2.1495 & 2.0630 & 2.1098\end{bmatrix}$$

$$A^{-4} = \begin{bmatrix}1105.729 & -758.93 & -389.008\\-758.93 & 585.047 & 201.867\\-389.008 & 201.867 & 203.917\end{bmatrix}$$

$\sqrt{A}$ifcalculated in,

Eigenvalues, $\lambda_1 = 0.1542,\ \lambda_2 = 0.2987\ and \lambda_3 = 2.5471$are,

$$a_0 + 0.1542\ a_1 + 0.023778\ a_2 = 0.3297$$
$$a_0 + 0.2987a_1 + 0.089222\ a_2 = 0.5465$$
$$a_0 + 2.5471\ a_1 + 6.487718\ a_2 = 1.596$$

Fromthesolution of thisequation, a_0=0.0785, a_1=1.696ana$_2$=-0.4319areobtained.

$$a_0I + a_1A + a_2A^2 = 0.0785\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + 1.696\begin{bmatrix}1 & 0.831 & 0.781\\0.831 & 1 & 0.707\\0.781 & 0.707 & 1\end{bmatrix} - 0.4319\begin{bmatrix}2.3005 & 2.2142 & 2.1495\\2.2142 & 2.1904 & 2.0630\\2.1495 & 2.0630 & 2.1098\end{bmatrix}$$

$$\sqrt{A} = \begin{bmatrix}0.82 & 0.4255 & 0.3828\\0.4255 & 0.8475 & 0.3172\\0.3828 & 0.3172 & 0.8677\end{bmatrix}$$

It is found. Theexp(A) e^Amatrix is alsocalculated as follows. Eigenvalues, $\lambda_1 = 0.1542,\ \lambda_2 = 0.2987\ and \lambda_3 = 2.5471$are,

$$a_0 + 0.1542\ a_1 + 0.023778\ a_2 = 1.1667$$
$$a_0 + 0.2987a_1 + 0.089222\ a_2 = 1.3481$$
$$a_0 + 2.5471\ a_1 + 6.487718\ a_2 = 12.77$$

whentheequationsystem issolved, a_0=1.0467, a_1=0.5315anda_2=1.5983 areobtained. Allvaluesobtainedareplaced in theequation.

$$a_0I + a_1A + a_2A^2 = 1.0467\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + 0.5315\begin{bmatrix}1 & 0.831 & 0.781\\0.831 & 1 & 0.707\\0.781 & 0.707 & 1\end{bmatrix} + 1.5983\begin{bmatrix}2.3005 & 2.2142 & 2.1495\\2.2142 & 2.1904 & 2.0630\\2.1495 & 2.0630 & 2.1098\end{bmatrix}$$

$$e^A = \begin{bmatrix}5.255 & 3.981 & 3.851\\3.981 & 5.079 & 3.673\\3.851 & 3.673 & 4.950\end{bmatrix}$$

isfound.

In a studyconducted in Iran, thecorrelationcoefficientbetween body weightandwitherheight in theMehrabanisheepbreedwas 0.91, thecorrelationbetween body weightandchestcircumferencewas 0.97, andthecorrelationcoefficientbetweenwitherheightandbreastcircumferencewas 0.85 [20]. Therelated (3*3) dimensionalsquarematrix is as follows.

$$A = \begin{bmatrix}1 & 0.91 & 0.97\\0.91 & 1 & 0.85\\0.97 & 0.85 & 1\end{bmatrix}$$

A^7 an$\sqrt[3]{A}$matrices of thismatrix can be calculated.First, theeigenvalues of thematrixarefoundfromthedet$(A - I\lambda=0$equation.

$$|A - I\lambda| = \begin{vmatrix}1-\lambda & 0.91 & 0.97\\0.91 & 1-\lambda & 0.85\\0.97 & 0.85 & 1-\lambda\end{vmatrix} = 0$$

From here$\lambda_1 = 0.0203$, $\lambda_2 = 0.1588$and$\lambda_3 = 2.8209$eigenvaluesareobtained.IfA^7 is calculated,

$$a_0 + 0.0203\,a_1 + 0.000412\,a_2 = 0.00000000000142$$
$$a_0 + 0.1588a_1 + 0.025217\,a_2 = 0.00000255$$
$$a_0 + 2.8209\,a_1 + 7.957477\,a_2 = 1421.392$$

whentheequationsystem is solved, a_0=0.6146, a_1=-34.145anda_2=190.650 areobtained. Thesevaluesareplaced in theequation.

$$a_0I + a_1A + a_2A^2 = 1.6146\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - 34.145\begin{bmatrix}1 & 0.91 & 0.97\\0.91 & 1 & 0.85\\0.97 & 0.85 & 1\end{bmatrix} + 190.6505\begin{bmatrix}2.7690 & 2.6445 & 2.7135\\2.6445 & 2.5506 & 2.5827\\2.7135 & 2.5827 & 2.6634\end{bmatrix}$$

$$A^7 = \begin{bmatrix}1.6146 & 0 & 0\\0 & 1.6146 & 0\\0 & 0 & 1.6146\end{bmatrix} + \begin{bmatrix}-34.1450 & -31.0720 & -33.1207\\-31.0720 & -34.1450 & -29.0233\\-33.1207 & -29.0233 & -34.1450\end{bmatrix} + \begin{bmatrix}527.9112 & 504.1752 & 517.3301\\504.1752 & 486.2732 & 492.3930\\517.3301 & 492.3930 & 507.7785\end{bmatrix}$$

$$A^7 = \begin{bmatrix}494.3628 & 473.0865 & 484.1921\\473.0865 & 452.7258 & 463.3535\\484.1921 & 463.3535 & 474.2306\end{bmatrix}$$

isfound.

$\sqrt[3]{A}$tocalculate$\lambda_1 = 0.0203$, $\lambda_2 = 0.1588$andusingtheir$\lambda_3 = 2.8209$eigenvalues

$$a_0 + 0.0203\,a_1 + 0.000412\,a_2 = 0.2728$$
$$a_0 + 0.1588a_1 + 0.025217\,a_2 = 0.5415$$
$$a_0 + 2.8209\,a_1 + 7.957477\,a_2 = 1.4130$$

theequationsystem is solved. Theroots of thisequationarea$_0$=0.2316, a_1=2.0432anda$_2$=-0.5758.

$$a_0I + a_1A + a_2A^2 = 0.2316\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + 2.0432\begin{bmatrix}1 & 0.91 & 0.97\\0.91 & 1 & 0.85\\0.97 & 0.85 & 1\end{bmatrix} - 0.5758\begin{bmatrix}2.7690 & 2.6445 & 2.7135\\2.6445 & 2.5506 & 2.5827\\2.7135 & 2.5827 & 2.6634\end{bmatrix}$$

$$\sqrt[3]{A} = \begin{bmatrix} 0.2316 & 0 & 0 \\ 0 & 0.2316 & 0 \\ 0 & 0 & 0.2316 \end{bmatrix} + \begin{bmatrix} 2.0432 & 1.8593 & 1.98197 \\ 1.8593 & 2.0432 & 1.7367 \\ 1.9819 & 1.7367 & 2.0432 \end{bmatrix} + \begin{bmatrix} -1.5944 & -1.5227 & -1.5624 \\ -1.5227 & -1.4686 & -1.4871 \\ -1.5624 & -1.4871 & -1.5336 \end{bmatrix}$$

$$\sqrt[3]{A} = \begin{bmatrix} 0.6802 & 0.3365 & 0.4194 \\ 0.3365 & 0.8060 & 0.2495 \\ 0.4194 & 0.2495 & 0.7410 \end{bmatrix}$$

Body weight-body lengthcorrelationwas 0.64, body weight-withersheightcorrelationwas 0.61, and body length-withersheightcorrelationwas 0.62 in dairycattle in Western Kenya [21].Thematrixcreatedaccordingtothisinformation is

$$A = \begin{bmatrix} 1 & 0.64 & 0.61 \\ 0.64 & 1 & 0.62 \\ 0.61 & 0.62 & 1 \end{bmatrix}$$

A^{-6}andexp(A) matrices of thismatrix can be calculated. Again, eigenvalues of thematrixarefoundfromthe$\det(A - I\lambda) = 0$equation.

$$|A - I\lambda| = \begin{vmatrix} 1-\lambda & 0.64 & 0.61 \\ 0.64 & 1-\lambda & 0.62 \\ 0.61 & 0.62 & 1-\lambda \end{vmatrix} = 0$$

From here$\lambda_1 = 0.359$, $\lambda_2 = 0.3942$and$\lambda_3 = 2.2468$eigenvaluesareobtained.If$A^{-6}$is calculated,

$$a_0 + 0.359\ a_1 + 0.128888\ a_2 = 467.0523$$
$$a_0 + 0.3942\ a_1 + 0.155425\ a_2 = 266.3381$$
$$a_0 + 2.2468\ a_1 + 5.047887\ a_2 = 0.007774$$

whentheequationsystem is solved, a_0=2931.682, a_1=-7922.81anda_2=2945.646areobtained.Thesevaluesarereplaced in theequation.

$$a_0 I + a_1 A + a_2 A^2$$
$$= 2931.682 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 7922.81 \begin{bmatrix} 1 & 0.64 & 0.61 \\ 0.64 & 1 & 0.62 \\ 0.61 & 0.62 & 1 \end{bmatrix}$$
$$+ 2945.646 \begin{bmatrix} 1.7817 & 1.6582 & 1.6168 \\ 1.6582 & 1.7940 & 1.6304 \\ 1.6168 & 1.6304 & 1.7565 \end{bmatrix}$$

$$A^{-6} = \begin{bmatrix} 2931.682 & 0 & 0 \\ 0 & 2931.682 & 0 \\ 0 & 0 & 2931.682 \end{bmatrix} + \begin{bmatrix} -7922.81 & -5070.6 & -4832.91 \\ -5070.6 & -7922.81 & -4912.14 \\ -4832.91 & -4912.14 & -7922.81 \end{bmatrix}$$
$$+ \begin{bmatrix} 5248.257 & 4884.470 & 4762.520 \\ 4884.470 & 5284.489 & 4802.581 \\ 4762.520 & 4802.581 & 5174.027 \end{bmatrix}$$

$$A^{-6} = \begin{bmatrix} 257.1295 & -186.1282 & -70.3936 \\ -186.1282 & 293.3609 & -109.5610 \\ -70.3936 & -109.5610 & 182.8992 \end{bmatrix}$$

isfound.

e^Atocalculate, $\lambda_1 = 0.359$, $\lambda_2 = 0.3942$and$\lambda_3 = 2.2468$eigenvalues,

$$a_0 + 0.359\ a_1 + 0.128888\ a_2 = 1.4319$$
$$a_0 + 0.3942\ a_1 + 0.155425\ a_2 = 1.4833$$
$$a_0 + 2.2468\ a_1 + 5.047887\ a_2 = 9.457$$

whentheequationsystem is solved, a_0=1.1213, a_1=0.324anda_2=1.5071areobtained.Thesevaluesarereplaced in thefunction.

$$a_0 I + a_1 A + a_2 A^2 = 1.1213 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.324 \begin{bmatrix} 1 & 0.64 & 0.61 \\ 0.64 & 1 & 0.62 \\ 0.61 & 0.62 & 1 \end{bmatrix} + 1.5071 \begin{bmatrix} 1.7817 & 1.6582 & 1.6168 \\ 1.6582 & 1.7940 & 1.6304 \\ 1.6168 & 1.6304 & 1.7565 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 1.1213 & 0 & 0 \\ 0 & 1.1213 & 0 \\ 0 & 0 & 1.1213 \end{bmatrix} + \begin{bmatrix} 0.3240 & 0.2074 & 0.1976 \\ 0.2074 & 0.3240 & 0.2009 \\ 0.1976 & 0.2009 & 0.3240 \end{bmatrix} + \begin{bmatrix} 2.6852 & 2.4991 & 2.4367 \\ 2.4991 & 2.7037 & 2.4572 \\ 2.4367 & 2.4572 & 2.6472 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 4.1305 & 2.7064 & 2.6343 \\ 2.7064 & 4.1490 & 2.6581 \\ 2.6343 & 2.6581 & 4.0925 \end{bmatrix}$$

isfound.

In these applications, different strengths, different polynomials and functions of some matrices and correlation matrices consisting of relationships between body properties in animals were obtained by Cayley-Hamilton theorem.

In one of the studies made with this theorem, Cayley-Hamilton theorem has been extended to the Drazin inverse matrices and standard inverse matrices. The theorems can be extended to any integer powers k = 2, 3, … of the matrices [22]. (Kaczorek, 2016). In a research, Cayley-Hamilton theorem was generalized to any polynomial matrix of arbitrary degree with coefficients as square matrices of any order [23]. In another study, the Cayley-Hamilton theorem was formulated in the min-plus algebra. The result of the study display a slight difference from conventional algebra to the min-plus algebra, where the addition and multiplication operation are replaced by minimum and plus operation. Besides that, the formulation in conventional algebra is equal to zero whereas in the min-plus algebra cannot be equated to zero (Siswanto et al., 2021).

IV. Conclusion

Cayley–Hamilton theorem has been extended for real polynomial matrices in 2 and 3 variables. It has been displayed that the known extensions of the Cayley–Hamilton theorem are particularcases of the proposed extension. Application of the desired extension has been characterized by instances in correlation matrices from livestock studies.

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