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Research Paper

Common fixed point theorems for interpolative Meir-Keeler type contractions in (α, c)-interpolative metric space

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Abstract
In the present article, we shall give the new notion of interpolative Kannan-Meir-Keeler type contractions and interpolative Jaggi-Meir-Keeler type contractions for two maps in the setting of (α, c)-interpolative metric space. Then, we shall prove the common fixed point theorems for these two contractions. Many examples are given to illustrate the proved results. Some results for single map are also deduced from main results in the form of corollaries. Finally, in the form of application a boundary value problem is solved to show the real existence of proved results.
Keywords: *common fixed point,(α, c)-interpolative metric space, self map.*

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I. INTRODUCTION

Banach [1] contraction principle states that every contraction map on a complete metric space has a unique fixed point. It is the central idea of metrical fixed point theory. Then, by seeing the utility of Banach contraction principle many authors generalize the notion of this principle as well as metric space to get new fixed point theorems. It has various applications in applied mathematics and other branch of sciences.

To get new fixed point theorems Karapinar [5] in 2023, generalize the notion of metric space and gave (α, c)-interpolative metric space.

Definition 1 [5] Let $X \neq \emptyset$. A function $d : X \times X \to [0, \infty)$ is called an (α, c)–interpolative metric if for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. there exist $\alpha \in (0, 1)$ and $c \geq 0$ such that

$$d(x, y) \leq d(x, z) + d(z, y) + c\,[d(x, z)]^{\alpha}[d(z, y)]^{1-\alpha}.$$

In this case, (X, d) is called an (α, c)–interpolative metric space.

Definition 2 [5] Let (X, d) be an (α, c)-interpolative metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges to $x \in X$ if and only if

$$d(x_n, x) \to 0 \quad \text{as } n \to \infty.$$

Definition 3 [5] Let (X, d) be an (α, c)-interpolative metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence if and only if

$$\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0.$$

Definition 4 [5] Let (X, d) be an (α, c)-interpolative metric space. We say that (X, d) is complete if every Cauchy sequence in X converges to a point in X.

Again in 2024, Karapinar [6] proved the following theorem in complete (α, c)-interpolative metric space.

Theorem 1 *[6] Let (X, d) be an (α, c)-interpolative metric space and let $T : X \to X$ be a mapping. Suppose that there exists q with $0 < q < 1$ such that*

$$d(Tx, Ty) \leq q\, M_d(x, y), \tag{1}$$

for all $x, y \in X$, where

$$M_d(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then T possesses a unique fixed point in X.

Several fixed point results in (α, c)-interpolative metric space can be find in ([2],[7]).

In 1969, Meir and Keeler [11] gave the new type of contraction as defined below:

Definition 5 [11] Let (X, ρ) be a metric space. A mapping $S : X \to X$ is said to satisfy Meir-Keeler type contraction such that:
for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < \rho(x, y) < \varepsilon + \delta \;\Rightarrow\; \rho(Sx, Sy) \leq \varepsilon.$$

Then, they show that if S is a Meir-Keeler type contraction then it has a unique fixed point.
After that many authors generalize the notion of Meir-Keeler contractions in various spaces see ([10],[12]).
In the sequence of generalization the notion of Meir-Keeler contraction in 2018, Karapinar [9] gave the following definition of interpolative Kannan-Meir-Keeler contraction in metric space.

Definition 6 [9] Let (X, ρ) be a metric space. A mapping $S : X \to X$ is said to satisfy Meir-Keeler type contraction if there exists $\gamma \in (0, 1)$ such that:
for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < [d(x, Sx)]^{\gamma} [d(y, Sy)]^{1-\gamma} < \varepsilon + \delta \;\Rightarrow\; \rho(Sx, Sy) \leq \varepsilon$$

Then, he proved the existence of fixed point of S.

Jaggi [3] in 1977 proved the following theorem.

Theorem 2 *[3] Let (X, ρ) be a complete metric space and $S : X \to X$ be a continuous map. Assume that there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that:*

$$\rho(Sx, Sy) \leq \alpha \frac{\rho(x, Sx).\rho(y, Sy)}{\rho(x, y)} + \beta \rho(x, y)$$

holds for every distinct $x, y \in X$.
Then, S has a unique fixed point.

Many researchers generalized the notion of Jaggi type contraction see ([4],[8],[13]).

Inspired from Definition 6 and Theorem 2, we present the new notion of interpolative Kannan-Meir-Keeler and interpolative Jaggi-Meir-Keeler contractions for two maps in the next section.

2 Results

Here, we present the main results and give some examples in the support of proven results.

Definition 7 Let (X,d) be an (α,c)–interpolative metric space. Two mappings $S,T : X \to X$ are said to satisfy an interpolative Kannan–Meir–Keeler type contraction if there exists $\gamma \in (0,1)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < [d(x,Sx)]^{\gamma}[d(y,Ty)]^{1-\gamma} < \varepsilon+\delta \Rightarrow d(Sx,Ty) \le \varepsilon, \quad for \quad all \quad x,y \in X. \tag{2}$$

From equation (2), we conclude that

$$d(Sx,Ty) < [d(x,Sx)]^{\gamma}[d(y,Ty)]^{1-\gamma}, \tag{3}$$

for all $x,y \in X$ with $x \neq Sx$, $y \neq Ty$.

Theorem 3 *Let (X,d) be a complete (α,c)–interpolative metric space and let $S,T : X \to X$ are Kannan-Meir-Keeler type contraction. Assume that one of S and T is continuous. Then, T has a common fixed point that is there exists $x^* \in X$ such that*

$$Sx^* = Tx^* = x^*.$$

Proof Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n \ge 0.$$

Using equation (3), we obtain

$$d(x_{2n+1},x_{2n+2}) = d(Sx_{2n},Tx_{2n+1}) < [d(x_{2n},x_{2n+1})]^{\gamma}[d(x_{2n+1},x_{2n+2})]^{1-\gamma}. \tag{4}$$

Now, if possible assume that $d(x_{2n+1},x_{2n+2}) > d(x_{2n},x_{2n+1})$, then equation (4) becomes $d(x_{2n},x_{2n+1}) < d(x_{2n},x_{2n+1})$.
a contradiction.
Hence,

$$d(x_{2n+1},x_{2n+2}) < d(x_{2n},x_{2n+1}). \tag{5}$$

Thus, $\{d(x_{2n},x_{2n+1})\}$ is decreasing sequence of positive numbers so bounded below by 0. Hence, it converges to some $\omega \ge 0$.
Assume $\omega > 0$.
Then, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\omega < d(x_{2n},x_{2n+1}) < \omega+\delta(\omega).$$

By the use of equation (2), we get

$$d(Sx_{2n},Tx_{2n+2}) = d(x_{2n+1},x_{2n+2}) \le \omega,$$

By choosing $\varepsilon = \omega$.
Thus, a contradiction is arrived.

So, $\lim_{n\to\infty} d(x_{2n},x_{2n+1}) = 0$.
Hence, we can say that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{6}$$

Now, our claim is that the sequence $\{x_n\}$ is a Cauchy sequence.
For each $\varepsilon.\frac{2}{2+c} > 0$ choose δ in such a way that $\delta < \varepsilon.\frac{2}{2+c}$.
From equation (6), we can say that there exists $k \in \mathbb{N}$ such that

$$d(x_n,x_{n+1}) < \frac{\varepsilon}{2}.\frac{2}{2+c} = \frac{\varepsilon}{2+c}, \tag{7}$$

for all $n \ge k$.
Next we show that

$$d(x_n,x_{n+q}) < \varepsilon, \tag{8}$$

for any $q \in \mathbb{N}$.
From equation (7), we conclude that equation (8) holds for $q = 1$. Now, we shall assume that result is true for any q. So, its enough to prove the result for $q+1$.
Next, using the (α,c)–interpolative inequality

$$d(x_n,x_{n+q+1}) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+q+1}) + c[d(x_n,x_{n+1})]^{\alpha}[d(x_{n+1},x_{n+q+1})]^{1-\alpha} \tag{9}$$

Now, by equation (3), we get

$$d(x_{n+1}, x_{n+q+1}) = d(Sx_n, Tx_{n+q}) \leq [d(x_n, Sx_n)]^{\gamma}[x_{n+q}, Tx_{n+q}]^{1-\gamma},$$
$$= [d(x_n, x_{n+1})]^{\gamma}[x_{n+q}, x_{n+q+1}]^{1-\gamma},$$
$$< \frac{\varepsilon}{2+c}. \quad (10)$$

With the help of equations (3) and (10), equation (9) becomes

$$d(x_n, x_{n+q+1}) < \frac{\varepsilon}{2+c} + \frac{\varepsilon}{2+c} + \frac{c.\varepsilon}{2+c} = \varepsilon.$$

It follows that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Assume that S is continuous then, $x_{2n} \to x^*$ implies that $Sx_{2n} = x_{2n+1} \to Sx^*$. Now, uniqueness of limit implies that
$Sx^* = x^*$.
Again, from equation (2) we conclude that

$$d(Sx^*, Tx^*) = d(x^*, Tx^*) = 0.$$

So, $Sx^* = Tx^* = x^*$. □

Example 1 Let $X = [0, 1]$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = |x - y|^3, \quad x, y \in X.$$

Define mappings $S, T : X \to X$ by

$$S(x) = \frac{x}{2}, \qquad T(x) = \frac{x}{3}.$$

Then (X, d) is a complete (α, c)-interpolative metric space and S, T satisfy the interpolative Kannan–Meir–Keeler contraction condition of equation (2). Consequently, S and T have a unique common fixed point in X.

Proof For all $x, y, z \in X$,

$$|x - y|^3 = |(x - z) + (z - y)|^3 \leq 4\big(|x - z|^3 + |z - y|^3\big).$$

Hence,

$$d(x, y) \leq 4\big(d(x, z) + d(z, y)\big).$$

Using $a + b \leq a + b + 2\sqrt{ab}$ for $a, b \geq 0$, we obtain

$$d(x, y) \leq d(x, z) + d(z, y) + 6[d(x, z)]^{1/2}[d(z, y)]^{1/2}.$$

Thus, (X, d) is an (α, c)-interpolative metric space with

$$\alpha = \tfrac{1}{2}, \quad c = 6.$$

Now, let (x_n) be a Cauchy sequence in (X, d). Then

$$d(x_n, x_m) = |x_n - x_m|^3 \to 0 \quad \text{as } n, m \to \infty,$$

which implies $|x_n - x_m| \to 0$.
Hence $\{x_n\}$ is Cauchy in the usual metric on $[0, 1]$. Since $[0, 1]$ is complete, there exists $x \in [0, 1]$ such that $x_n \to x$. Moreover,

$$d(x_n, x) = |x_n - x|^3 \to 0,$$

so (X, d) is complete.
Now, to verify equation (2), we compute:

$$d(x, Sx) = |x - x/2|^3 = \frac{x^3}{8}, \qquad d(y, Ty) = |y - y/3|^3 = \frac{8y^3}{27}.$$

Thus,

$$[d(x, Sx)]^{1/2}[d(y, Ty)]^{1/2} = \frac{(xy)^{3/2}}{\sqrt{27}}. \quad (11)$$

Also,

$$d(Sx, Ty) = \left|\frac{x}{2} - \frac{y}{3}\right|^3. \quad (12)$$

Using

$$\left|\frac{x}{2} - \frac{y}{3}\right| \leq x + y \leq 2\sqrt{xy},$$

we obtain

$$d(Sx, Ty) \leq 8(xy)^{3/2}. \quad (13)$$

Let $\varepsilon > 0$ and assume

$$\varepsilon < \frac{(xy)^{3/2}}{\sqrt{27}} < \varepsilon + \delta.$$

Then

$$(xy)^{3/2} < \sqrt{27}(\varepsilon + \delta).$$

Hence, equation (13) becomes

$$d(Sx, Ty) \leq 8\sqrt{27}(\varepsilon + \delta).$$

Now, choose

$$\delta = \frac{\varepsilon}{16\sqrt{27}}.$$

Then,

$$d(Sx, Ty) \leq \varepsilon.$$

Thus, the equation (2) hold.
So, by Theorem 3, T and S has a common fixed point.

Solving

$$Sx = x \Rightarrow \frac{x}{2} = x \Rightarrow x = 0,$$
$$Tx = x \Rightarrow \frac{x}{3} = x \Rightarrow x = 0,$$

we obtain the common fixed point $x^* = 0$.

□

Definition 8 Let (X, d) be an (α, c)–interpolative metric space. Mappings $S, T : X \to X$ are said to satisfy a two-map Jaggi-Meir–Keeler type contraction if there exist $\gamma \in (0,1)$ and $\lambda \in (0,1)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < \lambda \frac{d(x, Sx)\, d(y, Ty)}{1 + d(x, y)} + (1-\lambda)[d(x, Sx)]^{\gamma}[d(y, Ty)]^{1-\gamma} < \varepsilon + \delta \tag{14}$$

implies

$$d(Sx, Ty) \leq \varepsilon,$$

for all $x, y \in X$.
From equation (14), we conclude that

$$d(Sx, Ty) < \lambda \frac{d(x, Sx)\, d(y, Ty)}{1 + d(x, y)} + (1-\lambda)[d(x, Sx)]^{\gamma}[d(y, Ty)]^{1-\gamma},$$

for all $x, y \in X$ with $x \neq Sx,\ y \neq Ty$.

Theorem 4 *Let (X, d) be a complete (α, c)–interpolative metric space and let $S, T : X \to X$ are Jaggi-Meir-Keeler type contraction. Assume that one of S and T is continuous. Then there exists $x^* \in X$ such that*

$$Sx^* = Tx^* = x^*.$$

Proof Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n \geq 0.$$

From equation (14), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &< \lambda \frac{d(x_{2n}, Sx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \\ &+ (1-\lambda)[d(x_{2n}, Sx_{2n})]^{\gamma}[d(x_{2n+1}, Tx_{2n+1})]^{1-\gamma}, \\ &= \lambda \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \\ &+ (1-\lambda)[d(x_{2n}, x_{2n+1})]^{\gamma}[d(x_{2n+1}, x_{2n+2})]^{1-\gamma}. \end{aligned} \tag{15}$$

Now, if possible assume that $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$.
Hence, equation (15) implies that

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2}).$$

a contradiction.
Thus, $\{d(x_{2n}, x_{2n+1})\}$ is decreasing sequence of positive numbers so bounded below by 0.
Hence, it converges to some $\omega \geq 0$.
Assume $\omega > 0$.
Then, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\omega < d(x_{2n}, x_{2n+1}) < \omega + \delta(\omega).$$

By the use of equation (14), we get

$$d(Sx_{2n}, Tx_{2n+2}) = d(x_{2n+1}, x_{2n+2}) \leq \omega,$$

By choosing $\varepsilon = \omega$.
Thus, a contradiction is arrived.

So, $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$.
Hence, we can say that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{16}$$

Now, our claim is that the sequence $\{x_n\}$ is a Cauchy sequence.
For each $\varepsilon.\frac{2}{2+c} > 0$ choose δ in such a way that $\delta < \varepsilon.\frac{2}{2+c}$.
From equation (16), we can say that there exists $k \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \frac{\varepsilon}{2}.\frac{2}{2+c} = \frac{\varepsilon}{2+c}, \tag{17}$$

for all $n \geq k$.
Assume $\zeta = \frac{\varepsilon}{2+c}$.

Next we show that

$$d(x_n, x_{n+q}) < \varepsilon, \tag{18}$$

for any $q \in \mathbb{N}$.
From equation (17), we conclude that equation (18) holds for $q = 1$. Now, we shall assume that result is true for any q. So, its enough to prove the result for $q + 1$.
Next, using the (α, c)-interpolative inequality

$$d(x_n, x_{n+q+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+q+1}) + c[d(x_n, x_{n+1})]^{\alpha}[d(x_{n+1}, x_{n+q+1})]^{1-\alpha} \tag{19}$$

Now, by equation (14), we get

$$\begin{aligned} d(x_{n+1}, x_{n+q+1}) = d(Sx_n, Tx_{n+q}) &\leq \lambda\frac{d(x_n, Sx_n)\, d(x_{n+q}, Tx_{n+q})}{1 + d(x_n, x_{n+q})} \\ &+ (1-\lambda)[d(x_n, Sx_n)]^{\gamma}[d(x_{n+q}, Tx_{n+q})]^{1-\gamma}, \\ &= \lambda\frac{d(x_n, x_{n+1})\, d(x_{n+q}, x_{n+q+1})}{1 + d(x_n, x_{n+q})} \\ &+ (1-\lambda)[d(x_n, x_{n+1})]^{\gamma}[d(x_{n+q}, x_{n+q+1})]^{1-\gamma}, \\ &= \lambda\frac{\zeta.\zeta}{1+\zeta} + (1-\lambda)\zeta, \\ &< \lambda.\zeta + (1-\lambda)\zeta, \\ &= \zeta, \\ &= \frac{\varepsilon}{2+c}. \end{aligned} \tag{20}$$

With the help of equations (17) and (20), equation (19) becomes

$$d(x_n, x_{n+q+1}) < \frac{\varepsilon}{2+c} + \frac{\varepsilon}{2+c} + \frac{c.\varepsilon}{2+c} = \varepsilon.$$

It follows that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Assume that S is continuous then, $x_{2n} \to x^*$ implies that $Sx_{2n} = x_{2n+1} \to Sx^*$. Now, uniqueness of limit implies that
$Sx^* = x^*$.
Again, from equation (14) we conclude that

$$d(Sx^*, Tx^*) = d(x^*, Tx^*) = 0,$$

So, $Sx^* = Tx^* = x^*$. □

Example 2 Let $X = [0, 1]$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = |x - y|^3.$$

Then (X, d) is a complete (α, c)-interpolative metric space with $\alpha = \frac{1}{2}$ and $c = 6$.
Define mappings $S, T : X \to X$ by

$$S(x) = 0, \qquad T(x) = \frac{x}{8}, \quad x \in X.$$

Then, for all $x, y \in X$,

$$d(x, Sx) = x^3, \qquad d(y, Ty) = \left|y - \frac{y}{8}\right|^3 = \left(\frac{7y}{8}\right)^3 = \frac{343}{512}y^3,$$

and

$$d(Sx, Ty) = \left|\frac{y}{8}\right|^3 = \frac{y^3}{512}.$$

Let $\lambda = \frac{1}{2}$ and $\gamma = \frac{1}{2}$. Consider

$$A(x, y) = \lambda\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + (1 - \lambda)[d(x, Sx)]^\gamma [d(y, Ty)]^{1-\gamma}.$$

Then

$$A(x, y) \geq \frac{1}{2}x^{3/2}\left(\frac{343}{512}\right)^{1/2} y^{3/2} = \frac{7\sqrt{7}}{32\sqrt{2}}x^{3/2}y^{3/2}.$$

Let $\varepsilon > 0$ and suppose that

$$\varepsilon < A(x, y) < \varepsilon + \delta.$$

Then

$$x^{3/2}y^{3/2} < \frac{32\sqrt{2}}{7\sqrt{7}}(\varepsilon + \delta).$$

Since $x \in [0, 1]$, we have $x^{3/2} \leq 1$, and hence

$$y^{3/2} < \frac{32\sqrt{2}}{7\sqrt{7}}(\varepsilon + \delta).$$

Using $0 \leq y \leq 1$, it follows that

$$y^3 = (y^{3/2})^2 \leq y^{3/2} < \frac{32\sqrt{2}}{7\sqrt{7}}(\varepsilon + \delta).$$

Therefore,

$$d(Sx, Ty) = \frac{y^3}{512} < \frac{1}{512} \cdot \frac{32\sqrt{2}}{7\sqrt{7}}(\varepsilon + \delta) = \frac{\sqrt{2}}{112\sqrt{7}}(\varepsilon + \delta).$$

Set

$$C = \frac{\sqrt{2}}{112\sqrt{7}} < 1.$$

Choose

$$\delta = \frac{1}{2}\varepsilon\left(\frac{1}{C} - 1\right) > 0.$$

Then

$$d(Sx, Ty) \leq \varepsilon.$$

Thus, S and T satisfy the Jaggi-Meir–Keeler type contraction condition of equation (14). Thus, all the conditions of Theorem 4. Hence, S and T have common fixed point.

Finally, solving

$$Sx = x \Rightarrow x = 0, \qquad Tx = x \Rightarrow x = 0,$$

we conclude that $x^* = 0$ is the unique common fixed point of S and T.

3 Consequences

Definition 9 Let (X, d) be an (α, c)–interpolative metric space. A self map $S : X \to X$ is said to satisfy an interpolative Kannan–Meir–Keeler type contraction if there exists $\gamma \in (0, 1)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < [d(x, Sx)]^\gamma [d(y, Sy)]^{1-\gamma} < \varepsilon + \delta \;\Rightarrow\; d(Sx, Sy) \leq \varepsilon, \quad for \quad all \quad x, y \in X. \tag{21}$$

From equation (21), we conclude that

$$d(Sx, Sy) < [d(x, Sx)]^\gamma [d(y, Sy)]^{1-\gamma}, \tag{22}$$

for all $x, y \in X$ with $x \neq Sx, y \neq Sy$.

Corollary 1 Let (X, d) be a complete (α, c)–interpolative metric space and let $S : X \to X$ is interpolative Kannan-Meir-Keeler type contraction. Then, T has a fixed point.

Proof By taking $T = S$ in Theorem 3, one can obtain the proof. □

Definition 10 Let (X, d) be an (α, c)–interpolative metric space. Mappings $S : X \to X$ is said to satisfy a Jaggi-Meir–Keeler type contraction if there exist $\gamma \in (0, 1)$ and $\lambda \in (0, 1)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < \lambda \frac{d(x, Sx)\, d(y, Ty)}{1 + d(x, y)} + (1 - \lambda)[d(x, Sx)]^{\gamma}[d(y, Ty)]^{1-\gamma} < \varepsilon + \delta \tag{23}$$

implies

$$d(Sx, Ty) \le \varepsilon,$$

for all $x, y \in X$.
From equation (23), we conclude that

$$d(Sx, Ty) < \lambda \frac{d(x, Sx)\, d(y, Ty)}{1 + d(x, y)} + (1 - \lambda)[d(x, Sx)]^{\gamma}[d(y, Ty)]^{1-\gamma},$$

for all $x, y \in X$ with $x \neq Sx,\ y \neq Sy$.

Corollary 2 Let (X, d) be a complete (α, c)–interpolative metric space and let $S : X \to X$ satisfy equation (23). Then, S has a fixed point.

Proof By inserting $T = S$ in Theorem 4, we can get the proof. □

4 Solution of Boundary Value Problem

Theorem 5 *Let $f : [0, 1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist constants $L \in (0, 8)$ and $\gamma \in (0, 1)$ such that*

$$|f(t, u) - f(t, v)| \le L\, |u - \mathcal{S}u|^{\gamma} |v - \mathcal{T}v|^{1-\gamma},$$

for all $t \in [0, 1]$ and $u, v \in \mathbb{R}$, where the operators $S : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ is defined by

$$Sx(t) = \int_0^1 G(t, s) f(s, x(s))\, ds$$

and

$$G(t, s) = \begin{cases} t(1 - s), & 0 \le t \le s \le 1, \\ s(1 - t), & 0 \le s \le t \le 1. \end{cases}$$

Then the boundary value problem

$$\begin{cases} x''(t) = f(t, x(t)), & t \in [0, 1], \\ x(0) = x(1) = 0, \end{cases}$$

admits a unique solution in (α, c)-interpolative metric space $C([0, 1], \mathbb{R})$.

Proof Let

$$X = C([0, 1], \mathbb{R})$$

and define

$$d(x, y) = \|x - y\|_{\infty}^{3}, \qquad x, y \in X,$$

where

$$\|x\|_{\infty} = \sup_{t\in[0,1]} |x(t)|.$$

By Example 1, (X, d) is a complete (α, c)-interpolative metric space with

$$\alpha = \frac{1}{2}, \qquad c = 6.$$

The given boundary value problem is equivalent to the Hammerstein integral equation

$$x(t) = \int_0^1 G(t, s) f(s, x(s))\, ds. \tag{24}$$

Therefore, a solution of (24) is precisely a fixed point of S.

First, we show that S is continuous. Let $\{x_n\} \subset X$ be such that $x_n \to x$ uniformly on $[0, 1]$. Since f is continuous, it follows that $f(s, x_n(s)) \to f(s, x(s))$ uniformly in $s \in [0, 1]$. Consequently,

$$\|Sx_n - Sx\|_{\infty} = \sup_{t\in[0,1]} \left| \int_0^1 G(t, s)\big(f(s, x_n(s)) - f(s, x(s))\big)\, ds \right|$$

$$\le \sup_{t\in[0,1]} \int_0^1 G(t, s)\, ds \sup_{s\in[0,1]} |f(s, x_n(s)) - f(s, x(s))|.$$

Letting $n \to \infty$, we obtain

$$Sx_n \to Sx.$$

Hence, S is continuous.
Let $x, y \in X$ and $t \in [0,1]$. Then

$$\begin{aligned}|Sx(t) - Sy(t)| &= \left|\int_0^1 G(t,s)\big(f(s,x(s)) - f(s,y(s))\big)\,ds\right| \\ &\le \int_0^1 G(t,s)\,|f(s,x(s)) - f(s,y(s))|\,ds.\end{aligned}$$

Using the hypothesis, we get

$$\begin{aligned}|Sx(t) - Ty(t)| &\le L\int_0^1 G(t,s)|x(s) - Sx(s)|^{\gamma}|y(s) - Sy(s)|^{1-\gamma}\,ds \\ &\le L\,\|x - Sx\|_\infty^{\gamma}\|y - Ty\|_\infty^{1-\gamma}\int_0^1 G(t,s)\,ds.\end{aligned}$$

Taking supremum over $t \in [0,1]$, we obtain

$$\|Sx - Sy\|_\infty \le L\,\|x - Sx\|_\infty^{\gamma}\|y - Sy\|_\infty^{1-\gamma}\sup_{t\in[0,1]}\int_0^1 G(t,s)\,ds. \tag{25}$$

Now,

$$\int_0^1 G(t,s)\,ds = \int_0^t s(1-t)\,ds + \int_t^1 t(1-s)\,ds = \frac{t(1-t)}{2}.$$

Therefore,

$$\sup_{t\in[0,1]}\int_0^1 G(t,s)\,ds = \sup_{t\in[0,1]}\frac{t(1-t)}{2} = \frac{1}{8}.$$

Substituting this into (25), we get

$$\|Sx - Sy\|_\infty \le \frac{L}{8}\|x - Sx\|_\infty^{\gamma}\|y - Sy\|_\infty^{1-\gamma}. \tag{26}$$

Cubing both sides of (26), we obtain

$$\begin{aligned}d(Sx, Sy) &= \|Sx - Ty\|_\infty^3 \\ &\le \left(\frac{L}{8}\right)^3\|x - Sx\|_\infty^{3\gamma}\|y - Sy\|_\infty^{3(1-\gamma)} \\ &= \left(\frac{L}{8}\right)^3[d(x,Sx)]^{\gamma}[d(y,Sy)]^{1-\gamma}.\end{aligned}$$

Since $L < 8$, we have

$$\left(\frac{L}{8}\right)^3 < 1.$$

Hence,

$$d(Sx, Sy) < [d(x,Sx)]^{\gamma}[d(y,Sy)]^{1-\gamma}.$$

Thus, S satisfy the interpolative Kannan–Meir–Keeler contractive condition.

All the hypotheses of Corollary 1 are therefore satisfied. Hence, the nonlinear boundary value problem admits a solution in $C([0,1],\mathbb{R})$. □

Example 3 Consider the nonlinear boundary value problem

$$\begin{cases} x''(t) = 1 + \frac{1}{20}\sin(x(t)), & 0 \le t \le 1, \\ x(0) = 0, \quad x(1) = 0. \end{cases} \tag{27}$$

We shall show that problem (27) possesses a unique nontrivial solution by applying Corollary 1.

Step 1: Integral formulation. The Green function associated with the linear problem

$$x''(t) = h(t), \qquad x(0) = x(1) = 0,$$

is given by

$$G(t,s) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

Define the operator $S : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$Sx(t) = \int_0^1 G(t,s)\left(1 + \frac{1}{20}\sin(x(s))\right)ds.$$

Then a fixed point of S is precisely a solution of (27).

Step 2: The underlying metric space. Let

$$X = C([0,1],\mathbb{R}),$$

equipped with the metric

$$d(x,y) = \|x-y\|_\infty^3, \qquad \|x\|_\infty = \sup_{t\in[0,1]} |x(t)|.$$

Then (X,d) is a complete (α,c)-interpolative metric space.

Step 3: Verification of the contractive condition. Let $x, y \in X$. For any $t \in [0,1]$, we have

$$|(Sx)(t) - (Ty)(t)| = \left|\int_0^1 G(t,s)\frac{\sin(x(s)) - \sin(y(s))}{20}\,ds\right|$$
$$\leq \frac{1}{20}\int_0^1 G(t,s)\,|\sin(x(s)) - \sin(y(s))|\,ds.$$

Using the elementary inequality

$$|\sin a - \sin b| \leq |a-b|,$$

we obtain

$$|(Sx)(t) - (Ty)(t)| \leq \frac{1}{20}\int_0^1 G(t,s)\,|x(s)-y(s)|\,ds$$
$$\leq \frac{1}{20}\|x-y\|_\infty \int_0^1 G(t,s)\,ds.$$

Now,

$$\int_0^1 G(t,s)\,ds = \frac{t(1-t)}{2} \leq \frac{1}{8}, \qquad t\in[0,1].$$

Hence,

$$|(Sx)(t) - (Ty)(t)| \leq \frac{1}{160}\|x-y\|_\infty.$$

Taking supremum over $t \in [0,1]$, we get

$$\|Sx - Ty\|_\infty \leq \frac{1}{160}\|x-y\|_\infty.$$

Therefore,

$$d(Sx,Ty) = \|Sx-Ty\|_\infty^3 \leq \left(\frac{1}{160}\right)^3 d(x,y).$$

Thus, S is a strict contraction and consequently satisfies the contractive condition of Corollary 1.

Step 4: Existence and uniqueness. All the hypotheses of Corollary 1 are satisfied. Therefore, S possess a fixed point $x^* \in X$. Hence, problem (27) admits a solution.

Step 5: Non triviality of the solution. Suppose that $x(t) \equiv 0$ on $[0,1]$. Then

$$x''(t) = 0,$$

whereas equation (27) yields

$$x''(t) = 1 + \frac{1}{20}\sin 0 = 1,$$

which is impossible. Therefore, the solution is nontrivial.

Consequently, the boundary value problem (27) has a nonzero solution in $C([0,1],\mathbb{R})$.

5 Conclusion

In this article, we have introduced the new notions of interpolative Kannan-Meir-Keeler and Jaggi-Meir-Keeler type contractions for two maps in the setting of (α,c)-interpolative metric space. Then, related fixed point theorems are proved and examples are given to show the real existence of results. Some consequences of the proved theorems are also given. Finally, a boundary value problem is solved with the aid of proved result.

In future, the authors can extend these results in the setting of generalized (α,c)-interpolative metric space.

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