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Research Paper

On the Fermat Quartic Equation $544x^4 + y^4 = z^4$

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Abstract: *In this paper, we show that the only primitive non-zero integer solutions to the Fermat quartic equation* $544x^4 + y^4 = z^4$ *are* $(x, y, z) = (\pm 1, \pm 3, \pm 5)$.

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I. Introduction

A Diophantine equation of the form $ax^4 + by^4 = cz^4$, where a, b and c are fixed non-zero fourth power free integers, is also known as Fermat quartic equation. We define a nonzero integer solution to this equation as a solution (x_0, y_0, z_0) where ax_0, by_0 and cz_0 are pairwise relatively prime with $x_0 y_0 z_0 \neq 0$. Darmon and Gravillen gave the theorem to conclude that a Fermat quartic equation has only a finite number of primitive non-zero solutions [2]. When $a = b = c = 1$ and $n = 4$, we recognize Fermat's last theorem.

II. Main Results

Theorem 1. The only Primitive non-zero integer solutions to the Fermat quartic equation $544x^4 + y^4 = z^4$ are $(x, y, z) = (\pm 1, \pm 3, \pm 5)$.

Proof. We have $544x^4 + y^4 = z^4 \Longrightarrow (z^2 + y^2).(z + y).(z - y)$

After the congruence considerations, we recognize that, x, y and z are odd.

Furthermore, $544\ x$, y and z are pairwise relatively prime. Let $z = p + q$ and $y = p - q$ where $p \not\equiv q(mod2)$ and $(p, q) = 1$ since $(y, z) = 1$

Hence we get,

$$544x^4 = 2.(p^2 + q^2).2p.2q$$

$$68x^4 = (p^2 + q^2).p.q$$

<u>Case I.</u> p is even and q is odd.

Hence $p = 2^2.t$ since x is even and we get

$$17x^4 = (p^2 + q^2).t.q \quad (1)$$

Since $p^2 + q^2$, t and q are pairwise relatively prime, we can distinguish three different cases.

Subcase 1. $17|q \Rightarrow q = 17d$ and from (1), we have

$$x^4 = (p^2 + q^2).t.d$$

Hence,

$$p^2 + q^2 = e^4 \quad (2)$$

Since p and q are squared in equation (2), we may assume that p and q are positive. Hence, we have $p = 4t$ and $t = f^4 \Rightarrow p = 4f^4$

Thus from (2), we have

$$(4f^4)^2 + q^2 = e^4 \Rightarrow q^2 = e^4 - (2f^2)^4$$

which has no non-zero solution according to [6].

Subcase 2. $17|t \Rightarrow t = 17g$ and insert this t value in equation (1), we have

$$x^4 = (p^2 + q^2).g.q$$

We have,

$$p^2 + q^2 = h^4 \qquad (3)$$

Since p and q are squared in equation (2), we may assume that p and q are positive. Hence, we have $q = j^4$ and from (3), we get $p^2 = h^4 - (j^2)^4$, which has no non-zero solution according to [6].

Subcase 3. $17|p^2 + q^2$. Thus, we have

$$p^2 + q^2 = 17k^4 \qquad (4)$$

From (1), we get $x^4 = k^4.t.q$ and since the left-hand side is positive, we must have $t = l^4$ *and* $q = m^4$ or $t = -l^4$ *and* $q = -m^4$. If these substitutions are inserted in (4), we get since $p = 4t$

$$(\pm l^4)^2 + (\pm m^4)^2 = 17k^4$$

$$(2l^2)^4 + (m^2)^4 = 17k^4 \qquad (5)$$

Moreover, the equation $x^4 + y^4 = 17z^4$ has according to [3] has only primitive non-zero solutions $(x, y, z) = (\pm 1, \pm 2, \pm 1)$ and $(x, y, z) = (\pm 2, \pm 1, \pm 1)$. Hence $(2l^2, m^2, k) = (\pm 1, \pm 2, \pm 1)$ and $(2l^2, m^2, k) = (\pm 2, \pm 1, \pm 1)$. Since $2l^2 \neq \pm 1$ and $m^2 \neq \pm 2$, so that only the second alternative must be applicable on (5). Hence $2l^2 = 2 \Rightarrow l = \pm 1$ and $m^2 = 1 \Rightarrow m = \pm 1$. Thus, by the previous expressions of t, p and q we have $t = (\pm 1)^4 = 1$ and $q = (\pm 1)^4 = 1$ or $t = -(\pm 1)^4 = -1$ and $q = -(\pm 1)^4 = -1$. Since $p = 4t$, we get $p = 4$ and $q = 1$ or $p = -4$ and $q = -1$. Since $z = p + q$ and $y = p - q$, we get $z = 5$ and $y = 3$ or $z = -5$ and $y = -3$. Thus, we have $(z, y) = (5,3)$ and $(z, y) = (-5, -3)$.

<u>Case II.</u> p is odd and q is even.

Hence $q = 2^2.t$ since x is even and we get

$$17x^4 = (p^2 + q^2).t.p \qquad (6)$$

Since $p^2 + q^2$, t and p are pairwise relatively prime, we can classify three different cases.

Subcase 4. $17|t \Rightarrow t = 17A$ and from (6), we have

$$x^4 = (p^2 + q^2).A.p$$

Hence,

$$p^2 + q^2 = B^4 \qquad (7)$$

Since p and q are squared in equation (2), we may assume that p and q are positive. Hence we have $q = 4t$ and $t = C^4 \Rightarrow q = 4C^4$

Thus from (7), we have

$$p^2 + (4C^4)^2 = B^4 \Rightarrow p^2 = B^4 - (2C^2)^4$$

which has no non-zero solution according to [4].

Subcase 5. $17|p \Rightarrow p = 17D$ and from (6), we have

$$x^4 = (p^2 + q^2).D.t$$

We have,

$$p^2 + q^2 = E^4 \qquad (8)$$

As in subcase 4, we may assume that p is positive. Hence, we have $p = F^4$ and from (8), we get $q^2 = E^4 - (F^2)^4$, which has no non-zero solution according to [4].

Subcase 6. $17|p^2 + q^2$. Thus we have

$$p^2 + q^2 = 17G^4 \quad (9)$$

From (6), we get $x^4 = G^4.p.t$ and since the left hand side is positive, we must have $t = H^4 \, and \, p = J^4$ or $t = -H^4 \, and \, p = -J^4$. If these substitutions are inserted in (9), we get since $q = 4t$

$$(\pm J^4)^2 + (\pm 4H^4)^2 = 17G^4$$

$$(J^2)^4 + (2H^2)^4 = 17G^4 \quad (10)$$

Moreover, the equation $x^4 + y^4 = 17z^4$ has according to [3] has only primitive non-zero solutions $(x, y, z) = (\pm 1, \pm 2, \pm 1)$ and $(x, y, z) = (\pm 2, \pm 1, \pm 1)$. Hence $(J^2, 2H^2, G) = (\pm 1, \pm 2, \pm 1)$ and $(J^2, 2H^2, G) = (\pm 2, \pm 1, \pm 1)$. Since $2H^2 \neq \pm 1$ and $J^2 \neq \pm 2$, so that only the first alternative must be applicable on (10). Hence $2H^2 = 2 \Rightarrow H = \pm 1$ and $J^2 = 1 \Rightarrow J = \pm 1$. Thus, by the previous expressions of t, p and q we have $t = (\pm 1)^4 = 1$ and $p = (\pm 1)^4 = 1$ or $t = -(\pm 1)^4 = -1$ and $p = -(\pm 1)^4 = -1$. Since $q = 4t$, we get $q = 4$ and $p = 1$ or $q = -4$ and $p = -1$. Since $z = p + q$ and $y = p - q$, we get $z = 1 + 4 = 5$ and $y = 1 - 4 = -3$ or $z = -1 - 4 = -5$ and $y = -1 - (-4) = 3$. Thus, we have $(z, y) = (5, -3)$ and $(z, y) = (-5, 3)$.

Finally, from the cases I and II, we see that $544x^4 = z^4 - y^4 = (\pm 5)^4 - (\pm 3)^4 \Rightarrow x = \pm 1$ and this completes the proof of Theorem 1.

III. Conclusion

The primitive non-zero integer solutions to the Diophantine equation $ax^m + by^n = cz^p$ is a matter of great concern. By using elementary number theory methods, we solved the primitive non-zero integer solution on the Diophantine equation when $a = 544, b = c = 1, m = n = p = 4$ has the only solution $(x, y, z) = (\pm 1, \pm 3, \pm 5)$.

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