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Research Paper

To the spectral theory of polynomial Keldysh bundle.

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The solving of Caushy problem for the operator- differential equation with the several initial conditions of the type

$$(E - A_0 - A_1 \frac{d}{dt} - \ldots - A_n \frac{d^n}{dt^n})x = 0 \qquad (1)$$

led to the research of polynomial operator bundle

$$A(\lambda) = A_0 + \lambda A_1 B + \lambda^2 A_2 B^2 + \ldots + \lambda^{n-1} A_{n-1} B^{n-1} + \lambda^n B^n, \qquad (2)$$

which is known as Keldysh bundle. It is known that the solving of the Caushy problem is closely connected with the questions of multiple completeness of eigen and associated vectors of the bundle (2)[1],[5]. We give some necessary definitions.

1.If for some nonzero vector y_0 we have $A(c)y_{0=} = y_0$,then y_0 is called an eigenvector of operator $A(\lambda)$,corresponding to eigenvalue c [4],[5].

2. Vector y_k is called an $k-th$ -associated vector to the

Eigenvector y_0 if the following equalities

$$y = A(c)y$$

$$y_1 = A(c)y_1 + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y$$

....

$$y_k = A(c)y_k + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y_{k-1} + \ldots + \frac{1}{k!}\frac{\partial^k A(c)}{\partial c^k}y \qquad (3)$$

are fulfilled[1]

The system of linear independent eigen and associated vectors is called a chain of eigen and associated vectors of operator $A(\lambda)$,corresponding to eigenvalue c [1]. The number of eigen and associated vectors in the chain is called a length

of eigenvector y_0 .[1]. The set of all independent eigen and associated vectors ,corresponding to all eigenvectors with the eigenvalue c ,is called the multiplicity of the eigenvalue c .

3.M.V.Keldysh built the derivative systems with the help of the formulas[1]

$$\left[\frac{d^k}{dt^k}e^{\lambda t}\left(x_k+\frac{1}{1!}x_{k-1}+\ldots+\frac{1}{k!}x_0\right)\right](t=0) \quad k=1,2,\ldots,s \qquad (4)$$

4. System of eigen and associated vectors of operator bundle $A(\lambda)$ in space H

forms the n – multiple complete system if any n elements $f_0, f_1, \ldots, f_{n-1}$ of the space H can be approximated with the help of linear combinations of elements $\{x_i^{(j)}\}_{i=1}^{\infty}, j=0,1,2,\ldots,n-1$ in accordance with predetermined accuracy and the same coefficients, not depending on indices of elements $f_0, f_1, \ldots, f_{n-1}$ [1],[5].

5.The system of subspaces $\{M_k\}_{k=1}$ forms n -multiple basis if any element x may be presented in the form $x=\sum_{j=1}^{\infty} {}_1x_j$ where x_j from M_j [5].

6.The completely continuous operator A has a finite order if the series of eigenvalues of operator $\left(A^*A\right)^{\frac{1}{2}}$ in some positive degrees converge. Lower boundary of such degrees is called an order of operator $= A$. Set of completely continuous operators of finite order p forms of class.... G_p . In [12] it was introduced the class G_{-p} of operators of negative order

$-p \quad (p>0)$ .Really, operator A has a negative order $-p$ if for A

$KerA=0$, and A^{-1}, is a completely continuous operator of order p . Thus, the formula for the product $A_1 A_2$ of two completely continuous of finite orders p_1 and p_2 , operators A_1 and A_2 can be extended on the case when one or boith of the operators A_1 and A_2 have the negative orders. For example, let operators A_1 and A_2 have p_1 and $-p_2$ orders, correspondingly,(p_1 and p_2 may be positive or negative numbers) , then the order of operator $A_1 A_2$ is equal to $\frac{1}{p_1}-\frac{1}{p_2}=\frac{1}{p}$,or $p=\frac{p_1\, p_2}{p_1-p_2}$ It should be need that the order of $A_1 A_2$ may be both positive and negative number.

We consider s polynomial bundles

$$A_{0,k} + \lambda A_{1,k} B_k + \lambda^2 A_{2,k} B_k{}^2 + \ldots + \lambda^{n-1} A_{n-1,k} B_k{}^{n-1} + \lambda^n B_k{}^n,$$
$$k = 1, 2, \ldots, s \quad (5)$$

and

$$A(\lambda) = A_1(\lambda) + A_2(\lambda) + \ldots + A_s(\lambda) \quad (6)$$

Theorem 1. Let the following conditions:

a)operators $B_k\ (k = 1, 2, \ldots, s)$ are completely continuous of finite orders $p_k\ (p_s \le p_{s-1} \le \ldots < p_k)$.

Operator B_1 is self-adjoint

b) operators $A_{k,i}\ (k = 1, \ldots, n-1 : i = 2, 3, \ldots, s)$ is a are bounded, and operators $A_{k,1}\ (k = 0, 1, \ldots, n-1)$ are completely continuous) $Ker\{E + \sum_{k=2}^{s} B_k{}^i B_1{}^{-i} = 0$. Then the multiple completeness of eigen and associated vectors of polynomial bundle (6) takes place.

Proof.

Because all the conditions of Keldysh theorem for operators $A_{k,1}\ (k = 0, 1, \ldots, n-1)$ are completely continuousand B_1 of the operator bundle $A_1(\lambda)$ are fulfilled ,then the multiple completeness of eigen and associated vectors of operator $A_1(\lambda)$ takes place. From the conditions of the Theorem 1 it follows also

$$Gp_s \subseteq Gp_{s-1} \subseteq \ldots \subseteq Gp_1$$

We transform the operator $A_{i,k} B_k{}^i$, standing at λ^i parameter

in the bundle $A_k(\lambda)$ .We have $A_{i,k} B_k{}^i = A_{i,k} B_k{}^i, B_1{}^{-i}, B_1{}^i$.

Further, we use the result , obtained in [12] . Operator $B_k{}^i B_1{}^{-i}$ has the order $(\frac{i}{p_2} - \frac{i}{p_1})^{-1}$ or

$$p = \frac{p_1\ p_k}{p_1 - p_k} > 0$$
.

Consequently, ${B_k}^i{B_1}^{-i}$ is a completely continuous operator of finite positive order because $p_k < p_1$ ... /.......

.(Operator ${B_k}^i{B_1}^{-i}$ is bounded on closure $\overline{R\left(B_1^{-i}\right) = H}$,then ${B_k}^i{B_1}^{-i}$ can be extended as bounded operator on all space H).

Let's introduce the notations:

$$T_{ik} = {B_k}^i{B_{,1}}^{-i} \quad (i = 1,...,n-1, \quad k = 1,2,...,s) \qquad (7)$$

Taking into account the expressions for the operators
$A_{i,k} = A_{i,k}{B_k}^i, {B_1}^{-i} = A_{i,k}T_{i,k}(i = 1,...,n-1, \quad k = 1,2,...,s)$

,we can write (6) in the form:

$$A(\lambda) = \sum_{k=1}^{s}\{A_{0,k} + \lambda A_{1,1}T_{1,k} \ B_1 + \ ...+$$
$$...+\lambda^{n-1}A_{1n-1,k} \ T_{1n-1,k} \ {B_1}^{n-1} + T_{n,k} + B_1^n \quad \} \qquad (8)$$

Because the operator, standing at $\lambda^i(i = 1,2,...,n)$ parameter ,is equal to the $\sum_{k=1}^{s}\{A_{i,k} \ \ T_{i,k} \ \}{B_1}^i$ where $A_{i,k} \ T_{i,k}$ are completely continuous operators, then $A_{i,k}$ may be bounded. . If $Ker\{E + \sum_{k=2}^{s} \ T_{n,k} \ \} = 0$,then the operator $\{E + \sum_{k=2}^{s} \ T_{n,k}\}^{-1}$ exists and bounded. Really, operator $\{E + \sum_{k=2}^{s} \ T_{n,k}\}^{-1}$ is the sum of operator E and somecompletely continuous operator.

$$\{E + \sum_{k=2}^{s} \ T_{n,k} \ \}^{-1} =$$
$$= E - \{E + \sum_{k=2}^{s} \ T_{n,k} \ \}^{-1}\}\sum_{k=2}^{s} \ T_{n,k} \ \}^{-1}$$

Thus,

Acting on the both parts of the equation (7) by the operator $\{E + \sum_{k=2}^{s} \ T_{n,k} \ \}^{-1}$,we get the Keldysh bundle for which all conditions of famous Keldysh's theorem [1] are fulfilled. Really, theaction of bounded operator $\{E + \sum_{k=2}^{s} \ T_{n,k} \ \}^{-1}$ on the right part of (7), does not change the boundness and completely continuous of all operators in right side of (7) .We have

$A_{i,1}(i=0,1,...,n)$ and $A_{o,k}\quad(k=1,2,...,s)$ are completely continuous operators, and in all other cases it is sufficient the bounded ness of all operators

$A_{i,k}(i=1,...,n,k=1,2,...,s)$ Thus, all conditions of Keldysh's theorem are fulfilled, then the multiple completeness of eigen and associated vectors of the bundle(7) or (6) is valid.

Theorem 1 is proven.

Analyzing the above statements ,we consider a bundle

$$A_{0,}+\lambda A_{1,}+\lambda^2 A_2+\ldots+$$
$$\lambda^{n-1}A_{n-1}+\lambda^n(B+\ A_n)\quad(9)$$

The Theorem 1 can be formulate as follows:

Theorem2. Let the following conditions are fulfilled :

B is a completely continuous operator of finite order p , $KerB=0$, $A_i\quad(i=0,1,...,n)$ are completely continuous operators which have the orders $\frac{np}{i}(i=1,2,...,n)$ .Then the multiple continuous of eigen and associated vectors of polynomial bundle (9) is true.

References

[1]. Keldysh M.V. About completeness of eigen functions of some class of not self-adjoint linear operators.UMN,1971,vol.27,issue 4,pp.15-41

[2]. Allakhhverdiev. J.E., About completeness of system of eigen and associated elements of not self-adjoint operator bundle closed to normal.DAN SSSR,vol.115, 2, 1957, 207-210(in Russian)

[3]. Gasymov M.G.The theory of polynomial operator bundles.DAN SSSR,1971,v.199,N.4,pp.747-750(in Russian)

[4]. N.I. Akhiezer andI.M.Glazman The theory of linear operators in Hilbert space. Science Publishng. Chief reduction of physical and mathematical literature. Moscow 1966 (in Russian)

[5]. I.Ts.Gokhberq, M.G.Krein.Introduction to the theory of linear not self-adjoint operators in Hilbert space ,Publishing House "Nauka",Moskva,1964 (in Russian)

[6]. Dzhabarzadeh R.M. About expansions on eigen and associated elements of bundle polynomial depending on parameter Scientific notes of Az. University ,seria fiz.-nath. sc.ience ,1964,issue.3,pp.75-81 (in Russian)

[7]. Vizitey V.N., MarkusA.S. About convergence of multible resolutions on system of eigen and associated vectors of operator bundles.Mathematical collection 1965,v.66,issue.2,pp.287-320 (in Russian)

[8]. Rakhshanda Dzhabarzadeh To the spectral theory of polynomial bundles in Hilbert spacce. Quest Journal of Research in Applied MathematicsVol.10,issue 7,pp.15-19,2024

[9]. R.M.Dzhabarzadeh, A.N. Jabrailova. Spectral theory of operator bundles The scientific heritage(Budapesht,Hungary)The journal registered and published in Hungary. Vol. 2,No 86, 2022, pp.30-

[10]. Rakhshanda Dzhabarrzadeh. About common eigenvalues and eigenvectors of two and more completely continuous poilynomial operator bundles in Hilbert space.Quest journal of Research amd Applied Mathematics,vol.10, issue 8,pp,40-46,2024

[11]. Rakhshanda Dzhabarzadeh Spectral decomposition of operator polynomial bundle,v.11,issue7,pp.13-18,2025

[12]. Dzhabarzadeh R.M. To the spectral theory of operator bundles. About G_p class of operators.Collection:. Some questions 0f theory of nonlinear analyses. Baku 1990,issue.2, pp..76-86. (in Russian)

[13]. Rakhshanda Dzhabarzadeh. On the completeness of roots vectors of completely continuous operator in Hilbert space Quesst journal of Reasearch in Applied Mathematicsvol.12, issue2 ,pp.124-127

[14]. Rakhshanda Dhabarzadeh. On the multiple completeness of eigen and associated vectors of the polynomial operator bundles in Hilbert space. Quesst journal of Reasearch in Applied MathematicsVol 12, issue 3, pp01-05, 2026