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Research Paper

A New Family of Optimal Eighth-Order Iterative Method for Solving Nonlinear Equations

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ABSTRACT :*In this paper, a new family of optimal eighth-order iterative methods for finding the roots of nonlinear equations is presented. This method is developed by modifying Bawazir's method, using a linear combination to combine Bawzir and Ostrowski methods, and adding Newton's method as the third step with an approximation of the derivatives. Some numerical comparisons were considered to demonstrate the performance of the proposed method.*
KEYWORDS: *Order of convergence, Optimal eighth-order, Iterative methods, Nonlinear equations, Efficiency index.*

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I. INTRODUCTION

Finding a simple root of nonlinear equations is one of the most important and the oldest problems in numerical analysis. These problems are common in engineering, applied mathematics, and scientific computing. In scientific departments, a need arises for solving nonlinear equations

$$f(x)=0, \qquad (1)$$

where $f:\mathbf{D}\subset \mathbf{R}\rightarrow\mathbf{R}$ for an open interval $\mathbf{D}$ [1-12].
Newton's method (NM) is a fundamental method for solving the nonlinear equation, given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}. \qquad (2)$$

Newton's method converges quadratically. The efficiency index [1] can be defined by $I=p^{\frac{1}{d}}$, such that p is the order of convergence and d is the number of total function evaluations per iteration. The optimal order can be calculated using Kung-Traub[2] conjectured which is given by 2^{d-1}.The efficiency index of the optimal method (2) is $I=2^{\frac{1}{2}}\approx 1.4142$.
There are many methods of the optimal two-step fourth-order for solving nonlinear equations which depend on Newton's method in the first step. An optimal two steps method is proposed by Ostrowski[3], given by

$$x_{n+1}=x_n-\frac{f(x_n)\left[f(x_n)-f(y_n)\right]}{f'(x_n)\left[f(x_n)-2f(y_n)\right]}. \qquad (3)$$

Ostrowski's method is an optimal fourth order of convergence, and it has an efficiency index $I=4^{\frac{1}{3}}\approx 1.5874$.
Recently, a fifth-order method depending on double Newton's method has been developed by Bawazir [4] given by

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1}=y_n-\frac{f(y_n)}{f'(y_n)}\left[1+\frac{f(y_n)\left(f'(x_n)-f'(y_n)\right)}{2f(x_n)f'(y_n)}\right]. \qquad (4)$$

In recent years, many eight-order schemes for finding the roots of nonlinear equations were improved see [5–9] and the references therein.
In this paper, we will present a new family of optimal eighth-order methods which depends on Newton's, Bawzir [4], Ostrowski [3], and derivative approximations.

II. CONSTRUCTION OF NEW ITERATIVE METHOD

Our aim is to construct a new family of optimal eighth-order of convergence, we designed this method by using a composition of three steps. In the first step, we will decrease the functional evaluations number of (4) by approximating $f'(y_n)$[10], so the order will be reduced to four instead of five.

$$f'(y_n)\approx\frac{f'(x_n)f(x_n)^2}{f(x_n)^2+2f(x_n)f(y_n)+f(y_n)^2}. \quad (5)$$

By substituting (5) in (4) we will get

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1}=y_n-\frac{f(y_n)\left(f(x_n)^3+f(y_n)^2f(x_n)+\frac{1}{2}f(y_n)^3\right)\left(f(x_n)+f(y_n)\right)^2}{f'(x_n)f(x_n)^5}. \quad (6)$$

This is considered as an optimal fourth-order of convergence. Then, we will combine (3) and (6) using a linear combination.

$$z_n=x_n+(\beta-1)\left(\frac{f(x_n)[f(x_n)-f(y_n)]}{f'(x_n)[f(x_n)-2f(y_n)]}\right)-\beta\left(\frac{f(x_n)}{f'(x_n)}+\frac{f(y_n)\left(f(x_n)^3+f(y_n)^2f(x_n)+\frac{1}{2}f(y_n)^3\right)\left(f(x_n)+f(y_n)\right)^2}{f'(x_n)f(x_n)^5}\right). \quad (7)$$

Where $\beta \in R$ is the adjusting parameter. When $\beta = 0$, the method proposed above reduces to the method (3), and when $\beta = 1$, it gives the method (6). It is noticeable that the methods which are given in equations (3) and (6) are fourth-order convergence methods. The above method (7) is an optimal fourth-order convergence, and its performance depends on a suitable choice of β.
To obtain an eight order of convergence we will add Newton's Method as a third step

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$

$$z_n=x_n+(\beta-1)\left(\frac{f(x_n)[f(x_n)-f(y_n)]}{f'(x_n)[f(x_n)-2f(y_n)]}\right)-\beta\left(\frac{f(x_n)}{f'(x_n)}+\frac{f(y_n)\left(f(x_n)^3+f(y_n)^2f(x_n)+\frac{1}{2}f(y_n)^3\right)\left(f(x_n)+f(y_n)\right)^2}{f'(x_n)f(x_n)^5}\right),$$

$$x_{n+1}=z_n-\frac{f(z_n)}{f'(z_n)}. \quad (8)$$

The above method (8) has eighth-order of convergence with five functions evaluations. Therefore, this method is not optimal. To decrease the functional evaluations number, we need to approximate$f'(z_n)$. Sivakumaret al[11] have proved an approximate value for $f'(z_n)$.

$$f'(z_n)\approx q'(z_n)=a_1+2a_2(z_n-x_n)+3a_3(z_n-x_n)^2. \quad (9)$$

Where:

$$a_1=f'(x_n), \quad (10)$$

$$a_2=\frac{f[y_n,x_n,x_n](z_n-x_n)-f[z_n,x_n,x_n](y_n-x_n)}{z_n-y_n}, \quad (11)$$

$$a_3=\frac{f[z_n,x_n,x_n]-f[y_n,x_n,x_n]}{z_n-y_n}. \qquad (12)$$

Where the divided differences can be calculated by:

$$\begin{cases} f[y_n,x_n]=\dfrac{f(y_n)-f(x_n)}{y_n-x_n}, & (13)\\ f[y_n,x_n,x_n]=\dfrac{f[y_n,x_n]-f'(x_n)}{y_n-x_n}, & (14)\\ f[z_n,x_n]=\dfrac{f(z_n)-f(x_n)}{z_n-x_n}, & (15)\\ f[z_n,x_n,x_n]=\dfrac{f[z_n,x_n]-f'(x_n)}{z_n-x_n}. & (16)\end{cases}$$

Finally, by substituting the approximation (9) in the last step of (8), we will obtain the new family of optimal eight-order methods.

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$

$$z_n=x_n+(\beta-1)\left(\frac{f(x_n)\left[f(x_n)-f(y_n)\right]}{f'(x_n)\left[f(x_n)-2f(y_n)\right]}\right)-\beta\left(\frac{f(x_n)}{f'(x_n)}+\frac{f(y_n)\left(f(x_n)^3+f(y_n)^2f(x_n)+\frac{1}{2}f(y_n)^3\right)\left(f(x_n)+f(y_n)\right)^2}{f'(x_n)f(x_n)^5}\right),$$

$$x_{n+1}=z_n-\frac{f(z_n)}{q'(z_n)}. \qquad (17)$$

The Efficiency Index of the new method (17) is $I=8^{\frac{1}{4}}\approx 1.682$.

III. CONVERGENCE ANALYSIS AND PROOF

In this section, we have used Maple (2018) scripts to prove the convergence order of the provided method (HSM), which is given by (17).

3.1 Theorem

Let α be a simple root of f(x)=0 in an open interval D. If x_0 is close enough to α then the method (17) has a convergence order of eight when $\beta \in R$, $\beta\neq 0$ and $\beta\neq 1$.

3.2 Proof of Theorem

Consider the $e_n= x_0-\alpha$ to be the error at nth iteration.

Expanding f(x) about α by Taylor expansion, we will have

$$f(x)=f'(\alpha)(O(e_n^9)+c_8e_n^8+c_7e_n^7+c_6e_n^6+c_5e_n^5+c_4e_n^4+c_3e_n^3+c_2e_n^2+e_n). \qquad (18)$$

Where $c_n=\frac{f^{(n)}(\alpha)}{n!\,f'(\alpha)}$, n=2,3,…

$$f'(x)=f'(\alpha)(O(e_n^9)+9c_9e_n^8+8c_8e_n^7+7c_7e_n^6+6c_6e_n^5+5c_5e_n^4+4c_4e_n^3+3c_3e_n^2+2c_2e_n+1). \qquad (19)$$

Dividing (18) by (19), we get

$$\frac{f(x)}{f'(x)}= e_n-c_2e_n^2+\ldots+\left(-64c_2^7+304c_2^5c_3-176c_2^4c_4-408c_2^3c_3^2+\ldots+31c_4c_5-7c_8\right)e_n^8+O(e_n^9). \qquad (20)$$

Substituting the last equation (20), into first step of (17), we have

$$y_n= \alpha+c_2e_n^2+\left(-2c_2^2+2c_3\right)e_n^3+\ldots+\left(64c_2^7-304c_2^5c_3+\ldots+7c_8\right)e_n^8+O(e_n^9). \qquad (21)$$

Expanding $f(y_n)$ about α to get

$$f(y)=f'(\alpha)\left[c_2e_n^2+\left(-2c_2^2+2c_3\right)e_n^3+\ldots+\left(144c_2^7-552c_2^5c_3+\ldots+7c_8\right)e_n^8+O(e_n^9)\right]. \qquad (22)$$

Substituting (18), (19), (22), into second step of (17), we have

$$z_n= \alpha+\left(2\beta c_2^3+c_2^3-c_2c_3\right)e_n{}^4+\ldots+\left(-792\beta c_2^2c_3c_4+\ldots+\left(\frac{1575}{2}\beta\right)c_2^4c_4\right)e_n{}^8+O(e_n{}^9). \quad (23)$$

From (23), we will have

$$f(z) =f'(\alpha)[\left(2\beta c_2^3+c_2^3-c_2c_3\right)e_n{}^4+\ldots+\left(-209c_2^2c_3c_4+\ldots+4\beta^2c_2^7\right)e_n{}^8+O(e_n{}^9)]. \quad (24)$$

From (18), and (21)-(24), we can find

$$f[y_n,x_n]= f'(\alpha)\left[1+c_2e_n+\ldots+\left(112c_2^5c_3-74c_2^4c_4-94c_2^3c_3^2+\ldots+92c_2^2c_3c_4-30c_2c_3c_5\right)e_n{}^7+O(e_n{}^8)\right]. \quad (25)$$

$$f[z_n,x_n]=f'(\alpha)\left[1+c_2e_n+\ldots+\left(1247\beta c_2^5c_3\left(\frac{1}{2}\right)-329\beta c_2^4c_4\left(\frac{1}{2}\right)-380\beta c_2^3c_3^2+\ldots+102\beta c_2^2c_3c_4\right)e_n{}^7+O(e_n{}^8)\right]. \quad (26)$$

From (19), (21), (23), (25), and (26), we have

$$f[y_n,x_n,x_n]= f'(\alpha)\left[c_2+2c_3e_n+\ldots+\left(16c_2^5c_3-4c_2^4c_4-52c_2^3c_3^2+\ldots+46c_2^2c_3c_4-26c_2c_3c_5\right)e_n{}^6+O(e_n{}^7)\right]. \quad (27)$$

$$f[z_n,x_n,x_n]=f'(\alpha)\left[c_2+2c_3e_n+\ldots+\left(133\beta c_2^5c_3\left(\frac{1}{2}\right)-\ldots+42\beta c_2^2c_3c_4\right)e_n{}^6+O(e_n{}^7)\right]. \quad (28)$$

In view of (21), (23), (27), and (28), we obtain

$$a_2= f'(\alpha)\left[c_2+3c_3e_n+5c_4e_n{}^2+(c_2c_4+7c_5)e_n{}^3+\ldots+\left(2\beta c_2^3c_4+5c_2^3c_4-3c_2^2c_5+\ldots+4c_3c_5\right)e_n{}^5+O(e_n{}^6)\right]. \quad (29)$$

$$a_3= f'(\alpha)\left[c_3+2c_4e_n+(c_2c_4+3c_5)e_n{}^2+\ldots+\left(2\beta c_2^3c_4+5c_2^3c_4-3c_2^2c_5+\ldots+4c_3c_5\right)e_n{}^4+O(e_n{}^5)\right]. \quad (30)$$

Substituting (10), (23), (29), and (30) into (9), we have

$$q'(z_n)=f'(\alpha)\left[1+\left(4\beta c_2^4+2c_2^4-2c_2^2c_3+c_2c_4\right)e_n{}^4+\ldots+\left(-60c_2^4c_3+\ldots+133\beta c_2^6-20c_2c_3c_4\right)e_n{}^6+O(e_n{}^7)\right]. \quad (31)$$

Finally, by substituting (23), (24), and (31) into the last step of (17), we have

$$x_{n+1}=\alpha+\left(4\beta^2c_2^7+4\beta c_2^7-4\beta c_2^5c_3+c_2^7+2\beta c_2^4c_4-2c_2^5c_3+c_2^4c_4+c_2^3c_3^2-c_2^2c_3c_4\right)e_n{}^8+O(e_n{}^9). \quad (32)$$

This completes the proof.

IV. NUMERICALEXAMPLES

In this section, we investigate the validity and efficiency of the new proposed family of optimal eighth-order of convergence methods (17) by considering some nonlinear equations. We compare the performance of two cases of the optimal eighth-order method (17) for $\beta = 2$ (HSM1) and $\beta = -2$ (HSM2), with the following eighth-order methods for the purpose of comparison:

Method proposed by Kung-Traub [2](KTM):

$$\begin{aligned} y_n&=x_n-\frac{f(x_n)}{f'(x_n)},\\ z_n&=y_n-\frac{f(y_n)f(x_n)}{\left(f(x_n)-f(y_n)\right)^2}\frac{f(x_n)}{f'(x_n)},\\ x_{n+1}&= z_n-\frac{f(x_n)}{f'(x_n)}\frac{f(x_n)f(y_n)f(z_n)}{\left(f(x_n)-f(y_n)\right)^2}\frac{f(x_n)^2+f(y_n)\left(f(y_n)-f(z_n)\right)}{\left(f(x_n)-f(z_n)\right)^2\left(f(y_n)-f(z_n)\right)}. \end{aligned} \quad (33)$$

Method given by Liu et al [7] (LWM):

$$\begin{aligned} y_n&=x_n-\frac{f(x_n)}{f'(x_n)},\\ z_n&=y_n-\frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)},\\ x_{n+1}&=z_n-\frac{f(z_n)}{f'(x_n)}\left(\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right)^2+\frac{f(z_n)}{f(y_n)-f(z_n)}+\frac{4f(z_n)}{f(x_n)+f(z_n)}\right). \end{aligned} \quad (34)$$

Method proposed by Al-Harbi et al [8] (TSM):

$$\begin{aligned} y_n&=x_n-\frac{f(x_n)}{f'(x_n)},\\ z_n&=y_n-\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \end{aligned}$$

$$x_{n+1}=z_n-\{-2\beta s_1^3 e^{s_1}(s_2^4+s_2^4\sin(s_3)+\sin(s_3)+1)+(s_2^4+1)(\sin(s_3)+1)\}\frac{f(z_n)f[x_n,y_n]}{f[y_n,z_n]f[x_n,z_n]}. \quad (35)$$

Where $s_1=\frac{f(y_n)}{f(x_n)}$, $s_2=\frac{f(z_n)}{f(y_n)}$,$s_3=\frac{f(z_n)}{f(x_n)}$, and β=0.

Method proposed by Sharma et al [5] (SAM):

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$
$$z_n=y_n-\left(3-2\frac{f[y_n,x_n]}{f'(x_n)}\right)\frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1}=z_n-\frac{f(z_n)}{f'(x_n)}\left(\frac{f'(x_n)-f[y_n,x_n]+f[z_n,y_n]}{2f[z_n,y_n]-f[z_n,x_n]}\right). \quad (36)$$

Method proposed by Parimala et al [6] (PMJ):

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$
$$z_n=x_n-\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1}=z_n-\frac{f(z_n)(z_n-y_n)}{f(z_n)-f(y_n)}(1+2\eta)\left(1+\tau^2+2\tau^3+\left(\frac{7}{24}\right)\tau^4\right). \quad (37)$$

Where $\eta=\frac{f(z_n)}{f(x_n)}$, and $\tau=\frac{f(y_n)}{f(x_n)}$.

The test functions and their exact root are shown in Table1with only fifteen decimal digits as follows.

Table 1. Test functions and their exact root

Functions	Roots
$f_1(x)=\sin x^2-x^2+1$	α = 1.40449164821534
$f_2(x)=(x-1)^3-1$	α =2.0
$f_3(x)=\sin^{-1}(x^2-1)-\frac{x}{2}+1$	α = -0.296550195139443
$f_4(x)=\cos(x)-x$	α =0.739085133215161
$f_5(x)=10xe^{(-x^2)}-1$	α =1.67963061042845
$f_6(x)=x^3+\log(1+x)$	α =0

All computations were done by Matlab (R2018a) software, using 1000 digits. The stopping criteria
$|x_n-\alpha| \leq 10^{-300}$,
$|f(x_n)| \leq 10^{-300}$.
As shown in Table2, the number of iterations (IT), the absolute value of the function $|f(x_n)|$, and the absolute error $|x_n-\alpha|$. Furthermore, the computational order of convergence (COC)[12] which approximated by

$$\rho=\frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n+1}-\alpha)|}.$$

Table 2. Comparison of various iterative methods

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n-\alpha\rvert$	COC
$f_1(x)=\sin x^2-x^2+1$	x_0=1.3			
PMJ	3	1.06388e-482	4.28557e-483	8
TSM	-	-	-	-
KTM	3	1.77865e-473	7.16484e-474	8
SAM	3	3.08327e-420	1.24202e-420	8

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - \alpha\rvert$	COC
LWM	3	1.25968e-470	5.0743e-471	8
HSM1	3	5.81726e-430	2.34333e-430	8
HSM2	3	8.67307e-448	3.49372e-448	8
$f_2(x)=(x-1)^3-1$	$x_0=2.2$			
PMJ	3	5.65132e-358	1.88377e-358	8
TSM	3	1.2849e-382	4.28301e-383	8
KTM	3	2.05589e-341	6.85298e-342	8
SAM	3	2.31099e-321	7.70331e-322	8
LWM	3	1.61685e-342	5.38949e-343	8
HSM1	3	4.28696e-308	1.42899e-308	8
HSM2	3	1.94936e-341	6.49788e-342	8
$f_3(x)=\sin^{-1}(x^2-1)-\frac{x}{2}+1$	$x_0=-0.48$			
PMJ	3	8.46194e-756	4.34755e-756	8
TSM	3	3.13618e-761	1.6113e-761	8
KTM	3	5.44064e-788	2.79528e-788	8
SAM	3	4.87452e-728	2.50441e-728	8
LWM	3	2.77271e-719	1.42455e-719	8
HSM1	3	9.12729e-743	4.68939e-743	8
HSM2	3	4.07989e-723	2.09615e-723	8
$f_4(x)=\cos(x)-x$	$x_0=0.6$			
PMJ	3	1.13203e-655	6.76397e-656	8
TSM	3	1.01004e-673	6.03511e-674	8
KTM	3	2.97301e-646	1.7764e-646	8
SAM	3	9.31564e-622	5.56619e-622	8
LWM	3	2.73271e-626	1.63282e-626	8
HSM1	3	4.28048e-644	2.55763e-644	8
HSM2	3	4.6892e-685	2.80184e-685	8
$f_5(x)=10xe^{(-x^2)}-1$	$x_0=1.5$			
PMJ	3	4.49239e-381	1.62539e-381	8
TSM	3	7.53701e-399	2.72696e-399	8
KTM	3	9.10897e-369	3.2957e-369	8
SAM	3	3.18862e-347	1.15367e-347	8
LWM	3	7.00539e-355	2.53461e-355	8
HSM1	3	7.26037e-357	2.62687e-357	8
HSM2	3	1.70714e-391	6.1766e-392	8
$f_6(x)= x^3+\log(1+x)$	$x_0=0.25$			
PMJ	3	1.70259e-422	1.70259e-422	8
TSM	3	1.98495e-426	1.98495e-426	8
KTM	3	1.29102e-432	1.29102e-432	8
SAM	3	1.34383e-358	1.34383e-358	8
LWM	3	1.28561e-415	1.28561e-415	8
HSM1	3	2.83948e-460	2.83948e-460	8
HSM2	3	3.72441e-434	3.72441e-434	8

V. CONCLUSION

In this paper, we have proposed a new family of optimal eighth-order of convergence for finding the roots of non-linear equations. The newly proposed family is obtained by approximating $f^{'}(y_n)$, using the technique of linear combination, and approximating $f^{'}(z_n)$. Numerical results were presented to illustrate the performance and efficiency of the newly proposed method.

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