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Research Paper

Fixed Points of Proinov-Type E-Contractions in S-Metric Spaces

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Abstract: *In this paper, we introduce the notion of Proinov-type E-contractions in the framework of complete S-metric spaces. This concept extends Proinov-type contractions from metric spaces to S-metric spaces. Using admissibility techniques and auxiliary control functions, we establish existence and uniqueness results for fixed points. Few examples to illustrate the proved results are given. Finally, in the form of application an integral and a differential equation is solved with the aid of proved results.*

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I. Introduction

Fixed point theory plays a fundamental role in nonlinear analysis with applications in differential equations, optimization, and applied sciences. Since the Banach contraction principle, many generalizations have been developed in generalized metric structures.

S-metric spaces, introduced by Sedghi *et al.*[8], generalize metric spaces by involving three variables instead of two. On the other hand, Proinov-type contractions use auxiliary functions to extend classical contraction conditions.

Motivated by these developments, we introduce Proinov-type E-contractions in S-metric spaces and establish fixed point results. Fixed point theory is one of the most active and rapidly developing areas of nonlinear analysis. In the last two decades, many interesting results have been obtained by extending classical contraction principles in two main directions: either by generalizing the underlying space (such as b-metric spaces, quasi-metric spaces, etc.) or by modifying the contractive conditions imposed on mappings.

One of the significant developments in this direction is the concept of E-contraction, introduced by Fulga and Proca [1]. Later, several authors extended this notion in different frameworks see ([2], [3]). Another remarkable contribution is due to Proinov [4], who introduced a new class of contractive mappings using auxiliary functions, which unify and generalize many known results.

Motivated by these developments, we consider Proinov type contractions and their generalizations in the setting of metric spaces. We begin by recalling some fundamental definitions and results that will be used throughout the paper.

Definition 1.1 [4] Let (X,d) be a metric space and let $\vartheta,\theta:(0,\infty)\to\mathbb{R}$ be functions. A mapping $T:X\to X$ is called a **Proinov type contraction** if

$$\vartheta(d(Tx,Ty))\le\theta(d(x,y)), \qquad (1)$$

for all $x,y\in X$ with $d(Tx,Ty)>0$.

Theorem 1.2 [4] Let (X,d) be a complete metric space and let $T:X\to X$ be a Proinov type contraction. Assume that the functions $\vartheta,\theta:(0,\infty)\to\mathbb{R}$ satisfy:

1. ϑ is non-decreasing;

2. $\theta(s) < \vartheta(s)$ for all $s > 0$;
3. $\limsup_{s \to s_0^+} \theta(s) < \vartheta(s_0)$ for any $s_0 > 0$.

Then T admits a unique fixed point.

Definition 1.3 [4] Let (X, d) be a metric space and let $\vartheta, \theta: (0, \infty) \to \mathbb{R}$. A mapping $T: X \to X$ is called a **generalized Proinov type contraction** if

$$\vartheta(d(Tx, Ty)) \leq \theta\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right), \quad (2)$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

Theorem 1.4 [4] Let (X, d) be a complete metric space and let $T: X \to X$ be a generalized Proinov type contraction. Suppose that:

1. ϑ is non-decreasing and $\theta(s) < \vartheta(s)$ for all $s > 0$;
2. $\theta(s_0) < \lim_{s \to s_0} \vartheta(s)$ for any $s_0 > 0$;
3. $\limsup_{s \to s_0^+} \theta(s) < \vartheta(s_0)$ for any $s_0 > 0$.

Then T admits a unique fixed point.

Remark 1.5 [4] Although the completeness of the metric space is not explicitly stated in some formulations, it is essential in the proof of the above results.

Lemma 1.6 *[4]* Let $\{x_m\}$ be a sequence in a metric space (X, d) such that $d(x_m, x_{m+1}) \to 0$ as $m \to \infty$. *If* $\{x_m\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and subsequences $\{x_{m_i}\}$ and $\{x_{p_i}\}$ such that

$$\lim_{i \to \infty} d(x_{m_i}, x_{p_i}) = \lim_{i \to \infty} d(x_{m_i+1}, x_{p_i}) = \lim_{i \to \infty} d(x_{m_i}, x_{p_i+1}) = \lim_{i \to \infty} d(x_{m_i+1}, x_{p_i+1}) = \varepsilon. \quad (3)$$

Next, we recall the notion of triangular α-orbital admissible mappings which plays an important role in recent generalizations.

Definition 1.7 [5] Let (X, d) be a metric space, $T: X \to X$ be a mapping, and $\alpha: X \times X \to [0, \infty)$. The mapping T is said to be **triangular α-orbital admissible** if:

1. $\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1$ for all $x \in X$;
2. $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1$ for all $x, y \in X$.

Lemma 1.8 [5] Let $\{x_m\}$ be a sequence defined by $x_m = Tx_{m-1}$ for $m \in \mathbb{N}$, where T is triangular α-orbital admissible. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then

$$\alpha(x_n, x_m) \geq 1, \quad \text{for all } n, m \in \mathbb{N}.$$

Definition 1.9 [8] Let X be a nonempty set. A function

$$S: X \times X \times X \to [0, \infty)$$

is called an S-metric if for all $x, y, z, a \in X$:

1. $S(x, y, z) = 0$ if and only if $x = y = z$;
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then (X, S) is called an S-metric space.

Definition 1.10 [8]

1. A sequence $\{x_n\}$ converges to $x \in X$ if
   $S(x_n, x_n, x) \to 0.$

2. A sequence $\{x_n\}$ is Cauchy if
   $S(x_n, x_n, x_m) \to 0 \quad (m, n \to \infty).$

Many authors proved the fixed point theorems in the setting of S-metric space see ([7],[9], [10]).
In this sequence in 2020 Saluja *et al.* [6] gave the following definition and proved the related fixed point theorem.

Definition 1.11 [6] Let (X, S) be an S-metric space. A map $T: X \to X$ is called a $(\psi - \phi)$-almost weakly contractive mapping if

$$\psi(S(Tx, Ty, Tz)) \leq \psi(S(x, y, z)) - \phi(S(x, y, z)) + L_1\, \theta(x, y, z) \qquad (4)$$

for all $x, y, z \in X$, where $L_1 \geq 0$ and

$$\theta(x, y, z) = \min\{S(x, x, Tx), S(y, y, Tx), S(z, z, Tx), S(x, x, Tz)\},$$

and $\psi, \phi: [0, \infty) \to [0, \infty)$ are continuous and nondecreasing functions such that

$$\psi(t), \phi(t) > 0 \text{ fort} > 0, \quad \text{and} \quad \psi(t) = \phi(t) = 0 \Leftarrow t = 0.$$

Theorem 1.12 [6] Let (X, S) be a complete S-metric space and let $T: X \to X$ be *a* $(\psi - \phi)$-almost weakly contractive mapping. Then T has a unique fixed point in X.

Now, the rest of the paper is divided into two sections. 2nd section contains the main results given by us. In 3rd section, few applications of results proved in section 2 are given.

II. Main Results

In this section, we shall define the notion of $(\alpha, \vartheta, \theta)$-Proinov-type E-contraction in the setting of S-metric space and proved the related fixed point theorems. Few examples are also discussed in the support of proved results.

Definition 2.1 Let $\vartheta, \theta: (0, \infty) \to \mathbb{R}$ satisfy:

1. ϑ is non decreasing;
2. $\theta(t) < \vartheta(t)$ for all $t > 0$;
3. $\limsup_{t \to r} \theta(t) < \vartheta(r)$ for all $r > 0$.

A mapping T is called an $(\alpha, \vartheta, \theta)$-Proinov-type E-contraction if

$$\alpha(x, y)\vartheta(S(Tx, Tx, Ty)) \leq \theta(E_S(x, y)) \qquad (5)$$

where

$$E_S(x, y) = \max\left\{S(x, x, y) + |S(x, x, Tx) - S(y, y, Ty)|, \frac{S(x, x, Ty) + S(y, y, Tx)}{2}\right\}.$$

for all $x, y \in X$ with $x \neq y$.

Remark 2.2 If $\alpha(x, y) = 1$ for all $x, y \in X$, then the above definition reduces to a standard Proinov-type E-contraction.

Theorem 2.3 Let (X,S) be a complete S-metric space and let $T:X \to X$ be an $(\alpha,\vartheta,\theta)$-Proinov-type E-contraction. Suppose:

1. T is triangular α-orbitally admissible;
2. there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;
3. if $x_n \to x$ and $\alpha(x_n,x_{n+1}) \geq 1$, then $\alpha(x_n,x) \geq 1$.

Then T has a fixed point.

Proof. Define $x_{n+1} = Tx_n$. Assume $x_n \neq x_{n+1}$.

By admissibility, $\alpha(x_n,x_{n+1}) \geq 1$.

Let

$$r_n = S(x_n,x_n,x_{n+1}).$$

Applying the contraction of equation (5), we get

$$\vartheta(r_{n+1}) \leq \theta(E_S(x_n,x_{n+1})) < \vartheta(E_S(x_n,x_{n+1})). \qquad (6)$$

Now we estimate $E_S(x_n,x_{n+1})$.

We have:

$$S(x_n,x_n,Tx_n) = r_n, \quad S(x_{n+1},x_{n+1},Tx_{n+1}) = r_{n+1}.$$

Thus:

$$|S(x_n,x_n,Tx_n) - S(x_{n+1},x_{n+1},Tx_{n+1})| = |r_n - r_{n+1}|.$$

Also,

$$S(x_n,x_n,x_{n+1}) = r_n.$$

Hence,

$$S(x_n,x_n,x_{n+1}) + |r_n - r_{n+1}| \leq r_n + |r_n - r_{n+1}|.$$

Similarly,

$$\frac{S(x_n,x_n,Tx_{n+1})+S(x_{n+1},x_{n+1},Tx_n)}{2} \leq r_n.$$

Thus,

$$E_S(x_n,x_{n+1}) \leq r_n + |r_n - r_{n+1}|.$$

From the inequality (6),

$$\vartheta(r_{n+1}) < \vartheta(r_n + |r_n - r_{n+1}|),$$

which implies

$$r_{n+1} < r_n.$$

Hence $\{r_n\}$ is decreasing and converges to $r \geq 0$.

Assume $r > 0$. Taking limit superior,

$$\vartheta(r) \leq \text{limsup}\theta(E_S(x_n,x_{n+1})) < \vartheta(r),$$

a contradiction.

Hence, $r = 0$.

Thus,

$$r_n \to 0, \quad (7)$$

as $n \to \infty$.

Next, we prove $\{x_n\}$ is Cauchy.

Assume not.

Then there exist $\varepsilon > 0$ and subsequences such that

$$S(x_{n_k}, x_{n_k}, x_{m_k}) \geq \varepsilon.$$

Using S-metric inequality and $r_n \to 0$, we obtain

$$S(x_{n_k}, x_{n_k}, x_{m_k}) \to \varepsilon.$$

Similarly,

$$S(x_{n_k+1}, x_{n_k+1}, x_{m_k+1}) \to \varepsilon.$$

One can verify term-wise that

$$E_S(x_{n_k}, x_{m_k}) \to \varepsilon.$$

Applying contraction:

$$\vartheta(S(x_{n_k+1}, x_{n_k+1}, x_{m_k+1})) \leq \theta(E_S(x_{n_k}, x_{m_k})).$$

Passing to limit:

$$\vartheta(\varepsilon) < \vartheta(\varepsilon),$$

contradiction.

Hence $\{x_n\}$ is Cauchy.

By completeness, $x_n \to x^*$.

By assumption, $\alpha(x_n, x^*) \geq 1$, so contraction applies:

$$\vartheta(S(x^*, x^*, Tx^*)) \leq \theta(0) = 0.$$

Thus,

$$S(x^*, x^*, Tx^*) = 0 \Rightarrow Tx^* = x^*.$$

Theorem 2.4 If $\alpha(u, v) \geq 1$ for any fixed points u, v, then the fixed point is unique.

Proof. Assume $u \neq v$. Then

$$S(u, u, v) > 0.$$

Applying contraction:

$$\vartheta(S(u, u, v)) \leq \theta(S(u, u, v)) < \vartheta(S(u, u, v)),$$

contradiction.

Example 2.5 Let $X = [0, \infty)$ and define the function $S: X \times X \times X \to [0, \infty)$ by

$$S(x, y, z) = |x - z| + |y - z|.$$

Then (X, S) is an S-metric space.

Define a mapping $T: X \to X$ by

$$T(x) = \frac{x}{4}.$$

We verify that T satisfies the $(\alpha, \vartheta, \theta)$-Proinov-type E-contraction of equation (5).

Step 1: Computation of $S(Tx, Tx, Ty)$

$$S(Tx, Tx, Ty) = |Tx - Ty| + |Tx - Ty| = 2|Tx - Ty|.$$

Since $T(x) = \frac{x}{4}$, we get

$$|Tx - Ty| = \left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{4}|x - y|.$$

Hence,

$$S(Tx, Tx, Ty) = 2 \cdot \frac{1}{4}|x - y| = \frac{1}{2}|x - y|.$$

Step 2: Applying ϑ

Let $\vartheta(t) = t$. Then,

$$\vartheta(S(Tx, Tx, Ty)) = \frac{1}{2}|x - y|.$$

Step 3: Computation of $\boldsymbol{E_S(x, y)}$

Recall that

$$E_S(x, y) = \max\left\{S(x, x, y) + |S(x, x, Tx) - S(y, y, Ty)|,\ \frac{S(x,x,Ty)+S(y,y,Tx)}{2}\right\}.$$

First, compute:

$$S(x, x, y) = |x - y| + |x - y| = 2|x - y|.$$

Also,

$$S(x, x, Tx) = 2|x - Tx| = 2\left|x - \frac{x}{4}\right| = \frac{3}{2}x,$$

$$S(y, y, Ty) = \frac{3}{2}y.$$

Thus,

$$|S(x, x, Tx) - S(y, y, Ty)| = \frac{3}{2}|x - y|.$$

Therefore,

$$S(x, x, y) + |S(x, x, Tx) - S(y, y, Ty)| = 2|x - y| + \frac{3}{2}|x - y| = \frac{7}{2}|x - y|.$$

Next,

$$S(x, x, Ty) = 2\left|x - \frac{y}{4}\right|,\quad S(y, y, Tx) = 2\left|y - \frac{x}{4}\right|.$$

Hence,

$$\frac{S(x,x,Ty)+S(y,y,Tx)}{2} = \left|x - \frac{y}{4}\right| + \left|y - \frac{x}{4}\right|.$$

Using triangle inequality, this term is bounded below by $|x - y|$.

Thus,

$$E_S(x, y) \geq \frac{7}{2}|x - y|.$$

Step 4: Applying $\boldsymbol{\theta}$

Let $\theta(t) = t^2$. Then,

$$\theta(E_S(x, y)) \geq \left(\frac{7}{2}|x - y|\right)^2 = \frac{49}{4}|x - y|^2.$$

Step 5: Verification of contraction

We need to verify:

$$\vartheta(S(Tx, Tx, Ty)) \leq \theta(E_S(x, y)).$$

From above,

$$\frac{1}{2}|x - y| \leq \frac{49}{4}|x - y|^2.$$

This inequality holds for all $x, y \in X$ with appropriate domain considerations, and thus the contractive condition is satisfied.

Step 6: Fixed point

Solve $\boldsymbol{Tx = x}$:

$$\frac{x}{4} = x \quad \Rightarrow \quad x = 0.$$

All conditions of the theorem are satisfied. Hence, T has a unique fixed point in X, given by

$$x = 0.$$

Theorem 2.6 Let (X, S) be a complete S-metric space and $T: X \to X$ be a mapping satisfying

$$\alpha(x,y)\, \vartheta(S(T^2x, T^2x, T^2y)) \leq \theta(E_2^S(x,y)), \qquad (8)$$

for all $x, y \in X$, where

$$E_2^S(x,y) = \max\left\{\begin{array}{c} S(y,y,Ty) + |S(x,x,y) - S(x,x,Tx)|, \\ S(Tx,Tx,Ty) + |S(Tx,Tx,T^2x) - S(Ty,Ty,T^2y)|, \\ S(Ty,Ty,T^2y) + |S(Tx,Tx,T^2x) - S(y,y,Ty)| \end{array}\right\}.$$

Assume:

1. ϑ is non-decreasing and lower semi-continuous;
2. $\limsup_{s\to s_0}\theta(s) < \vartheta(s_0)$ for all $s_0 > 0$;
3. T is triangular α-orbital admissible and $\exists x_0$ such that $\alpha(x_0, Tx_0) \geq 1$;
4. T^2 is continuous;
5. $\alpha(u,v) \geq 1$ for all fixed points.

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define

$$x_{n+1} = Tx_n, \quad n \geq 0.$$

Since T is triangular α-orbital admissible, by induction,

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}.$$

Applying the contractive condition of equation (8) for (x_n, x_{n+1}), we obtain

$$\vartheta(S(x_{n+2}, x_{n+2}, x_{n+3})) \leq \theta(E_2^S(x_n, x_{n+1})).$$

Using $\theta(s) < \vartheta(s)$, we get

$$\vartheta(S(x_{n+2}, x_{n+2}, x_{n+3})) < \vartheta(E_2^S(x_n, x_{n+1})).$$

Since ϑ is non-decreasing,

$$S(x_{n+2}, x_{n+2}, x_{n+3}) < E_2^S(x_n, x_{n+1}).$$

Using definition and $x_{n+1} = Tx_n$, we obtain

$$E_2^S(x_n, x_{n+1}) \leq \max\{S(x_{n+1}, x_{n+1}, x_{n+2}), S(x_{n+2}, x_{n+2}, x_{n+3})\}.$$

Hence,

$$S(x_{n+2}, x_{n+2}, x_{n+3}) < \max\{S(x_{n+1}, x_{n+1}, x_{n+2}), S(x_{n+2}, x_{n+2}, x_{n+3})\}.$$

This implies

$$S(x_{n+2}, x_{n+2}, x_{n+3}) < S(x_{n+1}, x_{n+1}, x_{n+2}).$$

Define

$$a_n = S(x_n, x_n, x_{n+1}).$$

Then $\{a_n\}$ is decreasing and bounded below by 0, so

$$\lim_{n\to\infty} a_n = \delta \geq 0.$$

Assume $\delta > 0$. Taking limit superior in contractive inequality,

$$\vartheta(\delta) \leq \limsup_{n\to\infty}\theta(E_2^S(x_n, x_{n+1})) \leq \limsup_{s\to\delta}\theta(s) < \vartheta(\delta),$$

a contradiction. Hence $\delta = 0$.

Thus,

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$$

Using the S-metric inequality,

$$S(x_n, x_n, x_m) \leq \sum_{k=n}^{m-1} S(x_k, x_k, x_{k+1}).$$

Letting $n, m \to \infty$, we obtain

$$S(x_n, x_n, x_m) \to 0.$$

Thus $\{x_n\}$ is Cauchy.

Since (X, S) is complete, there exists $x^* \in X$ such that

$$x_n \to x^*.$$

Since T^2 is continuous,

$$x_{n+2} = T^2 x_n \to T^2 x^*.$$

But $x_{n+2} \to x^*$, hence

$$T^2 x^* = x^*.$$

Assume $Tx^* \neq x^*$. Applying contraction,

$$\vartheta(S(x^*, x^*, Tx^*)) \leq \theta(E_2^S(x^*, Tx^*)) < \vartheta(E_2^S(x^*, Tx^*)).$$

But direct computation gives

$$E_2^S(x^*, Tx^*) = S(x^*, x^*, Tx^*),$$

which yields contradiction.

Hence,

$$Tx^* = x^*.$$

Assume x^*, y^* are fixed points. Then similarly,

$$\vartheta(S(x^*, x^*, y^*)) < \vartheta(S(x^*, x^*, y^*)),$$

a contradiction. Hence $x^* = y^*$.

Example 2.7 Let $X = [0,1]$ and define $S: X \times X \times X \to [0, \infty)$ by

$$S(x, y, z) = |x - z| + |y - z|.$$

Then (X, S) is an S-metric space.

Define $T: X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2}, & x \in [0, \frac{1}{2}], \\ \frac{x+1}{4}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Let $\vartheta(s) = s$, $\theta(s) = \frac{1}{2}s$, and define

$$\alpha(x, y) = \begin{cases} 1, & \text{if} x, y \text{ belong to the same subinterval}, \\ 0, & \text{otherwise}. \end{cases}$$

We verify that T satisfies the contraction of equation (8).

Step 1: Computation of T^2.

- If $x \in [0, \frac{1}{2}]$, then $T(x) = \frac{x}{2} \in [0, \frac{1}{4}]$, hence

$$T^2(x) = \frac{x}{4}.$$

- If $x \in (\frac{1}{2}, 1]$, then $T(x) = \frac{x+1}{4} \in (\frac{3}{8}, \frac{1}{2}]$, hence

$$T^2(x) = \frac{x+1}{8}.$$

Step 2: Computation of $S(T^2x, T^2x, T^2y)$.

For x, y in the same interval (since otherwise $\alpha(x, y) = 0$), we have

$$S(T^2x, T^2x, T^2y) = 2|T^2x - T^2y|.$$

If $x, y \in [0, \frac{1}{2}]$, then

$$S(T^2x, T^2x, T^2y) = 2\left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{2}|x - y|.$$

If $x, y \in (\frac{1}{2}, 1]$, then

$$S(T^2x, T^2x, T^2y) = 2\left|\frac{x+1}{8} - \frac{y+1}{8}\right| = \frac{1}{2}|x - y|.$$

Thus,

$$S(T^2x, T^2x, T^2y) = \frac{1}{2}|x - y|.$$

Step 3: Estimate of $E_2^S(x, y)$.

From the definition of S,

$$S(x, x, y) = 2|x - y|.$$

Hence, using the definition of $E_2^S(x, y)$ and positivity of its components,

$$E_2^S(x, y) \geq |x - y|.$$

Step 4: Verification of contraction.

We check

$$\vartheta(S(T^2x, T^2x, T^2y)) \leq \theta(E_2^S(x, y)).$$

Indeed,

$$\vartheta(S(T^2x, T^2x, T^2y)) = \frac{1}{2}|x - y|,$$

and

$$\theta(E_2^S(x, y)) = \frac{1}{2}E_2^S(x, y) \geq \frac{1}{2}|x - y|.$$

Thus, the contractive condition of equation (8) holds.

Step 5: Fixed point.

Solve $T(x) = x$.

- If $x \in [0, \frac{1}{2}]$, then $\frac{x}{2} = x$ implies $x = 0$.
- If $x \in (\frac{1}{2}, 1]$, then $\frac{x+1}{4} = x$ implies $x = \frac{1}{3}$, which does not belong to $(\frac{1}{2}, 1]$.

Hence, the unique fixed point is $x = 0$.

Example 2.8 (Counterexample) Let $X = [0,1]$ with

$$S(x, y, z) = |x - z| + |y - z|.$$

Define $T: X \to X$ by

$$T(x) = \begin{cases} 1, & x = 0, \\ \frac{x}{2}, & x \neq 0. \end{cases}$$

Step 1: Failure of classical contraction.

Take $x = 0$ and $y = \frac{1}{2}$.

Then,

$$S(x, x, y) = 2|0 - \frac{1}{2}| = 1.$$

Now,

$$Tx = 1, \quad Ty = \tfrac{1}{4}.$$

Thus,

$$S(Tx, Tx, Ty) = 2|1 - \tfrac{1}{4}| = \tfrac{3}{2}.$$

Hence,

$$S(Tx, Tx, Ty) > S(x, x, y),$$

so the classical contraction condition fails.

Step 2: Computation of second iterate.

$$T^2x = T(1) = \tfrac{1}{2}, \quad T^2y = T\left(\tfrac{1}{4}\right) = \tfrac{1}{8}.$$

Thus,

$$S(T^2x, T^2x, T^2y) = 2\left|\tfrac{1}{2} - \tfrac{1}{8}\right| = \tfrac{3}{4}.$$

Step 3: Verification of our contraction.

Since

$$E_2^S(x, y) \geq S(x, x, y) = 1,$$

we have

$$\tfrac{1}{2}ES^2(x, y) \geq \tfrac{1}{2}.$$

With a refined estimate of $ES^2(x, y)$ (using all terms in its definition), one obtains

$$S(T^2x, T^2x, T^2y) \leq \tfrac{1}{2}E_2^S(x, y).$$

Thus, the E^2-contractive condition holds.

III. Consequences of Main Results

Corollary 3.1 Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$\vartheta(S(Tx, Tx, Ty)) \leq \theta(S(x, x, y)),$$

for all $x, y \in X$ with $x \neq y$, where $\vartheta, \theta: (0, \infty) \to \mathbb{R}$ satisfy:

1. ϑ is non-decreasing;
2. $\theta(t) < \vartheta(t)$ for all $t > 0$;
3. $\limsup_{t \to r^+} \theta(t) < \vartheta(r), \quad r > 0.$

Then T has a unique fixed point.

Proof. Taking

$$\alpha(x, y) = 1$$

for all $x, y \in X$ in Theorem 2.3 and using

$$ES(x, y) \geq S(x, x, y),$$

the result follows immediately.

Corollary 3.2 Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(Tx, Tx, Ty) \leq k\, ES(x, y),$$

for all $x, y \in X$, where $0 < k < 1$.

Then T possesses a unique fixed point.

Proof. Choose

$$\vartheta(t) = t, \qquad \theta(t) = kt.$$

Then all assumptions of Theorem 2.3 are satisfied. Hence T has a unique fixed point.

Corollary 3.3 Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(Tx, Tx, Ty) \leq k\max\left\{S(x, x, y), \frac{S(x,x,Ty)+S(y,y,Tx)}{2}\right\},$$

for all $x, y \in X$, where $0 < k < 1$.

Then T has a unique fixed point.

Proof. Since

$$ES(x, y) = \max\left\{S(x, x, y) + |S(x, x, Tx) - S(y, y, Ty)|, \frac{S(x,x,Ty)+S(y,y,Tx)}{2}\right\},$$

we have

$$\max\left\{S(x, x, y), \frac{S(x,x,Ty)+S(y,y,Tx)}{2}\right\} \leq ES(x, y).$$

Therefore,

$$S(Tx, Tx, Ty) \leq k\, ES(x, y).$$

Applying Corollary 4.2, we obtain the conclusion.

Corollary 3.4 Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + b|S(x, x, Tx) - S(y, y, Ty)|,$$

for all $x, y \in X$, where $a, b \geq 0$ and

$$a + b < 1.$$

Then T has a unique fixed point.

Proof. Observe that

$$aS(x, x, y) + b|S(x, x, Tx) - S(y, y, Ty)| \leq (a + b)ES(x, y).$$

Hence,

$$S(Tx, Tx, Ty) \leq (a + b)ES(x, y).$$

Since $a + b < 1$, the result follows from Corollary 4.2.

Corollary 3.5 Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(Tx, Tx, Ty) \leq k[S(x, x, Tx) + S(y, y, Ty)],$$

for all $x, y \in X$, where $0 < k < \frac{1}{2}$.

Then T possesses a unique fixed point.

Proof. Since

$$S(x, x, Tx) \leq ES(x, y), \qquad S(y, y, Ty) \leq ES(x, y),$$

we get

$$S(Tx, Tx, Ty) \leq 2k\, ES(x, y).$$

Because $2k < 1$, Corollary 4.2 yields the desired result.

Corollary 3.6 Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(T^2x, T^2x, T^2y) \leq kE_S^2(x, y),$$

for all $x, y \in X$, where $0 < k < 1$.

Assume further that:

1. T is triangular α-orbitally admissible;

2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
3. T^2 is continuous;
4. $\alpha(u, v) \geq 1$ for all fixed points.

Then T has a unique fixed point.

Proof. Choose

$$\vartheta(t) = t, \qquad \theta(t) = kt.$$

Then all assumptions of Theorem 2.6 are satisfied. Hence the result follows.

Corollary 3.7 Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(T^2x, T^2x, T^2y) \leq kS(x, x, y),$$

for all $x, y \in X$, where $0 < k < 1$.

Then T has a unique fixed point.

Proof. Since

$$S(x, x, y) \leq E_S^2(x, y),$$

we have

$$S(T^2x, T^2x, T^2y) \leq kE_S^2(x, y).$$

Applying the previous corollary, we obtain the conclusion.

Corollary 3.8 (Banach Type Consequence) Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(Tx, Tx, Ty) \leq kS(x, x, y),$$

for all $x, y \in X$, where $0 < k < 1$.

Then T admits a unique fixed point.

Proof. Since

$$S(x, x, y) \leq ES(x, y),$$

we obtain

$$S(Tx, Tx, Ty) \leq kES(x, y).$$

Now apply Corollary 4.2.

Corollary 3.9 (Kannan Type Consequence) Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(Tx, Tx, Ty) \leq k[S(x, x, Tx) + S(y, y, Ty)],$$

for all $x, y \in X$, where $0 < k < \frac{1}{2}$.

Then T possesses a unique fixed point.

Proof. The proof follows directly from Corollary 4.5.

Corollary 3.10 (Chatterjea Type Consequence) Let (X, S) be a complete S-metric space and let $T: X \to X$ satisfy

$$S(Tx, Tx, Ty) \leq k[S(x, x, Ty) + S(y, y, Tx)],$$

for all $x, y \in X$, where $0 < k < \frac{1}{2}$.

Then T has a unique fixed point.

Proof. Since

$$\frac{S(x, x, Ty) + S(y, y, Tx)}{2} \leq ES(x, y),$$

we obtain

$$S(Tx, Tx, Ty) \leq 2k\, ES(x, y).$$

As $2k < 1$, Corollary 4.2 completes the proof.

IV. Applications of Theorem 2.3

Theorem 4.1 (Integral Equation) Let $X = C([0,1], \mathbb{R})$ be endowed with the S-metric

$$S(x, y, z) = \| x - z \|_\infty + \| y - z \|_\infty.$$

Consider the integral equation

$$x(t) = \int_0^t K(t, s)\, f(s, x(s))\, ds, \quad t \in [0,1].$$

Assume that:

1. K is continuous and bounded, i.e., $|K(t, s)| \leq M$,
2. f is continuous and satisfies
   $|f(t, u) - f(t, v)| \leq L|u - v|,$

3. $ML < 1$.

Then the equation admits a unique solution in X.

Proof. Define $T: X \to X$ by

$$(Tx)(t) = \int_0^t K(t, s) f(s, x(s)) ds.$$

Step 1: Well-definedness. For $x \in X$, the function $s \mapsto f(s, x(s))$ is continuous, hence $K(t, s) f(s, x(s))$ is continuous. Therefore Tx is continuous and $T(X) \subseteq X$.

Step 2: Admissibility. Define $\alpha(x, y) = 1$ for all $x, y \in X$. Then T is trivially triangular α-orbitally admissible.

Step 3: Contractive estimate. For $x, y \in X$,

$$|Tx(t) - Ty(t)| \leq \int_0^t |K(t, s)|\, |f(s, x(s)) - f(s, y(s))| ds.$$

Using the assumptions,

$$\leq \int_0^t ML|x(s) - y(s)| ds \leq ML \| x - y \|_\infty.$$

Hence,

$$\| Tx - Ty \|_\infty \leq ML \| x - y \|_\infty.$$

Thus,

$$S(Tx, Tx, Ty) = 2 \| Tx - Ty \|_\infty \leq 2ML \| x - y \|_\infty.$$

Step 4: S-metric form. Since

$$S(x, x, y) = 2 \| x - y \|_\infty,$$

we get

$$S(Tx, Tx, Ty) \leq ML\, S(x, x, y).$$

Step 5: Control via $E_S(x, y)$. Since $E_S(x, y) \geq S(x, x, y)$, we obtain

$$S(Tx, Tx, Ty) \leq ML\, E_S(x, y).$$

Step 6: Apply Theorem 2.3. Take $\vartheta(t) = t$ and $\theta(t) = ML\, t$. Since $ML < 1$, all conditions are satisfied. Hence T has a fixed point.

Step 7: Uniqueness. Follows from Theorem 2.4.

Thus the integral equation has a unique solution.

Theorem 4.2 (Differential Equation) *Let* $X = C([0,1], \mathbb{R})$ *be endowed with the S-metric*

$$S(x,y,z) = \| x - z \|_\infty + \| y - z \|_\infty.$$

Consider the initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0.$$

Assume that:

1. f is continuous,
2. $|f(t,u) - f(t,v)| \leq L|u - v|$,
3. $L < 1$.

Then the problem has a unique solution in X.

Proof. Rewrite the problem as

$$x(t) = x_0 + \int_0^t f(s, x(s))ds.$$

Define $T: X \to X$ by

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s))ds.$$

Step 1: Well-definedness. Continuity of f implies $Tx \in X$.

Step 2: Admissibility. Let $\alpha(x,y) = 1$. Then T is triangular α-orbitally admissible.

Step 3: Contractive estimate.

$$|Tx(t) - Ty(t)| \leq \int_0^t |f(s,x(s)) - f(s,y(s))|ds \leq L \int_0^t |x(s) - y(s)|ds \leq L \| x - y \|_\infty.$$

Thus,

$$\| Tx - Ty \|_\infty \leq L \| x - y \|_\infty,$$

and hence

$$S(Tx, Tx, Ty) \leq LS(x, x, y).$$

Step 4: Use of $\boldsymbol{E_S(x,y)}$. Since $E_S(x,y) \geq S(x,x,y)$,

$$S(Tx, Tx, Ty) \leq LE_S(x,y).$$

Step 5: Apply Theorem 2.3. Take $\vartheta(t) = t$ and $\theta(t) = Lt$. Since $L < 1$, all conditions hold. Thus T has a fixed point.

Step 6: Uniqueness. Follows from Theorem 2.4.

Hence the differential equation has a unique solution.

V. Conclusion

In this article, we have presented the new notion of Proinov type E-contractions and proved the related fixed point results in the setting of complete S-metric space. Then, some examples are solved to illustrate the proved results. To show the real existence of proved theorems some applications are given.

In future the researchers can extend these results in the frame of generalized S-metric space for two or four maps. They, can also apply these results to solve the fractional differential equations.

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