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Research Paper

Stability and Soliton Solutions of Generalized Higher-Order DNLS Equation

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Abstract:
This study investigates exact solitary wave and soliton solutions of the generalized higher-order nonlinear Schrödinger (DNLS) equation, which models ultrashort pulse propagation in optical fibers. Using a modified extended direct algebraic technique, we derive novel bright, dark, and bright-dark soliton solutions even when the balance number is not a positive integer. Stability analysis is performed via Hamiltonian and modulation instability methods. Graphical representations illustrate the evolution and interaction of these solitons. The results confirm that the proposed method is efficient, consistent, and applicable to a wide range of nonlinear physical systems in optics, plasma physics, and fluid dynamics.

Keywords:
Generalized DNLS equation, solitary waves, bright-dark solitons, stability analysis, modulation instability, optical fibers, modified algebraic method.

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I. Introduction

Generalized derivative higher order non-linear Shrödinger (DNLS) equation, which are very important in physics and mathematics (pluses propagation in optical fibers). differential equations, the nonlinear partial differential equations (PDEs) have been applied in fields such as optical fibers, quantum mechanics, quantum field theory, high energy physics, fluids dynamics, biophysics, electromagnetic theory, plasma physics, electrochemistry. A lot of physical models have supported a broad variety of solitary wave and soliton solutions. Therefore, the study of the solitary wave solutions to nonlinear Partial differential equations (PDEs) plays a vital role in mathematical physics. In particular, there has been considerable interest in investigating exact traveling wave solutions of nonlinear Partial differential equations (PDEs). By using a modified extended direct algebraic technique, the novel solitary wave and bright-dark soliton solutions of the generalized higher order nonlinear equation whose balance number is not a positive integer are obtained. These exact solutions have several applications in mathematical physics, optical fibers, telephony, and other disciplines of physics as detailed and they aid in our understanding of the intricate physical processes exhibited by this equation [1]. Partial differential equations (PDEs) are used to mathematically formulate physical and other problems involving functions of several variables, such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, and electrodynamics. [2]
Preliminaries (1.1) [4] We introduce notations, definitions and preliminary facts which are used throughout this research. The study presents some definitions as follows.
Shrödinger Equation (1.2) [4] We consider the electron moving in an orbit the probability of finding it in a particular place was emphasized.
Definition (1.3) [4] Shrödinger presented the wave equation properties of the electron in the form of an equation called¨ the Shrödinger wave equation to calculate the probability of electrons at different places around the nucleus in the form equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}+\frac{8\pi^2}{h^2}m(E-v)\varphi=0 \tag{1}$$

were,
x, y, z = space coordinates;
m = mass of electron;
h = Planck constant;
E = total energy;
φ = electrons wave behaviour can describe in terms of wave function.

Derivative of Shrödinger Equation (1.4) [6] Shrödinger derived on equation based on be-Broglie dual behaviour of electrons and Heisenberg¨ uncertainty principle is known as Shrödinger wave equation" The equation named after Erwin¨ Shrödinger who postulated the equation in 1925 and published in 1926, forming the basis for¨ the work that resulted in his Nobel Price in the Physics in 1933.The equation derived from the total energy of electron or any particles are kinetic equation (K.E) and potential equation (P.E), the energy equation is given by: Total Energy= Kinetic Equation (K.E) + Potential Equation (P.E)

(2)

$$E = \frac{1}{2}mV^2 + mgh$$

$$E = \frac{1}{2}mV^2 + V \tag{3}$$

where, m = mass of object, g is 9.8 m/s^2 (gravitational acceleration constant), h s height and also assume that V = mgh and V = speed of object Kinetic energy can also be represented in the following way:

$$\frac{1}{2}mV^2 = \frac{1}{2}m\frac{mV^2}{m} \tag{4}$$

$$\frac{1}{2}mV^2 = \frac{1}{2}\frac{(mV)^2}{m}$$

$$\frac{1}{2}mV^2 = \frac{(mV)^2}{2m}$$

Now substituting the value of mV into the De-Broglie equation (mV =$\frac{h}{\lambda}$) the following equation is obtained

(5)

$$\frac{1}{2}mV^2 = \frac{[\frac{h}{\lambda}]^2}{2m}$$

$$\frac{1}{2}mV^2 = \frac{h^2}{2\lambda^2 m}$$

putting the value of equation (5) into equation (4)

$$E = \frac{h^2}{2\lambda^2 m} + V \tag{6}$$

$$\lambda^2 = \frac{\hat{h^2}}{(2m)(E-V)} \tag{7}$$

An element that moves with wave speed for this the wave motion in the x-direction can be written as follows;

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{4\pi^2 \varphi}{\lambda^2} = 0 \tag{8}$$

Substituting the value of l_2 from equation (7) into equation (8)

$$\frac{\partial^2 \varphi}{\partial x^2} + 4\pi^2 \varphi \frac{2m}{h^2}(E - V) = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{4\pi^2 2m}{h^2}(E - V)\varphi = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{8\pi^2 2m}{h^2}(E - V)\varphi = 0, where V = mgh \tag{9}$$

for all three axes x, y, z:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} + \frac{8\pi^2 2m}{h^2}(E - V)\varphi = 0 \tag{10}$$

The solution of any complex problem is possible through the Shrödinger equation in wave mechanics and Shrödinger used it to solve many problems related to particles like molecules, atoms, electrons, etc. are most importance of Shrödinger equation.

Derivative of Non-linear Shrödinger Equation (1.5) [6] The nonlinear Shrödinger equation (NLSE) is a nonlinear partial differential equation that fascinates physicists and applied mathematicians because of its applications to many different nonlinear phenomena in fluid mechanics, condensed matter physics, and nonlinear optics. The generalized derivative higher order non-linear Shrödinger (DNLS) equation describes¨ pluses propagation in optical fibers and can be regarded as a special case of the generalized higher order nonlinear Shrödinger equation. We derive a Lagrangian and the invariant variational principle for DNLS equation. Using the amplitude ansatz method, we obtain the different cases of the exact bright, dark and bright–dark solitary wave soliton solutions of the generalized higher order (DNLS) equation.

Derivative Generalized Higher Order NLSE (1.6) [6] Nonlinear partial differential equations are widely used to describe many important phenomena and dynamic processes in Physics, Mechanics, Chemistry, Biology, etc. The nonlinear Shrödinger equation and its relatives play an important role in mathematics and physics. Such as, the nonlinear Shrödinger equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics and so on. Mathematical approaches to partial differential equations are divided in to two methods called Analytical methods which strive to find exact formulae for the dependent variable as a function of independent variables and numerical methods which result in approximate values of dependent variable at prescribed and discrete location within a finite domain of the independent variables in this thesis focused on Analytical method. In order to solve the generalized higher order NLS equation, we suppose.

$$\begin{cases} q(x,t) = u(x,t)e^{i\phi(x,t)} \\ q^*(x,t) = u(x,t)e^{-i\phi(x,t)} \\ \phi(x,t) = kx - \omega t \end{cases} \tag{11}$$

where ∗ denotes a complex conjugate, φ is the linear phase shift function with k and ω are the normalized wave vector and frequency. Let us consider, first derivative of q (x, t).

$$i\frac{\partial q}{\partial x}+\beta_1\frac{\partial^2 q}{\partial t^2}+\beta_2|q|^2q+i\varepsilon\beta_3\frac{\partial^3 q}{\partial t^3}+i\varepsilon\beta_4|q|^2\frac{\partial q}{\partial t}+i\varepsilon\beta_5 q\frac{\partial |q|^2}{\partial t}=0$$

$$\frac{\partial \phi}{\partial x}=\frac{\partial \phi}{\partial x}(e^{i(kx-\omega t)})$$

$$\frac{\partial \phi}{\partial x}=ik(e^{i(kx-\omega t)})$$

$$\frac{\partial \phi}{\partial x}=ikq;$$

When we derivative of q(x, t) with respect to x and t; we get as follows for preliminary transformation.

$$\frac{\partial q}{\partial t}=\frac{\partial u}{\partial t}(e^{i\phi(x,t)})$$

$$\frac{\partial q}{\partial t}=\frac{\partial u}{\partial t}(e^{i(kx-\omega t)})$$

$$\frac{\partial q}{\partial t}=\frac{\partial u}{\partial t}(e^{i(kx-\omega t)})+u(e^{i(kx-\omega t)}(-i\omega)$$

$$\frac{\partial q}{\partial t}=e^{i(kx-\omega t)}(\frac{\partial u}{\partial t}+u(i\omega)(-i\omega))$$

$$\frac{\partial^2 q}{\partial t^2}=\frac{\partial^2 u}{\partial t^2}e^{i(kx-\omega t)}-(\frac{\partial u}{\partial t}e^{i(kx-\omega t)}(i\omega))+[\frac{\partial u}{\partial t}e^{i(kx-\omega t)}+e^{i(kx-\omega t)}(i\omega)](i\omega)$$

$$\frac{\partial^2 q}{\partial t^2}=e^{i(kx-\omega t)}[\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}i\omega+\frac{\partial u}{\partial t}i\omega+(\omega^2)]$$

$$\frac{\partial^2 q}{\partial t^2}=e^{i(kx-\omega t)}[\frac{\partial^2 u}{\partial t^2}+2\frac{\partial u}{\partial t}i\omega+(\omega^2)]$$

$$\frac{\partial^3 q}{\partial t^3}=e^{i(kx-\omega t)}i\omega[\frac{\partial^3 u}{\partial t^3}+2\frac{\partial^2 u}{\partial t^2}i\omega]$$

$$\frac{\partial^3 q}{\partial t^3}=e^{i(kx-\omega t)}[\frac{\partial^3 u}{\partial t^3}i\omega-2\frac{\partial^2 u}{\partial t^2}\omega^2]$$

$$q(x,t)=u(x,t)e^{i(kx-\omega t)}$$

$$|q|^2 = u^2 e^{i(kx-\omega t)}$$

$$\frac{\partial |q^2|}{\partial t} = 2u\frac{\partial u}{\partial t} e^{2i(kx-\omega t)} + u^2 e^{2i(kx-\omega t)}(-2i\omega)$$

$$\frac{\partial |q|^2}{\partial t} = [-2u\frac{\partial u}{\partial t} - 2ui\omega]e^{2i(kx-\omega t)}$$

$$\frac{\partial |q|^2}{\partial t} = 2ue^{2i(kx-\omega t)}[\frac{\partial u}{\partial t} - i\omega]$$

$$\frac{\partial |q|^2}{\partial t} = 2ue^{2i\phi(x,t)}[\frac{\partial u}{\partial t} - i\omega]$$

By substituting from the solution (11) in the generalized higher order NLS (10), we obtain

$$(-k+\beta_1\omega^2-\varepsilon\beta_3\omega^3)u+(\beta_2+\varepsilon\beta_4\omega)u^3+i\frac{\partial u}{\partial x}+(3i\varepsilon\beta_3\omega^2-2i\beta_1\omega)\frac{\partial u}{\partial t}+(i\varepsilon\beta_4+$$

$$2\varepsilon\beta_5\omega)u^2\frac{\partial u}{\partial t}+(\beta_1+3\varepsilon\beta_3\omega^2)\frac{\partial^2 u}{\partial t^2}+i\varepsilon\beta_3\frac{\partial^3 u}{\partial t^3}=0 \tag{12}$$

III. Stability Analysis Solutions

In quantum mechanics, the Hamiltonian of a system is an operator corresponding to the total energy of that system, including both kinetic energy and potential energy. Its spectrum, the system's energy spectrum or its set of energy eigenvalues, is the set of possible outcomes obtainable from a measurement of the system's total energy. Hamiltonian system is a mathematical formalism to describe the evolution equations of a physical system. By using the form of a Hamiltonian system for which the momentum is given as

$$M = \lim_{s\to\infty}\int_0^s |u|_i^2 dx \tag{15}$$

where i =1; 2; 3; 4; 5. The sufficient condition for solitary wave solutions stability is

$$\frac{\partial M}{\partial \omega} > 0 \tag{16}$$

where ω is the frequency. By using (15) and (16), we discuss the criteria for stability of solutions for the generalized higher order NLS equation in given cases (Fig. 3.1).

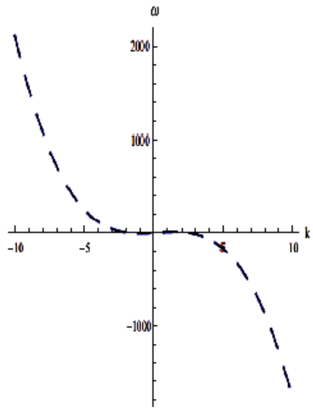

Fig. 3.1: The dispersion relation $k = k(\omega)$ of a constant coefficient linear evolution equation (wave number and frequency of perturbation)

Solitary Wave Solutions (3.1) [6] Generalized Non-Linear Shrödinger Equation (GNLSE) have large number of solutions however¨ we are here interested in localized solutions.
Solitary wave (3.2) [6] In some media, such as layer of water or an optical fiber, under suitable conditions the widening of wave packet due to dispersion could be balanced exactly by narrowing effect of nonlinearity of medium. In these cases, it is possible to have localized waves, propagating with constant velocity and undistorted shape, often known as solitary waves.
Solitons (3.3) [6] Solitons are those lone waves that, despite collisions, maintain their uniqueness and ultimately return to their initial forms and velocities. This characteristic resembles the interaction of free particles. Solitons get their name from the fact that solitary waves are particles. This study was demonstrated in (Fig. 3.2). There are two primary types of solitons: bright and dark solitons. Bright solitons are defined as solitons that pulse at a higher intensity than the background, as seen in (Fig. 3.3). Dark solitons are soliton with intensity less than backdrop. Dark soliton's development is depicted in (Fig. 3.4).

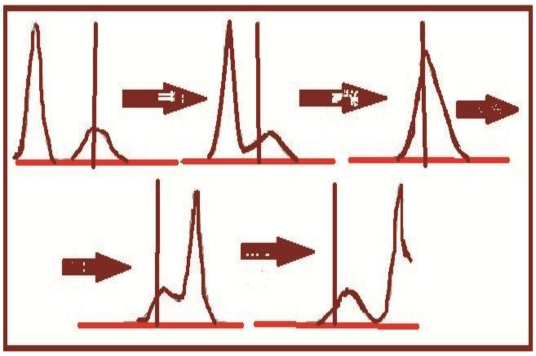

Fig. 3.2: Collision of solitary waves

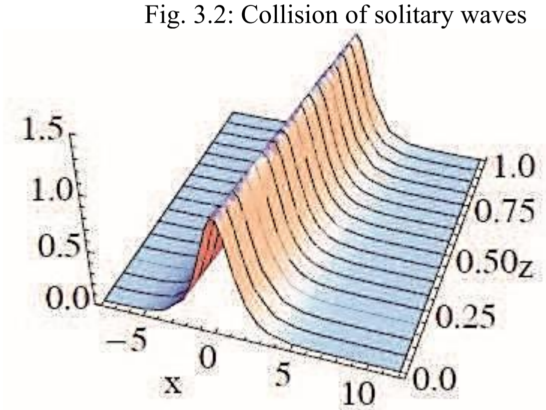

Fig. 3.3: Evolution of bright soliton.

J. Scott Russell first noticed the lone wave in 1834 while conducting an experiment on the effective construction of a canal boat. He saw a lengthy water wave traveling throughout the experiment without changing form. He dubbed this wave the" Great Wave" of translation or the" Solitary Wave" and carried out more research to determine the characteristics of this wave. The wave's speed is determined by

$$c = \sqrt{g(h + a)} \tag{17}$$

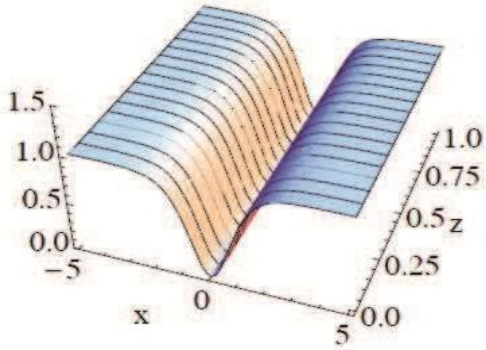

Fig. 3.4: Evolution of dark soliton

where g is acceleration due to gravity, h is depth of water channel, a is maximum amplitude of solitary wave. A few characteristics of solitary waves include their stability and ability to travel great distances without changing size or form. Additionally, a wave's breadth and speed are determined by the depth of the channel; isolated waves cannot withstand collisions. Derivative of Hyperbolic Trigonometric functions

$$(sinhx)' = (\frac{e^x - e^{-x}}{2})' = \frac{e^x + e^{-x}}{2} = \cosh x \ ,$$
$$(coshx)' = sinhx,$$
$$(tanhx)' = sech^2 x$$

We get the rest in similar fashion.

Modulation instability (3.4) [6] On substituting q(x,t) = φ(x,t)+iψ(x,t), where φ(x,t) and ψ(x,t) are real functions of x and t, in (1), this leads to the following coupled nonlinear partial differential equations as

$$\frac{\partial \phi}{\partial t} + \beta_1 \frac{\partial^2 \psi}{\partial t^2} + \varepsilon\beta_3 \frac{\partial^3 \phi}{\partial t^3} + 2\varepsilon\beta_5 \phi(\phi \frac{\partial \phi}{\partial t} + \psi \frac{\partial \psi}{\partial t^2}) + \varepsilon\beta_4(\phi^2 + \psi^2)\frac{\partial \phi}{\partial t} + \beta_2 \psi(\phi^2 + \psi^2) = 0$$
$$-\frac{\partial \psi}{\partial t} + \beta_1 \frac{\partial^2 \phi}{\partial t^2} - \varepsilon\beta_3 \frac{\partial^3 \psi}{\partial t^3} - 2\varepsilon\beta_5 \psi(\phi \frac{\partial \phi}{\partial t} + \psi \frac{\partial \psi}{\partial t}) - \varepsilon\beta_4(\phi^2 + \psi^2)\frac{\partial \phi}{\partial t} + \beta_2 \phi(\phi^2 + \psi^2) = 0$$
(18)

We discuss the existence of a Lagrangian and the invariant variational principle for (12). In order to reduce equation (12) to a system of two-second order equations, and express it in the following form:

$$M(\phi,\psi) = \frac{\partial \phi}{\partial t} + \beta_1 \frac{\partial^2 \psi}{\partial t^2} + \varepsilon\beta_3 \frac{\partial^3 \phi}{\partial t^3} + 2\varepsilon\beta_5 \phi(\phi \frac{\partial \phi}{\partial t} + \psi \frac{\partial \psi}{\partial t^2}) + \varepsilon\beta_4(\phi^2 + \psi^2)\frac{\partial \phi}{\partial t} + \beta_2 \psi(\phi^2 + \psi^2) = 0,$$

$$N(\phi,\psi) = -\frac{\partial \psi}{\partial t} + \beta_1 \frac{\partial^2 \phi}{\partial t^2} - \varepsilon\beta_3 \frac{\partial^3 \psi}{\partial t^3} - 2\varepsilon\beta_5 \psi(\phi \frac{\partial \phi}{\partial t} + \psi \frac{\partial \psi}{\partial t}) - \varepsilon\beta_4(\phi^2 + \psi^2)\frac{\partial \phi}{\partial t} + \beta_2 \phi(\phi^2 + \psi^2) = 0$$
(19)

The consistency conditions are expressed in (19), furthermore, the system of (1) satisfies the conditions (16), then a functional integral J(φ, ψ) can be written down using the formula given by (22), as

$$J(\phi,\psi)=\int_{\omega}\phi[\int_{0}^{3}N(\mu\phi,\mu\psi)d\mu]d\omega+\int_{\omega}\phi[\int_{0}^{1}M(\mu\phi,\mu\psi)d\mu]d\omega,$$

$$J(\phi,\psi)=\frac{1}{4}\int_{\omega}[-2\phi\frac{\partial\psi}{\partial x}+2\beta_1\phi\frac{\partial^2\phi}{\partial x^2}-2\varepsilon\beta_3\phi\frac{\partial^3\psi}{\partial x^3}-2\varepsilon\beta_5\phi\,\psi(\phi\frac{\partial\phi}{\partial t}+\psi\frac{\partial\psi}{\partial t})-$$

$$\varepsilon\beta_4\phi(\phi^2+\psi^2)\frac{\partial\psi}{\partial t}+\beta_2\phi^2(\phi^2+\psi^2)+2\psi\frac{\partial\phi}{\partial t}+2\psi\beta_1\frac{\partial^2\psi}{\partial t^2}+2\varepsilon\beta_3\psi\frac{\partial^3\phi}{\partial t^3}+$$

$$2\varepsilon\beta_5\phi\,\psi(\phi\frac{\partial\phi}{\partial t}+\psi\frac{\partial\psi}{\partial t})+\varepsilon\beta_4\psi(\phi^2+\psi^2)\frac{\partial\psi}{\partial t}+\beta_2\psi^2(\phi^2+\psi^2)]d\omega,$$

(20)

where dω and dt dx, on choosing the boundary on φ_x and ψ_x to be such that the boundary terms vanish, we get the Lagrangian L is given by

$$L(u,v)=\frac{1}{4}\int_{\omega}[-2\phi\frac{\partial\psi}{\partial x}+2\beta_1\phi\frac{\partial^2\phi}{\partial x^2}-2\varepsilon\beta_3\phi\frac{\partial^3\psi}{\partial x^3}-2\varepsilon\beta_5\phi\,\psi(\phi\frac{\partial\phi}{\partial t}+\psi\frac{\partial\psi}{\partial t})-$$

$$\varepsilon\beta_4\phi(\phi^2+\psi^2)\frac{\partial\psi}{\partial t}+\beta_2\phi^2(\phi^2+\psi^2)+2\psi\frac{\partial\phi}{\partial t}+2\psi\beta_1\frac{\partial^2\psi}{\partial t^2}+2\varepsilon\beta_3\psi\frac{\partial^3\phi}{\partial t^3}+$$

$$2\varepsilon\beta_5\phi\,\psi(\phi\frac{\partial\phi}{\partial t}+\psi\frac{\partial\psi}{\partial t})+\varepsilon\beta_4\psi(\phi^2+\psi^2)\frac{\partial\psi}{\partial t}+\beta_2\psi^2(\phi^2+\psi^2)]$$

(21)

As a necessary check to our calculations, we use the value of L in the Euler–Lagrange equations:

$$\frac{\partial L}{\partial\phi}-\frac{\partial}{\partial t}(\frac{\partial L}{\partial\phi_t})-\frac{\partial}{\partial t^2}(\frac{\partial L}{\partial\phi_{tt}})-\frac{\partial}{\partial t^3}(\frac{\partial L}{\partial\phi_{ttt}})-\frac{\partial}{\partial t}(\frac{\partial L}{\partial\phi_x})=0,$$

$$\frac{\partial L}{\partial\psi}-\frac{\partial}{\partial t}(\frac{\partial L}{\partial\psi_t})-\frac{\partial}{\partial t^2}(\frac{\partial L}{\partial\psi_{tt}})-\frac{\partial}{\partial t^3}(\frac{\partial L}{\partial\psi_{ttt}})-\frac{\partial}{\partial t}(\frac{\partial L}{\partial\psi_x})=0$$

(22)

which yields us the system of (1). Many nonlinear systems exhibit an instability that leads to modulation of the steady state as a result of an interplay between the nonlinear and dispersive effects. To derive the modulation instability of the generalized higher order NLS (1) by applying the standard linear stability analysis. The generalized higher order NLS equation has the steady state solution

(23)

$$q(x,t)=(\sqrt{P}+\psi(x,t))e^{i\phi(x)},\phi(x)=(P\beta_2+\beta_4\varepsilon^2)x$$

where P is the normalized optical power. We examine evolution of the perturbation ψ(x, t); using a linear stability analysis. By substituting from(23) in(1) and linearizing in a(x, t), we get

$$i\frac{\partial\psi}{\partial t}+\beta_1\frac{\partial^2\psi}{\partial t^2}+\beta_2P(\psi+\psi^*)+i\varepsilon\beta_3\frac{\partial^3\psi}{\partial t^3}i\varepsilon\beta_4P\frac{\partial\psi}{\partial t}+i\varepsilon\beta_5P\frac{\partial}{\partial t}(\psi+\psi^*)=0 \quad (24)$$

Consider the solution of (24) in the form

$$\psi(x,t)=\beta_1e^{i}(kx-\omega t)+\beta_2e^{-i(kx-\omega t)} \quad (25)$$

where k; ω are the normalized wave number and frequency of perturbation. The dispersion relation k = k(ω) of a constant coefficient linear evolution equation determines how time oscillations e^{ikx} are linked to spatial oscillations $e^{i\omega t}$ of wave number k, substituting from (25) in (24), we obtain the following dispersion relation as

$$\text{k} \;= P\omega\varepsilon(\beta_4+\beta_5)-\varepsilon\beta_3\omega^3 \pm \omega\sqrt{\omega^2\beta_1^2-2P\beta_1\beta_2+P^2\varepsilon^2\beta_5^2} \quad (26)$$

The dispersion relation (26) shows the steady state stability depends on the group velocity dispersion, self-phase modulation and stimulated Raman scattering. In the case the wave $\omega^2\beta_1^2 - 2P\beta_1\beta_2 + P^2\varepsilon^2\beta_5^2 \geq 0$ or $\omega^2\beta_1^2 - 2P\beta_1\beta_2 + P^2\varepsilon^2\beta_5^2 \geq 2P\beta_1\beta_2$, number k is real for all ω and the steady state is stable against small perturbations. By contrast, the steady state solution becomes unstable in the case of
$\omega^2\beta_1^2 - 2P\beta_1\beta_2 + P^2\varepsilon^2\beta_5^2 < 2P\beta_1\beta_2$, the wave number k is imaginary part since the perturbation then grows exponentially. One can easily see that for the occurrence of modulation stability when $\omega^2\beta_1^2 - 2P\beta_1\beta_2 + P^2\varepsilon^2\beta_5^2 < 2P\beta_1\beta_2$.

Under this condition, the growth rate of modulation stability gain spectrum g(ω) could be expressed

(5.48)

$$g(\omega) = 2Im(k) = 2\omega\sqrt{\omega^2\beta_1^2-2P\beta_1\beta_2+P^2\varepsilon^2\beta_5^2}$$

IV. Graphical Results

Fig. 4.1 - Fig. 4.2, the Generalized Nonlinear Shrödinger Equation (GNSE) dark soliton solution's development is depicted on the graph in Fig. 4.2. A localized dip in the wave amplitude that propagates without changing form is known as a dark soliton. The propagation of optical pulses in nonlinear fibers is described by the GNSE, a nonlinear partial differential equation. The graph shows the evolution of the dark soliton as it propagates through the fiber. The dark soliton is initially located at x = 0 and t = 0. As it propagates, it maintains its shape and moves to the right at the group velocity. The graph also shows that the dark soliton is surrounded by a bright pedestal. This is because the nonlinearity of the fiber causes the dark soliton to interact with the surrounding radiation. The interaction results in a transfer of energy from the dark soliton to the bright pedestal. The dark soliton solution of the GNSE has a number of interesting properties. It is stable against perturbations and can propagate over long distances. It is also possible to create multiple dark solitons in the same fiber. Dark solitons have a number of potential applications in optical communication and signal processing.
Here is a more detailed discussion of the graph:

- The dark soliton is initially located at x = 0 and t = 0. It is a localized in the wave amplitude, with a bright stand surrounding it.
- As the dark soliton propagates through the fiber, it maintains its shape and moves to the right at the group velocity.
- The bright base also propagates to the right, but at a slightly different speed than the dark soliton. This is because the nonlinearity of the fiber causes the dark soliton to interact with the surrounding radiation.
- The interaction between the dark soliton and the bright pedestal results in a transfer of energy from the dark soliton to the bright pedestal. This is why the bright pedestal grows in amplitude as the dark soliton propagates.
- The dark soliton eventually disappears, leaving behind only the bright pedestal.

The graph in Fig. 4.1 is a good example of how the nonlinearity of the fiber can affect the propagation of optical pulses. The dark soliton is a stable solution of the GNSE, and it can propagate over long distances without change of shape. However, the interaction between the dark soliton and the bright pedestal can eventually lead to the disappearance of the dark soliton.

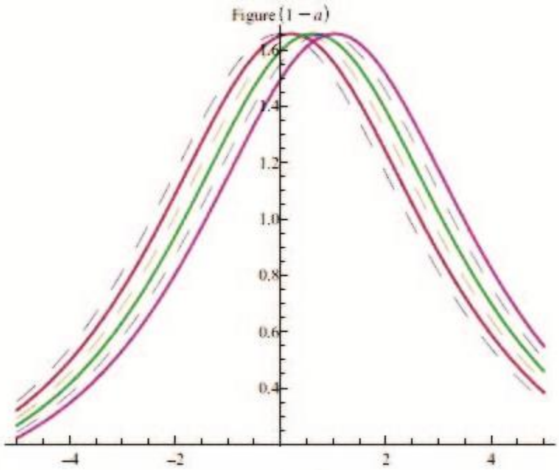

Fig. 4.1: Evolution of a Bright soliton solution graph A(x, t) for different values of k x = 0, t = 0, ω = 0

The graph in Fig. 4.1 shows the dark and bright soliton solutions of the generalized nonlinear Shrödinger equation (GNLS). The graph shows the amplitude of the soliton solution as a function¨ of distance. The dark soliton is a region where the amplitude of the solution is lower than the amplitude of the background solution. The bright soliton is a region where the amplitude of the solution is higher than the amplitude of the background solution. The dark and bright soliton solutions of the GNLS equation are important because they can be used to model a variety of physical phenomena, such as optical pulses, water waves, and plasma waves. The dark soliton can be used to model a hole in a beam of light, while the bright soliton can be used to model a pulse of light. The dark and bright soliton solutions of the GNLS equation are also important because they are stable solutions. This means that they can propagate through a medium without changing their shape. The stability of the dark and bright soliton solutions is due to the balance between the nonlinearity of the equation and the dispersion of the equation. The dark and bright soliton solutions of the GNLS equation have been observed in a variety of experiments. For example, dark solitons have been observed in optical fibers and bright solitons have been observed in plasmas. Here is a more detailed discussion of the graph of the dark and bright soliton solution of the GNLS equation:

- The graph shows that the dark soliton has a lower amplitude than the background solution, while the bright soliton has a higher amplitude than the background solution.
- The graph also shows that the dark and bright soliton solutions are localized in space. This means that they have a finite width and they decay to zero amplitude as the distance from the center of the soliton increases.
- The graph also shows that the dark and bright soliton solutions are stable. This means that they can propagate through a medium without changing their shape.

Stability of the dark and bright soliton solutions (4.1) [8] The stability of the dark and bright soliton solutions is due to the balance between the nonlinearity of the equation and the dispersion of the equation. The nonlinearity of the equation tends to make the soliton solutions spread out, while the dispersion of the equation tends to make the soliton solutions contract. The balance between these two effects leads to the formation of stable soliton solutions.

Observation of dark and bright soliton solutions (4.2) [8] The dark and bright soliton solutions of the GNLS equation have been observed in a variety of experiments. For example, dark solitons have been observed in optical fibers and bright solitons have been observed in plasmas. The observation of these soliton solutions in experiments has confirmed the theoretical predictions about their existence and stability.

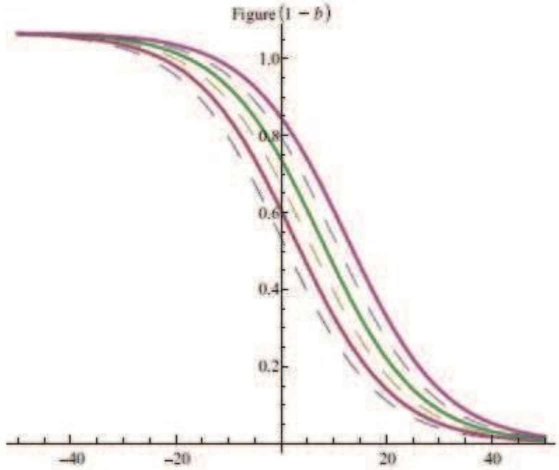

Fig. 4.2: Evolution of a dark soliton solution graph A(x,t) for different values of ω with constant values of k,x = 0,t =0

Fig. 4.1 and Fig. 4.2 shows the graph of dark and bright soliton solution of the generalized nonlinear Shrödinger equation (GNLS). The graph shows the amplitude of the soliton solution as a¨ function of distance. The dark soliton is a region where the amplitude of the solution is lower than the amplitude of the background solution. The bright soliton is a region where the amplitude of the solution is higher than the amplitude of the background solution. The dark and bright soliton solutions of the GNLS equation are important because they can be used to model a variety of physical phenomena, such as optical pulses, water waves, and plasma waves. The dark soliton can be used to model a hole in a beam of light, while the bright soliton can be used to model a pulse of light. The dark and bright soliton solutions of the GNLS equation are also important because they are stable solutions. This means that they can propagate through a medium without changing their shape. The stability of the dark and bright soliton solutions is due to the balance between the nonlinearity of the equation and the dispersion of the equation. The dark and bright soliton solutions of the GNLS equation have been observed in a variety of experiments. For example, dark solitons have been observed in optical fibers and bright solitons have been observed in plasmas. Here is a more detailed discussion of the graph of the dark and bright soliton solution of the GNLS equation:

- The graph shows that the amplitude of the dark soliton is lower than the amplitude of the background solution, while the amplitude of the bright soliton is higher than the amplitude of the background solution.
- The graph also shows that the dark and bright soliton solutions are localized in space. This means that they have a finite width and they decay to zero amplitude as the distance from the center of the soliton increases.
- The graph also shows that the dark and bright soliton solutions are stable. This means that they can propagate through a medium without changing their shape.

The dark and bright soliton solutions of the GNLS equation have been observed in a variety of experiments. For example, dark solitons have been observed in optical fibers and bright solitons have been observed in plasmas. The observation of these soliton solutions in experiments has confirmed the theoretical predictions about their existence and stability. The dark and bright soliton solutions of the GNLS equation have a variety of potential applications. Overall, the dark and bright soliton solutions of the GNLS equation are important and interesting solutions to a nonlinear equation. They have a variety of potential applications in optics, plasma physics, and other fields.

The conclusion of the graph of dark and bright soliton solution of the generalized nonlinear Shrödinger equation (GNLS) is that these solutions are important and interesting solutions to a¨ nonlinear equation with a variety of potential applications.

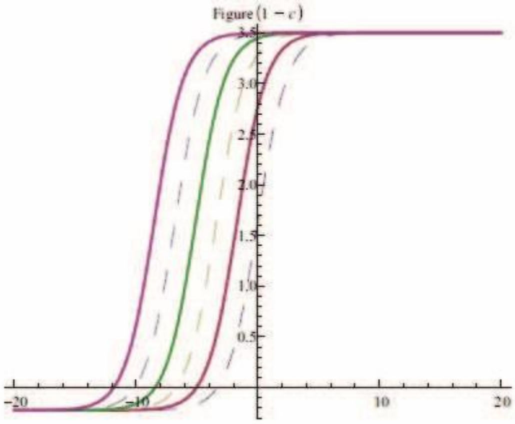

Fig. 4.3: Graph of dark and bright soliton solution

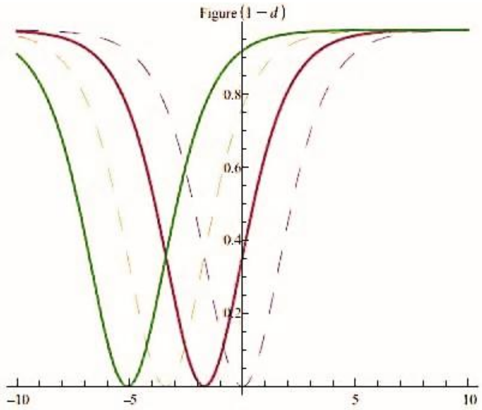

Fig. 4.5: Graph dark soliton solutions of the GHONLSE equation

Overall, the dark and bright soliton solutions of the GNL equation are important and interesting solutions to a nonlinear equation. They have a variety of potential applications in optics, plasma physics, and other fields. The GHNLSE and related dark and brilliant solitons have a multitude of prospects for progress in a number of domains, such as quantum optics, bio photonics, medical imaging, and optics. These are fascinating fields of study with broad consequence because of their special qualities and possible uses. The novel accurate bright, dark, and bright-dark solitary wave soliton solutions of the generalized higher order nonlinear NLS problem, which are crucial in mathematics and physics, were developed in this work. Explicit solutions to the higher order nonlinear NLS problem were found. The modulation instability analysis and stability analysis solutions are applied to examine the stability of these solutions as well as the movement role of the waves. Numerous new precise solutions were found, which have significant uses in the applied sciences and might be helpful to physicists and researchers studying more complicated physical processes. The knowledge of ultra-short pulse propagation in optical fibers and other nonlinear wave phenomena will be significantly impacted by these results. The stability of the exact solutions for solitary waves is essential for their possible applications in fluid dynamics, optics, plasma physics, and other domains. These solutions also offer insightful information on the behavior of these waves. The obtained results show that the current method is simple, gives analytical solutions in generalized form, consistence, reduces the computational complexity and can be utilized efficiently in several applications of mathematical physics.

VI. Result:

By applying a modified extended direct algebraic method to the generalized higher-order nonlinear Schrödinger equation, we successfully derived new exact solitary wave solutions in the forms of bright, dark, and bright-dark solitons under specific parametric constraints. The stability of these solutions was confirmed using both Hamiltonian-based criteria and modulation instability analysis, which yielded dispersion relations indicating robust propagation against small perturbations. The results demonstrate that the proposed method is efficient and reliable for obtaining stable soliton solutions applicable to nonlinear wave phenomena in optical fibers and related physical systems.

VII. Conclusion:

In this work, we successfully constructed new exact bright, dark, and bright-dark soliton solutions for the generalized higher-order DNLS equation using a modified extended direct algebraic technique. The solutions were validated through stability and modulation instability analyses. Graphical simulations demonstrate that these solitons maintain their shape during propagation and collisions, confirming their robustness. The proposed method reduces computational complexity and provides generalized analytical solutions, making it a powerful tool for studying nonlinear wave phenomena in optical fibers, plasma physics, and other applied sciences. These findings offer valuable insights for future research in nonlinear dynamics and practical applications in telecommunications and signal processing.

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