



Research Paper

The Considerations of toCalderón-Zygmund Singular IntegralOperators

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Abstract

The authors in [31] classify and study the boundedness of the Hardy-Littlewood maximal operator in abstract extrapolation Banach function lattices and their Köthe dual spaces. The extrapolation spaces are generated by compatible families of Banach function lattices on quasi-metric measure spaces with doubling measure. These results combined with a variant of the integral Coifman-Fefferman inequality imply that every Calderón-Zygmund singular integral operator is bounded in considered extrapolation spaces. We apply these results to extrapolation spaces determined by compatible families of Calderón-Lozanovskii spaces, in particular to compatible families of Orlicz spaces that are interpolation of weighted $L^{1+\epsilon}$ -spaces ($0 < \epsilon < \infty$) with $A_{1+\epsilon}$ weights defined on spaces of homogeneous type.

Keywords: Extrapolation spaces, Calderón-Zygmund singular operators, maximal operator, $A_{1+\epsilon}$ -weights, Orlicz spaces.

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I. Introduction

We study and determine as an application the boundedness of Calderón-Zygmund singular integral operators in general class of extrapolation Banach function lattices over the spaces of homogeneous type (as the way of [31]). The singular integral operators appear in applications of many areas of analysis, including harmonic analysis, complex function theory, and PDE. The most important examples of singular operators is the Hilbert transform, which arises in the Fourier analysis, in problems concerning almost everywhere convergence of Fourier series as well as in the potential theory. So various operators on different function spaces, maximal functions are used for controlling pointwise estimates of these operators including the Poisson integral operator or other even more complicated singular operators. The Hardy-Littlewood maximal operator in its various forms plays a fundamental role in Harmonic Analysis. So for a locally integrable function f_σ on $\mathbb{R}^{1+2\epsilon}$ the classical Hardy-Littlewood maximal operator is defined by

$$Mf_\sigma(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \sum_{\sigma} |f_\sigma(y)| dy, x \in \mathbb{R}^{1+2\epsilon},$$

where the supremum is taken over all cubes Q with sides parallel to the axes and Q is the volume of $|Q|$.

The most fundamental singular integral operator is the Hilbert transform H . Given a function f_σ in $L^{1+\epsilon}(\mathbb{R})$ with $0 < \epsilon < \infty$, Hf_σ is defined by the principal value integral

$$Hf_\sigma(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \sum_{\sigma} \frac{f_\sigma(y)}{x-y} dy = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{\{|x-y|>\epsilon\}} \sum_{\sigma} \frac{f_\sigma(y)}{x-y} dy, x \in \mathbb{R}$$

This limit exists in the sense of both $L^{1+\epsilon}$ norm and pointwise almost everywhere. An $(1+2\epsilon)$ -dimensional analogue of the Hilbert transform are the Riesz transforms $R_{1+\epsilon}$, $\epsilon \geq 0$ given by

$$R_{1+\epsilon} f_\sigma(x) := \frac{\Gamma((1+\epsilon))}{\pi^{(1+\epsilon)}} \lim_{\epsilon \rightarrow 0} \int_{\{|x-y|>\epsilon\}} \sum_{\sigma} \frac{x_{1+\epsilon} - y_{1+\epsilon}}{|x-y|^{2+2\epsilon}} f_\sigma(y) dy, x = (x_1, \dots, x_{1+2\epsilon}) \in \mathbb{R}^{1+2\epsilon}$$

for all f_σ in Schwartz space $\mathcal{S}(\mathbb{R}^{1+2\epsilon})$.

A natural generalization of the Hilbert transform is a Calderón-Zygmund singular integral operator on $\mathbb{R}^{1+2\epsilon}$ with kernels which satisfy decay and smoothness conditions. A Calderón-Zygmund singular operator T is a linear operator that is bounded in $L^2(\mathbb{R}^{1+2\epsilon})$ and

$$Tf_\sigma(x) := \int_{\mathbb{R}^{1+2\epsilon}} \sum_{\sigma} K(x, y) f_\sigma(y) dy, f_\sigma \in L_c^\infty(\mathbb{R}^{1+2\epsilon}), x \notin \text{supp} f_\sigma$$

where the kernel K satisfies the size and smoothness estimates for some $\epsilon \geq 0$ and $\gamma > 0$,

$$|K(x, y)| \leq \frac{1 + \epsilon}{|x - y|^{1+2\epsilon}}, x \neq y$$

and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq (1 + \epsilon) \frac{|x - x'|^\gamma}{|x - y|^{1+2\epsilon+\gamma}}$$

for all $x, x', y \in \mathbb{R}^{1+2\epsilon}$ with $|x - y| > 2|x - x'|$. We note that every Calderón-Zygmund operator is bounded in $L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})$ for all $0 < \epsilon < \infty$.

Recently there has been interest in the study of the boundedness of Calderón-Zygmund singular integral operators and the Hardy-Littlewood maximal operator in various function spaces over spaces of homogeneous type. This has been motivated by the applications to the nonlinear potential theory on metric spaces (see [6]). We note that homogeneous spaces extend Euclidean spaces, and include C^∞ compact Riemannian manifolds and Carleson curves.

Interpolation and extrapolation theories deliver powerful methods in the study of abstract spaces as well as operators on these spaces. Now, let us just point out that the general extrapolation theory studies natural limiting spaces associated with various interpolation scales and provides estimates for norms of appropriate operators. Many known constructions or results on boundedness of operators are the byproducts of abstract machinery connected with the interpolation or extrapolation methods. For some remarkable applications of theory extrapolation theory, see [17] and [5]. We mention also [5] and [9] regarding some open problems and latest results about grand Lebesgue spaces and extrapolation.

Nowadays, the theory of grand Lebesgue spaces $L^{1+\epsilon}$ introduced by [16] is one of the intensively developing directions of the modern analysis. As we will see, these spaces are in fact extrapolation spaces generated by a special family of $L_{1+\epsilon}$ -spaces. The necessity for the study of these spaces was recognized due to their rather essential role and applications in various fields. [16] prove that if $f_\sigma = ((f_\sigma)_1, \dots, (f_\sigma)_{1+2\epsilon}): \Omega \rightarrow \mathbb{R}^{1+2\epsilon}$ belongs to Sobolev space $W^{1,1}$, where Ω is an open subset in $\mathbb{R}^{1+2\epsilon}$ with $\epsilon \geq 0$, then the Jacobian determinant of $f_\sigma: J(f_\sigma, x) := \det Df_\sigma(x) \geq 0$ belongs to $L_{loc}^1(\Omega)$ whenever $g_\sigma \in L^{2+\epsilon}$, where $g_\sigma(x) := |Df_\sigma(x)|$ is the operator norm of $Df_\sigma(x)$, $x \in \mathbb{R}^{2+\epsilon}$.

The generalized grand Lebesgue spaces are appropriate in the theory of PDEs for studying the existence and uniqueness of solutions, and, the regularity problems for various nonlinear differential equations. The space $L^{1+\epsilon, \theta}$ (defined on bounded domains in $\mathbb{R}^{2+\epsilon}$) for arbitrary positive θ was introduced in [12], where the authors study the nonhomogeneous $(2 + \epsilon)$ -harmonic equation $\text{div} A(x, \nabla u) = \mu$.

We also note that the boundedness of the Hardy-Littlewood maximal operator in weighted grand spaces $L_{w_\sigma}^{1+\epsilon}([0,1])$, $0 < \epsilon < \infty$, which was studied in [10], is equivalent to the fact that the weight belongs the Muckenhoupt's class $A_{1+\epsilon}([0,1])$. The same statement is true for the boundedness of the Hilbert transform on $[0,1]$ (see [23]).

The authors in [31] establish the boundedness of Calderón-Zygmund singular integral operators in a general class of extrapolation Banach lattices over spaces of homogeneous type. We provide the definitions and basic facts that will be used. We investigate the boundedness of the Hardy-Littlewood maximal operator in extrapolation Banach function lattices and their Köthe dual spaces over spaces of homogeneous type. These results, combined with a variant of the integral Coifman-Fefferman inequality, imply that every Calderón-Zygmund singular operator is bounded in considered extrapolation spaces. We show the applications to extrapolation spaces generated by compatible families of Calderón-Lozanovskii spaces. We also define general variants of grand spaces generated by a special type of Calderón-Lozanovskii spaces called $(1 + \epsilon)$ -convexification of Banach function lattices. We show some general results on the boundedness of Calderón-Zygmund singular operators in extrapolation spaces generated by the families of Orlicz spaces with measure having the density weight function in the $A_{1+\epsilon}$ class for some $0 < \epsilon < \infty$. Hence, we prove that in the case of grand Orlicz-Zygmund spaces $L_{w_\sigma}^{1+\epsilon, \theta} \log^\alpha L$ over spaces of homogeneous type with integrable weight $w_\sigma \in A_{1+\epsilon}$ any Calderón-Zygmund singular operator is bounded in these spaces for all $\alpha, \theta > 0$ (see [31]).

II. Notation and background

For the standard main notations from and consequence results (see [31]), given two nonnegative functions f_σ and g_σ defined on a set A , we write $f_\sigma < g_\sigma$, if there is a constant $c > 0$ such that $f_\sigma(t) \leq c g_\sigma(t)$ for all $t \in A$, while $f_\sigma \approx g_\sigma$ means that $f_\sigma < g_\sigma$ and $g_\sigma < f_\sigma$ hold. If X and Y are topological linear spaces, then $X \hookrightarrow Y$ means that $X \subset Y$ and the inclusion map is continuous. In the case where X and Y are Banach spaces, we write $X \cong Y$ whenever $X = Y$ and the identity map $\text{id}: X \rightarrow Y$ is an isometry onto.

We begun with the definition of extrapolation spaces which has roots in interpolation theory (see [3], [25, pp. 16-18]). Let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of Banach spaces. A family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is said to be compatible if there exists a Hausdorff topological vector space \mathcal{X} such that $X_\alpha \hookrightarrow \mathcal{X}$ for all $\alpha \in \mathcal{A}$.

For given two compatible families $\{X_\alpha\} := \{X_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{Y_\alpha\} := \{Y_\alpha\}_{\alpha \in \mathcal{A}}$ with $X_\alpha \subset \mathcal{X}$ and $Y_\alpha \subset \mathcal{Y}$, we write $T: \{X_\alpha\} \rightarrow \{Y_\alpha\}$ if T is a linear mapping with domain $D(T) \subset \mathcal{X}$ and such that its restriction $T|_{X_\alpha}: X_\alpha \rightarrow Y_\alpha$ is bounded for every $\alpha \in \mathcal{A}$, and moreover, $\sup_{\alpha \in \mathcal{A}} \|T\|_{X_\alpha \rightarrow Y_\alpha} < \infty$. Banach spaces $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are called extrapolation spaces with respect to the families $\{X_\alpha\}$ and $\{Y_\alpha\}$ of compatible spaces if the condition $T: \{X_\alpha\} \rightarrow \{Y_\alpha\}$ implies that T is bounded from X into Y .

By an extrapolation method \mathcal{E} , we mean a functor defined on a class of compatible families such that $\mathcal{E}(\{X_\alpha\})$ and $\mathcal{E}(\{Y_\alpha\})$ are extrapolation spaces for all compatible families $\{X_\alpha\}$ and $\{Y_\alpha\}$ from that class. The most important extrapolation methods are the functors of the sum and the intersection of families of Banach spaces.

For $\{X_\alpha\} := \{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a compatible family of Banach spaces such that there exists a Banach space $X \subset X_\alpha$ for all $\alpha \in \mathcal{A}$ and $\sup_{\alpha \in \mathcal{A}} \|id : X \rightarrow X_\alpha\| < \infty$, we let

$$\Delta(\{X_\alpha\}) := \left\{ x \in \bigcap_{\alpha \in \mathcal{A}} X_\alpha; \|x\|_{\Delta(\{X_\alpha\})} := \sup_{\alpha \in \mathcal{A}} \|x\|_{X_\alpha} < \infty \right\}.$$

Then $(\Delta(\{X_\alpha\}), \|\cdot\|_{\Delta(\{X_\alpha\})})$ is a Banach space with the following properties:

- (a) $\sup_{\alpha \in \mathcal{A}} \|id : \Delta(\{X_\alpha\}) \rightarrow X_\alpha\| \leq 1$.
- (b) If F is a Banach space such that $F \hookrightarrow X_\alpha$ and $c = \sup_{\alpha \in \mathcal{A}} \|id : F \rightarrow X_\alpha\| < \infty$, then $\|id : F \rightarrow \Delta(\{X_\alpha\})\| \leq c$.

In the case where $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is such that there exists a Banach space Y with $X_\alpha \hookrightarrow Y$ and $\sup_{\alpha \in \mathcal{A}} \|id : X_\alpha \rightarrow Y\| < \infty$, then we let $\Sigma(\{X_\alpha\})$ to be a space of all $x \in \mathcal{X}$ representable in the form

$$x = \sum_{\alpha \in \mathcal{A}} x_\alpha (x_\alpha \in X_\alpha), \text{ where } \sum_{\alpha} \|x_\alpha\|_{X_\alpha} < \infty. \tag{*}$$

It follows from the last condition that there are only countably many summands in $\sum_{\alpha \in \mathcal{A}} x_\alpha$ different from zero. Note also that our hypothesis implies that the series $\sum_{\alpha \in \mathcal{A}} x_\alpha$ converges absolutely in Y . We equip the space $\Sigma(X_\alpha)$ with the norm

$$\|x\|_{\Sigma(\{X_\alpha\})} := \inf \left\{ \sum_{\alpha \in \mathcal{A}} \|x_\alpha\|_{X_\alpha}; x = \sum_{\alpha \in \mathcal{A}} x_\alpha \right\},$$

where the infimum is taken over all possible representations of x in the form (*). Then $\Sigma(\{X_\alpha\})$ is the smallest Banach space with the property $X_\alpha \hookrightarrow \Sigma(X_\alpha)$ for every $\alpha \in \mathcal{A}$.

Throughout, we often write $\Delta(X_\alpha)$ and $\Sigma(X_\alpha)$ shortly instead of $\Delta(\{X_\alpha\})$ and $\Sigma(\{X_\alpha\})$ whenever $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is a fixed compatible family of Banach spaces.

We consider functors Δ and Σ generated by families of Banach function lattices. We use standard notation from Banach space theory and operator theory. If (Ω, Σ, μ) is a σ -finite measure space, $L^0(\mu) = L^0(\Omega, \mu)$ denotes the space of (equivalence classes of) μ -measurable real-valued functions. As usual $L^0(\mu)$ is equipped with the topology τ_μ of convergence in measure on μ -finite sets. A linear subspace X of $L^0(\mu)$ is called an (order) ideal whenever $f_\sigma \in X, g_\sigma \in L^0(\mu)$, and $|g_\sigma| \leq |f_\sigma| \mu$ -a.e., imply that $g_\sigma \in X$. An order ideal $X \subset L^0(\mu)$ equipped with a monotone norm $\|\cdot\|$ is called a normed function lattice in $L^0(\mu)$. If the normed (function) lattice X is norm complete and there exists $u \in X$ such that $u > 0$ on Ω , then X is called a Banach function lattice in $L^0(\mu)$. A Banach lattice X is said to have the Fatou property (resp., weak Fatou property) if its unit ball B_X is closed in $L^0(\mu)$ (resp., in $(B_X, \tau_\mu|_{B_X})$). It is well known that the Fatou property (resp., the weak Fatou property) is equivalent to the following property: for any $f_\sigma \in L^0$ and a sequence $(f_\sigma)_{2+\epsilon} \in X$ such that $0 \leq (f_\sigma)_{2+\epsilon} \leq f_\sigma, (f_\sigma)_{2+\epsilon} \uparrow f_\sigma$ a.e. and $\sup \|(f_\sigma)_{2+\epsilon}\|_E < \infty$, we have that $f_\sigma \in X$ and $\|(f_\sigma)_{2+\epsilon}\|_X \rightarrow \|f_\sigma\|_X$ (resp., if $(f_\sigma)_{2+\epsilon}, f_\sigma \in X, 0 \leq (f_\sigma)_{2+\epsilon} \leq f_\sigma, (f_\sigma)_{2+\epsilon} \uparrow f_\sigma$ a.e., then $\|(f_\sigma)_{2+\epsilon}\|_X \rightarrow \|f_\sigma\|_X$).

So any family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of Banach function lattices in $L^0(\mu)$ forms a compatible family by any Banach lattice $X \hookrightarrow L^0(\mu)$ (see, e.g., [21, 25]).

The Köthe dual space (or associate space) X' of a Banach function lattice X on (Ω, Σ, μ) is defined as the space of all $f_\sigma \in L^0$ such that $\int_\Omega |f_\sigma g_\sigma| d\mu < \infty$ for every $g_\sigma \in X$. It is a Banach function space in $L^0(\mu)$ equipped with the norm

$$\|f_\sigma\|_{X'} = \sup_{g_\sigma \in B_X} \sum_{\sigma} \left| \int_\Omega f_\sigma g_\sigma d\mu \right|$$

We use the well known fact that a Banach function lattice $X \subset L^0(\mu)$ has the weak Fatou property if and only if (see, e.g., [18] or [21])

$$\|f_\sigma\|_X = \sup_{g_\sigma \in B_{X'}} \sum_{\sigma} \left| \int_{\Omega} f_\sigma g_\sigma d\mu \right|, f_\sigma \in X$$

We consider Banach lattices on a quasi-metric measure space with doubling measure. Let (Ω, d) be a quasi-metric (that is, d satisfies the axioms of a metric except for the triangle inequality, which holds in the weaker form $d(x, y) \leq \kappa(d(x, z) + d(z, y))$ for some $\kappa \geq 1$) and a positive measure μ that is defined on the σ -algebra generated by quasi-metric balls and open sets. We say that (Ω, d, μ) is a space of homogeneous type (SHT for short) if there exists a constant $D_\mu \geq 1$ such that for any $x \in \Omega$ and any $\epsilon \geq 0$,

$$\mu(B(x, 2(1 + \epsilon))) \leq D_\mu \mu(B(x, 1 + \epsilon)),$$

where $B(x, 1 + \epsilon) := \{y \in \Omega; d(x, y) < 1 + \epsilon\}$ is the ball centered at x with radius $(1 + \epsilon)$. The family of all balls in (Ω, d) is denoted by \mathcal{B} . To avoid trivial measures, we always assume that $0 < \mu(B) < \infty$ for every ball $B \in \mathcal{B}$. Consequently, μ is a σ -finite measure. We also assume that μ is a complete measure.

We also recall some fundamental definitions and results concerning the classes of weights that are related to our investigation. Let (Ω, d, μ) be an SHT and let $w_\sigma \in L^1(\Omega, \mu)$ be a weight function. We say that w_σ belongs to the class $A_{1+\epsilon}(\Omega)$ ($A_{1+\epsilon}$ shortly) for $0 < \epsilon < \infty$ if it satisfies the condition:

$$[w_\sigma]_{A_{1+\epsilon}} := \sup_{B \in \mathcal{B}} \sum_{\sigma} \left(\frac{1}{\mu(B)} \int_B w_\sigma d\mu \right) \left(\frac{1}{\mu(B)} \int_B w_\sigma^{-\frac{1}{\epsilon}} d\mu \right)^\epsilon < \infty$$

A weight w_σ is said to belong to the class A_1 if there exists a constant $\epsilon \geq 0$ such that

$$Mw_\sigma \leq (1 + \epsilon)w_\sigma \mu\text{-a.e.}$$

where M is the Hardy-Littlewood maximal operator defined for every $f_\sigma \in L^1_{loc}(\Omega, \mu)$ (i.e., f_σ is integrable over all balls B in Ω) by

$$Mf_\sigma(x) := \sup_{B \in \mathcal{B}} \frac{\chi_B(x)}{\mu(B)} \sum_{\sigma} \int_B |f_\sigma| d\mu, x \in \Omega$$

The smallest possible constant $(1 + \epsilon)$ is denoted by $[w_\sigma]_{A_1}$, i.e.,

$$[w_\sigma]_{A_1} := \text{esssup}_{x \in \Omega} \sum_{\sigma} \frac{Mw_\sigma(x)}{w_\sigma(x)}$$

Following [11], we also consider the class A'_∞ of weights w_σ such that

$$[w_\sigma]_{A'_\infty} := \sup_B \frac{1}{\mu(B)} \sum_{\sigma} \int_B M(w_\sigma \chi_B) d\mu < \infty$$

III. Calderón-Zygmund operators in extrapolation spaces

We investigate the boundedness of maximal and Calderón-Zygmund singular integral operators in extrapolation Banach function lattices on spaces of homogeneous type.

Before starting with the proofs, we recall that if (Ω, d, μ) is a space of homogeneous type, then $K: \Omega \times \Omega \setminus \{x = y\} \rightarrow \mathbb{R}$ is a Calderón-Zygmund kernel if there exist $\epsilon \geq 0$ such that it satisfies the decay condition:

$$|K(x_0, y)| \leq \frac{1 + \epsilon}{\mu(x_0, d(x_0, y))}$$

for all $x_0 \neq y \in \Omega, x_0 \in \Omega, x \in \Omega$, and the smoothness condition:

$$|K(x, y) - K(x_0, y)| \leq \left(\frac{d(x, x_0)}{d(x_0, y)} \right)^{1+\epsilon} \frac{1}{\mu(x_0, d(x_0, y))}$$

$$|K(y, x) - K(y, x_0)| \leq \left(\frac{d(x, x_0)}{d(x_0, y)} \right)^{1+\epsilon} \frac{1}{\mu(x_0, d(x_0, y))}$$

for $d(x_0, x) \leq (1 + \epsilon)d(x_0, y)$.

Let T be a singular integral operator associated with Calderón-Zygmund kernel K . In addition, if T is bounded in $L^2(\mu)$, then T is a Calderón-Zygmund singular operator.

We need a variant of Coifman-Fefferman type inequality, which estimates Calderón-Zygmund operator from above by a maximal function.

Proposition 3.1 (see [31]). Let T be a Calderón-Zygmund operator on an SHT (Ω, d, μ) and let $0 < \epsilon < \infty$. If $w_\sigma \in A_{1+\epsilon}(\Omega)$, then for all $f_\sigma \in L^1_{loc}(\Omega, \mu)$,

$$\int_{\Omega} \sum_{\sigma} |Tf_\sigma| w_\sigma d\mu \leq (1 + \epsilon) \sum_{\sigma} [w_\sigma]_{A_{1+\epsilon}} \int_{\Omega} (Mf_\sigma) w_\sigma d\mu$$

where $(1 + \epsilon)$ is a constant that depends on Ω, μ and T .

Proof. By using the formula of Lerner proved in the homogeneous setting for dyadic maximal operator M^D , in [1] it was shown in (see [2, Prop. 4.1]) that, for all $f_\sigma \in L^1_{loc}(\Omega, \mu)$, we have

$$\int_{\Omega} \sum_{\sigma} |Tf_\sigma| w_\sigma d\mu \leq C_1 \sum_{\sigma} [w_\sigma]_{A_{1+\epsilon}} \int_{\Omega} (M^D f_\sigma) w_\sigma d\mu$$

where C_1 is an absolute constant that, generally speaking, depends on T . To get the desired result, it is enough to apply the following pointwise inequality which concerns the maximal operator M and its dyadic counterpart M^D (see [13, Prop. 7.9]),

$$M^D f_\sigma(x) \leq C_2 M f_\sigma(x), x \in \Omega, f_\sigma \in L^1_{loc}(\Omega, \mu),$$

where the positive constant $C_2 = C_2(\Omega, \mu, T)$ depends only on Ω, μ and T .

The following Lemma is indeed well known in the case of Banach function spaces over $\mathbb{R}^{2+\epsilon}$ with Lebesgue measure; based on Proposition 3.1, we include a proof for the sake of completeness in the setting of Banach function lattices over spaces of homogeneous type.

Lemma 3.2 (see [31]). Let (Ω, d, μ) be an SHT and let $X \subset L^1_{loc}(\Omega, \mu)$ be a Banach function lattice with the weak Fatou property. Suppose that the maximal operator M is bounded in X and X' . Then Calderón-Zygmund singular integral operator T is bounded in X with

$$\|T\|_X \leq (1 + \epsilon) \|M\|_X \|M\|_{X'}$$

where $(1 + \epsilon)$ is a constant that depends on Ω, μ and also on T .

Proof. We use the well known Rubio de Francia iteration algorithm to construct A_1 weight with some properties, which also works in the setting of general Banach function lattices. Define a mapping R on X' by the formula:

$$Rg_\sigma := \sum_{\epsilon=-1}^{\infty} \sum_{\sigma} \frac{1}{2^{1+\epsilon}} \frac{M^{1+\epsilon} g_\sigma}{(\|M\|_{X'})^{1+\epsilon}}, g_\sigma \in X'$$

where, for each $\epsilon \geq 0, M^{1+\epsilon}$ denotes the $(1 + \epsilon)$ -th iteration of the maximal operator M and $M^0 g_\sigma := |g_\sigma|$.

Following the standard proof due to Rubio de Francia, we deduce (by boundedness of M in X') that R is bounded in X' with $\|Rg_\sigma\|_{X'} \leq 2 \|g_\sigma\|_{X'}, |g_\sigma| \leq Rg_\sigma$ and $Rh \in A_1$ with

$$[Rg_\sigma]_{A_1} \leq 2 \|M\|_{X'}$$

Fix $\epsilon > 0$. Since X has the weak Fatou property, for every $f_\sigma \in X$ there exists $g_\sigma \in B_{X'}$ such that

$$\|Tf_\sigma\|_X \leq (1 + \epsilon) \int_{\Omega} \sum_{\sigma} |Tf_\sigma| g_\sigma d\mu$$

The Rubio de Francia iteration algorithm combined with Proposition 3.1 yields (by the above properties of Rg_σ in A_1 and the fact that $[Rg_\sigma]_{A_{1+\epsilon}} \leq [Rg_\sigma]_{A_1}$ for all $0 < \epsilon < \infty$):

$$\begin{aligned} \|Tf_\sigma\|_X &\leq (1 + \epsilon) \int_{\Omega} \sum_{\sigma} |Tf_\sigma| Rg_\sigma d\mu \leq c(1 + \epsilon) \sum_{\sigma} [Rg_\sigma]_{A_1} \int_{\Omega} (Mf_\sigma) Rg_\sigma d\mu \\ &\leq 2c(1 + \epsilon) \|M\|_{X'} \sum_{\sigma} \int_{\Omega} (Mf_\sigma) Rg_\sigma d\mu \end{aligned}$$

where c is a constant that depends on Ω, μ and on T . Consequently, we obtain that for all $f_\sigma \in X$,

$$\|Tf_\sigma\|_X \leq 2c(1 + \epsilon) \sum_{\sigma} \|Mf_\sigma\|_X \|Rg_\sigma\|_{X'} \leq (1 + \epsilon^2) \sum_{\sigma} \|M\|_X \|M\|_{X'} \|f_\sigma\|_X,$$

where $(1 + \epsilon) = 4c$. Since $\epsilon > 0$ was arbitrary, the required statement follows.

We need the following proposition (see [31]).

Proposition 3.3. Let $X \subset L^0(\Omega_1, \Sigma_1, \mu)$ and $Y \subset L^0(\Omega, \Sigma_2, \nu)$ be Banach function lattices. Suppose that a mapping S with a domain $D(S) \supset X''$ is such that $|Sf_\sigma| \leq S|f_\sigma|$ a.e. for all $f_\sigma \in X$ and $0 \leq S(g_\sigma)_{2+\epsilon} \uparrow Sg_\sigma$ a.e. for any sequence $((g_\sigma)_{2+\epsilon})$ in X with $0 \leq (g_\sigma)_{2+\epsilon} \uparrow g_\sigma$ a.e. If S is bounded from X to Y with $\|Sf_\sigma\|_Y \leq (1 + \epsilon) \|f_\sigma\|_X$ for all $f_\sigma \in X$, then S is also bounded from X'' to Y'' with $\|Sf_\sigma\|_{Y''} \leq (1 + \epsilon) \|f_\sigma\|_{X''}$ for all $f_\sigma \in X''$.

Proof. We apply the following well known fact (cf. [30, pp. 451, 471]): If E is a Banach function lattice, then $g_\sigma \in E''$ if and only if there exists a sequence $((g_\sigma)_{2+\epsilon})$ of elements of E , such that $0 \leq (g_\sigma)_{2+\epsilon} \uparrow |g_\sigma|$ a.e. and $\sup_{\epsilon \geq -1} \|(g_\sigma)_{2+\epsilon}\|_E < \infty$. Moreover for $g_\sigma \in E''$ we have

$$\|g_\sigma\|_{E''} = \inf \left\{ \lim_{\epsilon \rightarrow \infty} \|(g_\sigma)_{2+\epsilon}\|_E; 0 \leq (g_\sigma)_{2+\epsilon} \uparrow g_\sigma \text{ a.e.} \right\}.$$

Next, we fix $f_\sigma \in X''$, and take any sequence $((f_\sigma)_{2+\epsilon})$ in X , such that $0 \leq (f_\sigma)_{2+\epsilon} \uparrow |f_\sigma|$ a.e. and $\sup_{\epsilon \geq -1} \|(f_\sigma)_{2+\epsilon}\|_X < \infty$. Then our hypotheses imply that $0 \leq S(f_\sigma)_{2+\epsilon} \uparrow Sf_\sigma$ a.e. (by $Y \hookrightarrow Y''$ with $\|\text{id}: Y \rightarrow Y''\| \leq 1$) and

$$\sup_{\epsilon \geq -1} \|S(f_\sigma)_{2+\epsilon}\|_{Y''} \leq \sup_{\epsilon \geq -1} \|S(f_\sigma)_{2+\epsilon}\|_Y \leq (1 + \epsilon) \sup_{\epsilon \geq -1} \|(f_\sigma)_{2+\epsilon}\|_X = (1 + \epsilon) \lim_{\epsilon \rightarrow \infty} \|(f_\sigma)_{2+\epsilon}\|_X.$$

Clearly, Y'' has the Fatou property. Thus $S|f_\sigma| \in Y''$ and, we get (by $|Sf_\sigma| \leq S|f_\sigma|$ a.e.)

$$\|Sf_\sigma\|_{Y''} \leq \|S|f_\sigma|\|_{Y''} = \lim_{\epsilon \rightarrow \infty} \|S(f_\sigma)_{2+\epsilon}\|_{Y''} \leq (1 + \epsilon) \lim_{\epsilon \rightarrow \infty} \|(f_\sigma)_{2+\epsilon}\|_X.$$

Since $((f_\sigma)_{2+\epsilon})$ in X was arbitrary, the required estimate follows from the mentioned fact. We have the following corollary(see [31]).

Corollary 3.4. Let (Ω, d, μ) be an SHT and let $X \subset L^1_{loc}(\Omega, \mu)$ be a Banach function lattice. Suppose that the maximal operator M is bounded in X with $\|M\|_X \leq (1 + \epsilon)$. Then M is also bounded in X'' with $\|M\|_{X''} \leq (1 + \epsilon)$.

Proof. Observe that for any sequence $((f_\sigma)_{2+\epsilon})$ in $L^1_{loc}(\Omega, \mu)$ with $0 \leq (f_\sigma)_{2+\epsilon} \uparrow f_\sigma$ a.e., it follows from the Lebesgue's monotone convergence theorem that

$$\begin{aligned} \lim_{\epsilon \rightarrow \infty} M((f_\sigma)_{2+\epsilon})(x) &= \sup_{\epsilon \geq -1} \sum_{\sigma} \left(\sup_{B \in \mathcal{B}} \frac{\chi_B(x)}{\mu(B)} \int_B |(f_\sigma)_{2+\epsilon}| d\mu \right) \\ &= \sup_{B \in \mathcal{B}} \sum_{\sigma} \left(\sup_{\epsilon \geq -1} \frac{\chi_B(x)}{\mu(B)} \int_B |(f_\sigma)_{2+\epsilon}| d\mu \right) = Mf_\sigma(x). \end{aligned}$$

for all $x \in \Omega$. Since the domain $D(M) \supset X''$ and $Mf_\sigma = M|f_\sigma|$ for all $f_\sigma \in D(M)$, Proposition 3.3 can be applied.

We are now ready to state extrapolation theorem for Calderón-Zygmund singular integral operators.

Theorem 3.5 (see [31]). Let (Ω, d, μ) be an SHT and let $\{X_\alpha\} := \{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a compatible family of Banach function lattices with the Fatou property, such that $X_\alpha \subset L^1_{loc}(\Omega, \mu)$ for every $\alpha \in \mathcal{A}$. Suppose that the Hardy-Littelwood maximal operator $M: \{X_\alpha\} \rightarrow \{X_\alpha\}$ and $M: \{X'_\alpha\} \rightarrow \{X'_\alpha\}$. Then every Calderón-Zygmund singular integral operator is bounded in the extrapolation space $\Delta(\{X_\alpha\})$.

Proof. It is easy to check that for any $f_\sigma \in \Sigma(\{X'_\alpha\})$, we have

$$\|f_\sigma\|_{\Sigma(\{X'_\alpha\})} = \inf \left\{ \sum_{\alpha \in \mathcal{A}} \sum_{\sigma} \|f_\sigma\|_{X'_\alpha}; |f_\sigma| \leq \sum_{\alpha \in \mathcal{A}} \sum_{\sigma} |f_\sigma| \text{ a.e.} \right\}.$$

This implies that M is bounded in $\Sigma(\{X'_\alpha\})$. Then by Corollary 3.4, it follows that M is bounded in $\Sigma(\{X'_\alpha\})''$. Combining the Köthe duality formula $\Delta(\{X'_\alpha\}) \cong \Sigma(\{X_\alpha\})'$ (see [27, Lemma 1]) with the Fatou property of X_α for all $\alpha \in \mathcal{A}$ yields

$$\Delta(\{X_\alpha\})' \cong \Delta(\{X'_\alpha\})' \cong \Sigma(\{X'_\alpha\})''.$$

Thus, we conclude that M is bounded in $\Delta(\{X_\alpha\})'$.

Clearly, M is bounded in $\Delta(\{X_\alpha\})$ and $\Delta(\{X_\alpha\})$ has the Fatou property. To complete the proof, we apply Lemma 3.2 to $X := \Delta(\{X_\alpha\})$.

IV. Applications to Calderón-Lozanovskii extrapolation spaces

We define variants of grand spaces generated by a special type of Calderón-Lozanovskii spaces called the $(1 + \epsilon)$ -convexification of Banach function lattices. (see [31]). Let $\vec{X} = (X_0, X_1)$ be a couple of Banach function lattices on a measure space (Ω, Σ, μ) and \mathcal{U} denote the set of all non-negative, concave and positively homogeneous continuous functions $\varphi_\sigma: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that $\varphi_\sigma(0,0) = 0$. Then the Calderón-Lozanovskii construction or the Calderón-Lozanovskii spaces $\varphi_\sigma(\vec{X}) = \varphi_\sigma(X_0, X_1)$ consists of all $f_\sigma \in L^0(\mu)$ such that $|f_\sigma| \leq \lambda \varphi_\sigma(|(f_\sigma)_0|, |(f_\sigma)_1|)$ for some $\lambda > 0$ and $(f_\sigma)_{1+\epsilon} \in X_{1+\epsilon}$ with $\|(f_\sigma)_{1+\epsilon}\|_{X_{1+\epsilon}} \leq 1, (1 + \epsilon) = 0,1$. The spaces $\varphi_\sigma(\vec{X})$ are Banach ideal spaces on Ω equipped with the norm

$$\|f_\sigma\|_{\varphi_\sigma(\vec{X})} := \inf \{ \lambda > 0; |f_\sigma| \leq \lambda \varphi_\sigma(|(f_\sigma)_0|, |(f_\sigma)_1|), \|(f_\sigma)_0\|_{X_0} \leq 1, \|(f_\sigma)_1\|_{X_1} \leq 1 \}$$

(see [19]). In the case of power functions $\varphi_\sigma(s, t) = s^{1-\theta} t^\theta$ for $s, t > 0$ with $0 < \theta < 1$, the corresponding spaces $\varphi_\sigma(\vec{X})$ are the well known Calderón spaces $X_0^{1-\theta} X_1^\theta$ (see [8]). In the particular case, we have $X^{1-\theta} (L^\infty)^\theta \cong X^{(1+\epsilon)}$ with $(1 + \epsilon) = 1/(1 - \theta)$, where $X^{(1+\epsilon)}$ is known as the $(1 + \epsilon)$ -convexification of X (see [18, p. 53]) equipped with the norm

$$\|f_\sigma\|_{X^{(1+\epsilon)}} = \| |f_\sigma|^{1+\epsilon} \|_X^{1/(1+\epsilon)}, f_\sigma \in X^{(1+\epsilon)}.$$

In what follows $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{Y_\alpha\}_{\alpha \in \mathcal{A}}$ are families of Banach lattices and S is a mapping (not necessarily linear) such that $S: X_\alpha \rightarrow Y_\alpha$ with $\sup_{\alpha \in \mathcal{A}} \|S\|_{X_\alpha \rightarrow Y_\alpha} < \infty$, where as usual $\|S\|_{X_\alpha \rightarrow Y_\alpha} := \sup_{x \in B_{X_\alpha}} \|Sx\|_{Y_\alpha}$.

It is well known and easy to verify that Calderón-Lozanovskii construction interpolates positive operators (see [29]). The similar proof verifies that every positive sublinear $T: X_0 + X_1 \rightarrow Y_0 + Y_1$ such that $T|_{X_{1+\epsilon}}: X_{1+\epsilon} \rightarrow Y_{1+\epsilon}$ is bounded for $(1 + \epsilon) = 0,1$ is bounded operator from $\varphi_\sigma(X_0, X_1)$ to $\varphi_\sigma(Y_0, Y_1)$ with

$$\|T\|_{\varphi_\sigma(X_0, X_1) \rightarrow \varphi_\sigma(Y_0, Y_1)} \leq \max_{(1+\epsilon)=0,1} \|T|_{X_{1+\epsilon}}\|_{X_{1+\epsilon} \rightarrow Y_{1+\epsilon}}$$

We state and prove the result on the boundedness of the Calderón-Zygmund singular operator in extrapolation spaces generated by an extrapolation functor Δ defined for compatible families of Calderón-Lozanovskii spaces over SHT.

Theorem 4.1 (see [31]). Let (Ω, d, μ) be an SHT and let $\{X_\alpha^{1+\epsilon}\}_{\alpha \in \mathcal{A}} := \{X_\alpha^{1+\epsilon}\}_{\alpha \in \mathcal{A}}$ be a compatible family of Banach function lattices with the Fatou property, such that $X_\alpha^{1+\epsilon} \subset L^1_{loc}(\Omega, \mu)$ for all $\alpha \in \mathcal{A}$ and each $(1 + \epsilon) \in \{0,1\}$. Suppose that the corresponding Hardy-Littlewood maximal operator is bounded, $M: \{X_\alpha^{1+\epsilon}\} \rightarrow \{X_\alpha^{1+\epsilon}\}$ and $M: \{(X_\alpha^{1+\epsilon})'\} \rightarrow \{(X_\alpha^{1+\epsilon})'\}$ for $(1 + \epsilon) \in \{0,1\}$. Then any Calderón-Zygmund singular integral operator T is bounded in $\Delta(\{(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)\})$ for any family $\{(\varphi_\sigma)_\alpha\}_{\alpha \in \mathcal{A}}$ of functions in \mathcal{U} with $\sup_{\alpha \in \mathcal{A}} (\varphi_\sigma)_\alpha(1,1) < \infty$.

Proof. Since the inclusion maps $\text{id}_\alpha: X_\alpha^0 \cap X_\alpha^1 \rightarrow (\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)$ are bounded with

$$\sup_{\alpha \in \mathcal{A}} \|\text{id}_\alpha\| \leq \sup_{\alpha \in \mathcal{A}} (\varphi_\sigma)_\alpha(1,1),$$

it follows that $\{(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)\}_{\alpha \in \mathcal{A}}$ forms a compatible family of Banach lattices. Next observe that our hypotheses combined with the interpolation property of Calderón-Lozanovskii spaces yields that the maximal operator M is bounded in $(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)$ for all $\alpha \in \mathcal{A}$ with

$$\sup_{\alpha \in \mathcal{A}} \|M\|_{(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)} \leq \max \left\{ \sup_{\alpha \in \mathcal{A}} \|M\|_{X_\alpha^0}, \sup_{\alpha \in \mathcal{A}} \|M\|_{X_\alpha^1} \right\} < \infty.$$

Hence M is bounded in the extrapolation space $\Delta(\{(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)\})$.

Now we use Köthe duality result due to Lozanovskii ([19]), which states that for any couple (X, Y) of Banach function lattices on measure space $(\mathcal{S}, \Sigma, \mu)$ and for any $\varphi_\sigma \in \mathcal{U}$, we have $\varphi_\sigma(X, Y)' = \widehat{\varphi_\sigma}(X', Y')$, with universal constants of equivalence norms that do not depend on φ_σ and (X, Y) . More precisely we have

$$\|\cdot\|_{\widehat{\varphi_\sigma}(X', Y')} \leq \|\cdot\|_{\varphi_\sigma(X, Y)'} \leq 2 \|\cdot\|_{\widehat{\varphi_\sigma}(X', Y')}.$$

Here $\widehat{\varphi_\sigma} \in \mathcal{U}$ is an involution of φ_σ defined by

$$\widehat{\varphi_\sigma}(s, t) = \inf \left\{ \frac{as + bt}{\varphi_\sigma(a, b)}; a, b > 0 \right\}, s, t \geq 0.$$

This duality result combined with our hypothesis $M: \{(X_\alpha^{1+\epsilon})'\} \rightarrow \{(X_\alpha^{1+\epsilon})'\}$ for both $\epsilon = -1$ and $\epsilon = 0$ implies that

$$M: \{(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)'\} \rightarrow \{(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)'\}.$$

To conclude we observe that the proof of Lemma 4.1 in [20] yields that, for any couple (X, Y) of Banach function lattices and $\varphi_\sigma \in \mathcal{U}$, $\varphi_\sigma(X, Y)$ has the weak Fatou (resp., Fatou) property whenever both X and Y have the weak Fatou (resp., Fatou) property. To finish it is enough to apply Theorem 3.5 for the family $\{X_\alpha\}_{\alpha \in \mathcal{A}} := \{(\varphi_\sigma)_\alpha(X_\alpha^0, X_\alpha^1)\}_{\alpha \in \mathcal{A}}$. \square

We also recall the definition of Orlicz spaces which are strictly connected with Calderón-Lozanovskii spaces. Let $\Phi: [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing, convex and left-continuous function, not identical 0 on $(0, \infty)$, with $\Phi(0) = 0$. Let $\varphi_\sigma \in \mathcal{U}$ be defined by $\varphi_\sigma(s, t) = t\Phi^{-1}(s/t)$ if $t > 0$ and 0 if $t = 0$, where Φ^{-1} is the right continuous inverse of Φ . Then, for every Banach function lattice E in $L^0(\Omega, \Sigma, \mu)$, the Calderón-Lozanovskii space $\varphi_\sigma(E, L^\infty)$ coincides isometrically with the space

$$E_\Phi := \{f_\sigma \in L^0(\mu); \Phi(|f_\sigma|/\lambda) \in E \text{ for some } \lambda > 0\}$$

equipped with the norm

$$\|f_\sigma\|_{E_\Phi} := \inf\{\lambda > 0; \|\Phi(|f_\sigma|/\lambda)\|_E \leq 1\}.$$

In particular, if $E = L^1(\mu)$, we recover the Orlicz space $L_\Phi(\mu)$ (L_Φ shortly).

In what follows for a given Banach space $(X, \|\cdot\|_X)$ and $t > 0$, we let tX to be the Banach space X equipped with the norm $t\|\cdot\|_X$. Now we give the definition of general variants of grand spaces.

Let $0 < \epsilon < \infty$ and let E be a Banach function lattice in $L^0(\Omega, \Sigma, \mu)$. Assume that $\omega_\sigma: (0, \epsilon) \rightarrow (0, \infty)$ is such a function that $\{\omega_\sigma(\epsilon)E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon)}$ forms a compatible family of Banach spaces. Then the extrapolation space

$\Delta\left(\{\omega_\sigma(\epsilon)E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon)}\right)$ is called a grand space (on (Ω, Σ, μ)) and is denoted by $\omega_\sigma E^{1+\epsilon}$. From our discussion in the previous section, we know that $\omega_\sigma E^{1+\epsilon}$ is a Banach function lattice in $L^0(\mu)$ equipped with the norm

$$\|f_\sigma\|_{\omega_\sigma E^{1+\epsilon}} = \sup_{\epsilon \in (0, \epsilon)} \omega_\sigma(\epsilon) \|f_\sigma\|_{E^{(1+\epsilon-\epsilon)}}, f_\sigma \in \omega_\sigma E^{1+\epsilon}$$

If w_σ is a weight on Ω , then the grand space defined with respect to the measure $w_\sigma d\mu$ is denoted by $\omega_\sigma E_{w_\sigma}^{1+\epsilon}$ (or by $\omega_\sigma E^{1+\epsilon}(w_\sigma d\mu)$). Whenever we mention the grand space $\omega_\sigma E^{1+\epsilon}$, we always assume that $\{\omega_\sigma(\epsilon)E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon)}$ forms a compatible family.

We note that if $\psi: (0, \epsilon) \rightarrow (0, \infty)$ is a continuous function that is non-decreasing on $(0, \epsilon_0]$ for some $\epsilon_0 < \epsilon$, and satisfies the condition $\psi(s) \rightarrow 0$ as $s \rightarrow 0+$ and $\omega_\sigma(\epsilon) = \psi(\epsilon)^{1/(1+\epsilon-\epsilon)}$ for all $\epsilon \in (0, \epsilon)$, then $\{\omega_\sigma(\epsilon)E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon)}$ form a compatible family of Banach spaces. In this case the grand space is denoted by $E^{1+\epsilon, h}$.

For $\psi(s) = s^\theta$, where θ is a positive number, we denote $E^{1+\epsilon, h}$ by $E^{1+\epsilon, \theta}$. In the case $E = L^{1+\epsilon}(\mu)$ on $\Omega = (0,1)$ or Ω is a domain in $\mathbb{R}^{2+\epsilon}$ where μ is the Lebesgue measure, we recover the classical grand Lebesgue spaces intensively studied in the recent time (see [24]).

We need the following lemma(see [31]).

Lemma 4.2. Let $0 < \epsilon < \infty$ and let E be a Banach lattice in $L^0(\Omega, \Sigma, \nu)$ such that $\chi_\Omega \in E$ and let $\omega_\sigma E^{1+\epsilon}$ be a grand space such that $\sup_{\epsilon \in (\epsilon_0, \epsilon)} \omega_\sigma(\epsilon) < \infty$. Suppose that S is a sublinear operator such that

$$\sup_{\epsilon \in (0, \epsilon_0)} \omega_\sigma(\epsilon) \|S\|_{E^{(1+\epsilon-\epsilon)}} < \infty. \text{ Then } S \text{ is bounded in the grand space } \omega_\sigma E^{1+\epsilon}.$$

Proof. We use the well known inequality (see, e.g., [18, Prop. 1.d.2, p. 43], [25, p. 40]) which is true for any Banach lattice E over any measure space and which states that for all $x, y \in E$ and $\theta \in (0, 1)$,

$$\| |x|^{1-\theta} |y|^\theta \|_E \leq \|x\|_E^{1-\theta} \|y\|_E^\theta.$$

If $u, v, 0 < \epsilon < \infty$ satisfy $1/1 + \epsilon = 1/u + 1/v$, we conclude (by taking $x = |f_\sigma|^u$ and $y = |g_\sigma|^v$) that $f_\sigma g_\sigma \in E^{(1+\epsilon)}$ for all $f_\sigma \in E^{(u)}$ and $g_\sigma \in E^{(v)}$ with

$$\|f_\sigma g_\sigma\|_{E^{(1+\epsilon)}} \leq \|f_\sigma\|_{E^{(u)}} \|g_\sigma\|_{E^{(v)}}.$$

This implies that for $v \in (1, \infty)$ defined by $\frac{1}{v} := \frac{1}{1+\epsilon} - \frac{1}{u}$ where $(1+\epsilon) := 1 + \epsilon - \epsilon$ and $u := 1 + \epsilon - \epsilon_0$, the following estimate holds:

$$\|f_\sigma\|_{E^{(1+\epsilon-\epsilon)}} \leq \|f_\sigma\|_{E^{(1+\epsilon-\epsilon_0)}} \|\chi_\Omega\|_{E^{(1/v)}}, f_\sigma \in E^{(1+\epsilon-\epsilon_0)}$$

This combined with $1/v - 1/u \leq \alpha := (\epsilon - \epsilon_0)/(1 + \epsilon - \epsilon_0)$ yields, in particular, that for all $f_\sigma \in \omega_\sigma E^{1+\epsilon}$,

$$\begin{aligned} \sup_{\epsilon \in (\epsilon_0, \epsilon)} \|f_\sigma\|_{E^{(1+\epsilon-\epsilon)}} &\leq (1+\epsilon)(1+\epsilon, \epsilon_0) \omega_\sigma(\epsilon_0) \|f_\sigma\|_{E^{(1+\epsilon-\epsilon_0)}} \\ &\leq (1+\epsilon)(1+\epsilon, \epsilon_0) \sup_{\epsilon \in (0, \epsilon_0]} \omega_\sigma(\epsilon) \|f_\sigma\|_{E^{(1+\epsilon-\epsilon)}} \end{aligned}$$

where $(1+\epsilon)(1+\epsilon, \epsilon_0) = \left(\sup_{\epsilon \in [\epsilon_0, \epsilon]} \frac{\omega_\sigma(\epsilon)}{\omega_\sigma(\epsilon_0)} \right) \max\{1, \|\chi_\Omega\|_E^\alpha\}$. As a consequence, it follows that

$$\omega_\sigma E^{1+\epsilon} := \Delta\left(\{\omega_\sigma(\epsilon) E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon)}\right) = \Delta\left(\{\omega_\sigma(\epsilon) E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon_0)}\right)$$

with

$$\|\cdot\|_{\Delta(\{\omega_\sigma(\epsilon) E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon_0)})} \leq \|\cdot\|_{\omega_\sigma E^{1+\epsilon}} \leq (1+\epsilon)(1+\epsilon, \epsilon_0) \|\cdot\|_{\Delta(\{\omega_\sigma(\epsilon) E^{(1+\epsilon-\epsilon)}\}_{\epsilon \in (0, \epsilon_0)})}.$$

Clearly, this yields the required statement.

We demonstrate the applications of obtained results for extrapolation spaces generated by Orlicz spaces. Before we state the first lemma, we recall that a function $\rho: [0, \infty) \rightarrow [0, \infty)$ is called quasi-concave if it is continuous and positive on $(0, \infty)$, and satisfies $\rho(s) \leq \max\left\{1, \frac{s}{t}\right\} \rho(t)$ for all $s, t > 0$. We use an obvious fact that every quasi-concave function ρ has a concave majorant $\tilde{\rho}$ given by $\tilde{\rho}(t) := \inf_{s>0} \left(1 + \frac{t}{s}\right) \rho(s)$, which satisfies $\rho(t) \leq \tilde{\rho}(t) \leq 2\rho(t)$ for all $t > 0$.

We need somewhat technical lemmas.

Lemma 4.3 (see [31]). Let $0 < \epsilon < \infty$ and let ϕ be an Orlicz function. The following statements are true for an Orlicz function Φ given by $\Phi(s) := \phi(s^{1+\epsilon})$ for all $s \geq 0$:

(i) The following formula holds:

$$X_\Phi \cong (X_\phi)^{(1+\epsilon)}.$$

(ii) There exists $\varphi_\sigma \in \mathcal{U}$ such that, for any Banach lattice X the following formula holds:

$$X_\Phi = \varphi_\sigma(X^{(1+\epsilon)}, L^\infty),$$

where the constants of equivalence of norms are universal and do not depend on $1 + \epsilon, \phi$ and X .

Proof. (i). The proof of this isometrical formula is obvious so we omit it.

(ii). Fix $0 < \epsilon < \infty$ and let $\Phi(s) := \phi(s^{1+\epsilon})$ for all $s \geq 0$. Then $\Phi^{-1}(s) = \phi^{-1}(s)^{1/1+\epsilon}$ for all $s \geq 0$. Since ϕ^{-1} is a concave function, $\rho(s) := \phi^{-1}(s^{1+\epsilon})^{1/1+\epsilon}$ is a quasi-concave function. This implies that $\varphi_\sigma \in \mathcal{U}$, where $\varphi_\sigma(0, 0) := 0$ and

$$\varphi_\sigma(s, t) := t\tilde{\rho}(s/t), s, t > 0.$$

We consider two functions $(\varphi_\sigma)_0, (\varphi_\sigma)_1 \in \mathcal{U}$ given by $(\varphi_\sigma)_0(s, t) := s^{1/1+\epsilon} t^{1-1/1+\epsilon}$ for all $s, t \geq 0$ and $(\varphi_\sigma)_1(0, 0) := 0$ and $(\varphi_\sigma)_1(s, t) := t\Phi^{-1}(s/t)$ for all $s, t > 0$. Then we have

$$\varphi_\sigma((\varphi_\sigma)_0(s, 1), 1) = \varphi_\sigma(s^{1/1+\epsilon}, 1) = \rho(s^{1/1+\epsilon}) = (\varphi_\sigma)_1(s, 1), s \geq 0$$

with universal constants of equivalence that do not depend on $(1 + \epsilon)$.

Now we use the special case of the so-called reiteration formulas (see [26]) which states: If $(\varphi_\sigma)_0, (\varphi_\sigma)_1, \varphi_\sigma \in \mathcal{U}$, then for any couple (X_0, X_1) of Banach function lattices

$$\varphi_\sigma((\varphi_\sigma)_0(X_0, X_1), X_1) = (\varphi_\sigma)_1(X_0, X_1)$$

with equivalence of norms depending only on constants of equivalence $(\varphi_\sigma)_1(\cdot, 1) \approx \varphi_\sigma((\varphi_\sigma)_0(\cdot, 1), 1)$. We use above formula with $X_0 = X, X_1 = L^\infty$ and constructed φ_σ to obtain (by $(\varphi_\sigma)_0(X, L^\infty) \cong X^{(1+\epsilon)}$)

$$X_\Phi \cong (\varphi_\sigma)_1(X, L^\infty) = \varphi_\sigma(X^{(1+\epsilon)}, L^\infty)$$

with universal constants of equivalence that do not depend on $(1 + \epsilon)$ and X . This completes the proof.

Corollary 4.4 [31]. Let ϕ be an Orlicz function and let, for a given $0 < \epsilon < \infty$, Φ be an Orlicz function given by $\Phi(s) = \phi(s^{1+\epsilon})$ for all $s \geq 0$. Then, there exists $\varphi_\sigma \in \mathcal{U}$ such that for any measure space (Ω, Σ, μ) the following formula holds:

$$L_\Phi(\mu) \cong L_\phi^{(1+\epsilon)} = \varphi_\sigma(L^{1+\epsilon}(\mu), L^\infty(\mu))$$

with constants of equivalence of norms that do not depend on $(1 + \epsilon)$.

We also need the following result.

Lemma 4.5 (see [31]). Let $0 \leq \epsilon < \infty$ and let Φ be an Orlicz function such that $s \mapsto \Phi(s)/s^{1+\epsilon}$ is non-decreasing function and $s \mapsto \Phi(s)/s^{1+2\epsilon}$ is non-increasing function. Then there exists $\varphi_\sigma \in \mathcal{U}$ such that, for any measure space (Ω, Σ, μ) ,

$$L_\Phi(\mu) = \varphi_\sigma(L^{1+\epsilon}(\mu), L^{1+2\epsilon}(\mu)),$$

with universal constants of equivalence of norms that do not depend on $(1 + \epsilon), (1 + 2\epsilon)$ and Φ .

Proof. By conditions imposed on Φ , it follows that $s \mapsto \Phi^{-1}(s)/s^{1/1+\epsilon}$ is a non-increasing and $s \mapsto \Phi^{-1}(s)/s^{1/1+2\epsilon}$ is a non-decreasing function. These assumptions easily imply that there exists a quasi-concave function ρ such that

$$\Phi^{-1}(s) = s^{1/1+\epsilon} \rho\left(\frac{1}{s^{1+2\epsilon} \frac{1}{1+\epsilon}}\right), s > 0.$$

(to see this it is enough to take $\rho(s) := s^{r/\rho} \Phi^{-1}(s^{-r})$ for $s > 0$, where $r := \frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}$). Therefore, by setting $\varphi_\sigma(0,0) := 0$ and $\varphi_\sigma(s, t) = s\tilde{\rho}(t/s)$ for all $s, t > 0$, we get that (by $\rho \leq \tilde{\rho} \leq 2\rho$)

$$\Phi^{-1}(s) \leq \varphi_\sigma(s^{1/1+\epsilon}, s^{1/1+2\epsilon}) \leq 2\Phi^{-1}(s) \text{ for all } s \geq 0.$$

Hence, for $\vec{X} := (L^{1+\epsilon}(\mu), L^{1+2\epsilon}(\mu))$, the standard calculations yields (see [28, pp. 460-461]) that $L_\Phi(\mu) = \varphi_\sigma(\vec{X})$ with $\|\cdot\|_{\varphi_\sigma(\vec{X})} \leq \|\cdot\|_{L_\Phi} \leq 2\|\cdot\|_{\varphi_\sigma(\vec{X})}$. This completes the proof.

In what follows we use two obvious properties of weights from the class $A_{1+\epsilon}$ with $0 \leq \epsilon < \infty$:

(i) If $0 < \epsilon < \infty$, then $w_\sigma \in A_{1+\epsilon}$ if and only if $w_\sigma^{-\frac{1}{\epsilon}} \in A_{\frac{1+\epsilon}{\epsilon}}$ and $\left[w_\sigma^{-\frac{1}{\epsilon}}\right]_{A_{\frac{1+\epsilon}{\epsilon}}} = [w_\sigma]_{A_{1+\epsilon}}^{\frac{1}{\epsilon}}$;

(ii) If $0 \leq \epsilon < \infty$, then $A_{1+\epsilon} \subset A_{1+2\epsilon}$ and $[w_\sigma]_{A_{1+2\epsilon}} \leq [w_\sigma]_{A_{1+\epsilon}}$.

The first property easily follows from the definitions and that second one by the Hölder inequality.

We apply the so-called "openness property" for $A_{1+\epsilon}$ weights in $L^0(\Omega, d, \mu)$, where (Ω, d, μ) is an SHT. In the remarkable paper [15], the authors explicitly compute an admissible value of $\epsilon > 0$ such that any $A_{1+\epsilon}$ weight w_σ belongs to $A_{1+\epsilon-\epsilon}$. This result states (see, [15, Thm. 1.2]): If $w_\sigma \in A_{1+\epsilon}$ for a given $0 < \epsilon < \infty$, then $w_\sigma \in A_{1+\epsilon-\epsilon}$ with

$$\epsilon = \frac{\epsilon}{1 + \tau_{(1+\epsilon)\mu}[\sigma]_{A'_\infty}},$$

where $\tau_{\kappa\mu} = 6(32\kappa^2(\kappa^2 + \kappa)^2)^{D\mu}$ and $\sigma = w_\sigma^{-\frac{1}{\epsilon}}$. Furthermore

$$[w_\sigma]_{A_{1+\epsilon-\epsilon}} \leq 2^\epsilon (4\kappa)^{(1+\epsilon)D\mu} [w_\sigma]_{A_{1+\epsilon}}.$$

We also use the following estimate (see [15, Thm. 1.3]) of the norm of M in $L^{1+\epsilon}(w_\sigma d\mu)$ where $w_\sigma \in A_{1+\epsilon}$,

$$\|M\|_{L^{1+\epsilon}(w_\sigma d\mu)} \leq C_\Omega \sum_\sigma \left(\frac{1}{\epsilon} [w_\sigma]_{A_{1+\epsilon}} [\sigma]_{A'_\infty}\right)^{1/1+\epsilon}$$

where the constant C_Ω depends only on the doubling constant of the measure μ and the geometric constant κ of the quasi-metric.

Notice that it follows by above estimate and property (ii) that, for every $w_\sigma \in A_{1+\epsilon}$ with $0 < \epsilon < \infty$, we have

$[\sigma]_{A'_\infty} \leq c_\Omega [\sigma]_{\frac{A_{1+\epsilon}}{\epsilon}} = c_\Omega [w_\sigma]_{A_{1+\epsilon}}^{\frac{1}{\epsilon}}$, where $c_\Omega \geq 1$ is also a constant that only depends on the doubling constant

of the measure μ and the geometric constant κ of the quasi-metric. In consequence, Buckley's type estimate follows (see [7] in the case of Euclidean space $\mathbb{R}^{2+\epsilon}$):

$$\|M\|_{L^{1+\epsilon}(\mu) \rightarrow L^{1+\epsilon}(w_\sigma(\mu))} \leq C_\Omega (c_\Omega)^{1+\epsilon} \left(\frac{1}{\epsilon}\right)^{1+\epsilon} [w_\sigma]_{A_{1+\epsilon}}^{\frac{1}{\epsilon}} \leq \tilde{c}_\Omega \frac{1+\epsilon}{\epsilon} [w_\sigma]_{A_{1+\epsilon}}^{\frac{1}{\epsilon}}$$

The following theorem shows the boundedness of the maximal operator in grand spaces generated by Orlicz spaces over spaces of homogenous type.

Theorem 4.6 (see [31]). Let (Ω, d, μ) be an SHT and let w_σ be a weight in $A_{1+\epsilon}(\Omega) \cap L^1(w_\sigma d\mu)$ for some $0 < \epsilon < \infty$. Then for every Orlicz function ϕ , the maximal Hardy-Littlewood operator M is bounded in the grand space $\omega_\sigma L_\phi(w_\sigma d\mu)^{1+\epsilon}$ whenever ω_σ is bounded on any subinterval $[\delta, \epsilon)$ of $(0, \epsilon)$.

Proof. For simplicity of notation, we define the measure ν by $d\nu := w_\sigma d\mu$. Based on the above discussion about properties of weights in the class $A_{1+\epsilon}$ weights, it follows that for any $w_\sigma \in A_{1+\epsilon}$ there exists $\epsilon_0 \in (0, \epsilon)$ such that

$$[w_\sigma]_{A_{1+\epsilon-\epsilon_0}} \leq 2^\epsilon (4\kappa)^{(1+\epsilon)D_\mu} [w_\sigma]_{A_{1+\epsilon}}$$

Moreover, the maximal operator M is bounded in $L^{1+\epsilon}(v)$ and

$$\|M\|_{L^{1+\epsilon}(v)} \leq \tilde{c}_\Omega \frac{1+\epsilon}{\epsilon} [w_\sigma]_{A_{1+\epsilon}}^{\frac{1}{\epsilon}}$$

We consider two cases: $\epsilon \in (0, \epsilon_0]$ and $\epsilon \in (\epsilon_0, \infty)$. In the first case we have $w_\sigma \in A_{1+\epsilon-\epsilon}$ with $1 \leq [w_\sigma]_{A_{1+\epsilon-\epsilon}} \leq [w_\sigma]_{A_{1+\epsilon-\epsilon_0}}$. Thus, the maximal operator M is bounded in $L_{1+\epsilon-\epsilon}(w_\sigma d\mu)$ and, for all $\epsilon \in (0, \epsilon_0]$, we have

$$\begin{aligned} \|M\|_{L^{1+\epsilon-\epsilon}(v)} &\leq \tilde{c}_\Omega (1+\epsilon-\epsilon)' [w_\sigma]_{A_{1+\epsilon-\epsilon}}^{\frac{1}{\epsilon-\epsilon}} \leq \tilde{c}_\Omega (1+\epsilon-\epsilon_0)' [w_\sigma]_{A_{1+\epsilon-\epsilon_0}}^{\frac{1}{\epsilon-\epsilon_0}} \\ &= C \left(\Omega, 1+\epsilon, [w_\sigma]_{A_{1+\epsilon-\epsilon_0}} \right) \end{aligned}$$

where $C \left(\Omega, 1+\epsilon, [w_\sigma]_{A_{1+\epsilon-\epsilon_0}} \right)$ is a constant that does not depend on $\epsilon \in (0, \epsilon_0)$. Consequently,

$$M: \{\omega_\sigma(\epsilon)L_\phi(v)\}_{\epsilon \in (0, \epsilon_0)} \rightarrow \{\omega_\sigma(\epsilon)L_\phi(v)\}_{\epsilon \in (0, \epsilon_0)}$$

By the conditions imposed on w_σ , we have $\chi_\Omega \in L_\phi(v)$. Thus, we can apply Lemma 4.2 to obtain that M is bounded in $\omega_\sigma L_\phi(v)^{1+\epsilon}$.

We prove the boundedness of Calderón-Zygmund singular operator in extrapolation spaces generated by Orlicz spaces over SHT.

Theorem 4.7 (see [31]). Let $0 < \epsilon < \infty$ and let (Ω, d, μ) be an SHT, and let w_σ be a weight in $A_{1+\epsilon}(\Omega) \cap L^1(\mu)$ for some $\epsilon > 0$. Suppose that a family of Orlicz functions $\{\Phi_\epsilon\}_{\epsilon \in (0, \epsilon)}$ with $\sup_{\epsilon \in (0, \epsilon)} \Phi_\epsilon^{-1}(1) < \infty$ such that

$t \mapsto \Phi_\epsilon/t^{1+\epsilon-\epsilon}$ is a non-decreasing and $t \mapsto \Phi_\epsilon(t)/t^{1+2\epsilon}$ is a non-increasing function for all $\epsilon \in (0, \epsilon)$. Then every Calderón-Zygmund singular operator T is bounded in the extrapolation space $\Delta \left(\{\omega_\sigma L_{\Phi_\epsilon}(w_\sigma d\mu)\}_{\epsilon \in (0, \epsilon)} \right)$ generated by any function $\omega_\sigma: (0, \epsilon) \rightarrow (0, \infty)$ that is bounded on any subinterval $[\delta, \epsilon)$ of $(0, \epsilon)$.

Proof. For simplicity of notation, we let $dv = w_\sigma d\mu$. We now apply Lemma 4.5 and its proof to find a family $\{(\varphi_\sigma)_\epsilon\}_{\epsilon \in (0, \epsilon)} \subset \mathcal{U}$ such that $\Phi_\epsilon^{-1}(s) = (\varphi_\sigma)_\epsilon(s^{1/(1+\epsilon-\epsilon)}, s^{1/(1+2\epsilon)})$ for all $s > 0$ and

$$L_{\Phi_\epsilon}(v) = \varphi_\sigma(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v)),$$

with universal constants of equivalence of norms that do not depend on $(1+\epsilon)$, $(1+2\epsilon)$ and ϵ . This implies that $\sup_{\epsilon \in (0, \epsilon)} (\varphi_\sigma)_\epsilon(1, 1) < \infty$ and moreover $\{\omega_\sigma(\epsilon)\varphi_\sigma(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))\}_{\epsilon \in (0, \epsilon)}$ forms an admissible family

with

$$\Delta \left(\{\omega_\sigma(\epsilon)L_{\Phi_\epsilon}(v)\}_{\epsilon \in (0, \epsilon)} \right) = \Delta \left(\{\omega_\sigma(\epsilon)\varphi_\sigma(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))\}_{\epsilon \in (0, \epsilon)} \right)$$

Now observe that it follows from the proof of Theorem 4.6 that there exists $\epsilon_0 \in (0, \epsilon)$ such that

$$\sup_{\epsilon \in (0, \epsilon_0)} \|M\|_{L^{1+\epsilon-\epsilon}(v)} < \infty$$

Since $\epsilon > -\epsilon_0$, we have $w_\sigma \in A_{1+2\epsilon}$ with $[w_\sigma]_{A_{1+2\epsilon}} \leq [w_\sigma]_{A_{1+\epsilon-\epsilon_0}}$,

$$\|M\|_{L^{1+2\epsilon}(v)} \leq C(\Omega, 1+\epsilon, [w_\sigma]_{A_{1+\epsilon}})$$

Thus, by interpolation property for Calderón-Lozanovskii space, it follows that

$$M: \Delta \left(\{\omega_\sigma(\epsilon)(\varphi_\sigma)_\epsilon(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))\}_{\epsilon \in (0, \epsilon_0)} \right) \rightarrow \Delta \left(\{\omega_\sigma(\epsilon)(\varphi_\sigma)_\epsilon(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))\}_{\epsilon \in (0, \epsilon_0)} \right)$$

Now observe that Calderón-Lozanovskii space $\varphi_\sigma(X_0, X_1)$ can be defined as the space of all functions $f_\sigma \in L^0(\mu)$ such that $|f_\sigma| = \varphi_\sigma(|(f_\sigma)_0|, |(f_\sigma)_1|)$ for some $(f_\sigma)_0 \in X_0, (f_\sigma)_1 \in X_1$ with the norm

$$\|f_\sigma\|_{\varphi_\sigma} = \inf \max \{ \|(f_\sigma)_0\|_{X_0}, \|(f_\sigma)_1\|_{X_1} \}$$

where the infimum is taken over all $(f_\sigma)_0 \in X_0, (f_\sigma)_1 \in X_1$ for which $|f_\sigma| = \varphi_\sigma(|(f_\sigma)_0|, |(f_\sigma)_1|)$. Combining this fact with proof of Lemma 4.2 (for $E = L_1$), we easily get that for all $f_\sigma \in \Delta \left(\{L_{\Phi_\epsilon}(v)\} \right)$,

$$\sup_{\epsilon \in [\epsilon_0, \epsilon)} \omega_\sigma(\epsilon) \|f_\sigma\|_{(\varphi_\sigma)_\epsilon(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))} \leq C(1+\epsilon, \epsilon_0) \sup_{\epsilon \in (0, \epsilon_0)} \omega_\sigma(\epsilon) \|f_\sigma\|_{(\varphi_\sigma)_\epsilon(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))}$$

and whence

$$\Delta \left(\{\omega_\sigma(\epsilon)L_{\Phi_\epsilon}(v)\}_{\epsilon \in (0, \epsilon)} \right) = \Delta \left(\{\omega_\sigma(\epsilon)(\varphi_\sigma)_\epsilon(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))\}_{\epsilon \in (0, \epsilon_0)} \right)$$

To complete the proof we recall that by property (i) of weights in the class $A_{1+\epsilon}$, for $0 < \epsilon < \infty$, we have that

$w_\sigma \in A_{1+\epsilon}$ is equivalent to $w_\sigma^{-\frac{1}{\epsilon}} \in A_{\frac{1+\epsilon}{\epsilon}}$ with $\left[w_\sigma^{-\frac{1}{\epsilon}} \right]_{A_{\frac{1+\epsilon}{\epsilon}}} = [w_\sigma]_{A_{1+\epsilon}}^{\frac{1}{\epsilon}}$. This implies that the maximal operator M is

bounded in $L^{1+\epsilon}(v)' \cong (L^{1+\epsilon}(w_\sigma d\mu))' \cong L^{\frac{1+\epsilon}{\epsilon}} \left(w_\sigma^{-\frac{1}{\epsilon}} d\mu \right)$ with

$$\|M\|_{L^{1+\epsilon}(v)'} \leq (1+\epsilon^2) [w_\sigma]_{A_{1+\epsilon}}$$

where $C = C(\mu, \kappa)$. Now if we take $C = 1 + \epsilon - \epsilon$ with $\epsilon \in (0, \epsilon_0)$, we conclude that

$$\|M\|_{L^{1+\epsilon-\epsilon}(v)'} \leq (1 + \epsilon)(1 + \epsilon - \epsilon)[w_\sigma]_{A_{1+\epsilon-\epsilon}} \leq (1 + \epsilon)[w_\sigma]_{1+\epsilon-\epsilon_0} \leq \tilde{C}(\Omega, 1 + \epsilon, [w_\sigma]_{A_{1+\epsilon}}).$$

This shows that $M: \{L^{1+\epsilon-\epsilon}(v)'\}_{\epsilon \in (0, \epsilon_0)} \rightarrow \{L^{1+\epsilon-\epsilon}(v)'\}_{\epsilon \in (0, \epsilon_0)}$. Similarly, for $0 \leq \epsilon < \infty$, we get (by $w_\sigma \in A_{1+2\epsilon}$ with $[w_\sigma]_{A_{1+2\epsilon}} \leq [w_\sigma]_{A_{1+\epsilon}}$),

$$\|M\|_{L^{1+2\epsilon}(w_\sigma d\mu)'} \leq (1 + 2\epsilon)[w_\sigma]_{A_{1+\epsilon}}$$

Now, by the Köthe duality formula for Calderón-Lozanovskii spaces, we have for all $\epsilon \in (0, \epsilon_0)$,

$$(\omega_\sigma(\epsilon)(\varphi_\sigma)_\epsilon(L^{1+\epsilon-\epsilon}(v), L^{1+2\epsilon}(v))' = \omega_\sigma(\epsilon)^{-1}(\overline{\varphi_\sigma})_\epsilon(L^{1+\epsilon-\epsilon}(v)', L^{1+2\epsilon}(v)')$$

with universal constants of equivalence of norms (that do not depend on ϵ). To finish, we apply Theorem 4.1 for the family of functions $\{\omega_\sigma(\epsilon)(\varphi_\sigma)_\epsilon\}_{\epsilon \in (0, \epsilon_0)} \subset \mathcal{U}$ to get the required statement.

By applying Theorem 4.7, we establish the boundedness of Calderón-Zygmund singular operator in extrapolation spaces generated by Orlicz-Zygmund spaces over SHT. We briefly fix some required notation. For $0 < \epsilon < \infty$ and $\alpha > 0$, we consider an Orlicz function $\Phi(t) = t^{1+\epsilon} \log^\alpha(1+t)$ for $t \geq 0$. Then Orlicz space L_Φ on any measure space (Ω, Σ, μ) is called an Orlicz-Zygmund space and is denoted by $L^{1+\epsilon} \log^\alpha L$.

Let $\{\Phi_\epsilon\}_{\epsilon \in (0, \epsilon)}$ be a family of Orlicz functions given by $\Phi_\epsilon(t) = t^{1+\epsilon-\epsilon} \log^\alpha(1+t)$ for all $t \geq 0$ and let $\omega_\sigma(\epsilon) := \epsilon^{\theta/(1+\epsilon-\epsilon)}$ for all $\epsilon \in (0, \epsilon)$ and some $\theta > 0$. Then the extrapolation space $\Delta(\{\omega_\sigma(\epsilon)L_{\Phi_\epsilon}(w_\sigma d\mu)\}_{\epsilon \in (0, \epsilon)})$ is well defined. In what follows this space is called grand Orlicz-Zygmund space and is denoted by $L_{w_\sigma}^{1+\epsilon, \theta} \log^\alpha L$.

We note that from a general result on the boundedness of the Hilbert transform H in Orlicz spaces $L_\Phi(w_\sigma dm)$ over $[0, 1]$ equipped with the Lebesgue measure m , it follows that H given by

$$Hf_\sigma(t) = \text{p.v.} \int_0^1 \sum \frac{f_\sigma(s)}{t-s} ds$$

is bounded in the Orlicz-Zygmund space $L^{1+\epsilon} \log^\alpha L(w_\sigma dm)$ if (and only if) $w_\sigma \in A_{1+\epsilon}([0, 1])$ (see [22, Thm. 3.4.1, p. 113]).

We have the following version of the theorem for any Calderón-Zygmund singular operator in the case of the grand Orlicz-Zygmund spaces over spaces of homogenous type.

Theorem 4.8 (see [31]). Let (Ω, d, μ) be an SHT and let w_σ be a weight in $A_{1+\epsilon}(\Omega) \cap L^1(\mu)$ for some $0 < \epsilon < \infty$. Then every Calderón-Zygmund singular operator T is bounded in the grand Orlicz-Zygmund space $L_{w_\sigma}^{1+\epsilon, \theta} \log^\alpha L$ over (Ω, d, μ) for all $\alpha, \theta > 0$.

Proof. Let $\{\Phi_\epsilon\}_{\epsilon \in (0, \epsilon)}$ be a family of Orlicz functions which generate our space. Clearly, $t \mapsto \Phi_\epsilon(t)/t^{(1+\epsilon-\epsilon)} = \log^\alpha(1+t)$ is an increasing and $t \mapsto \Phi_\epsilon(t)/t^{(1+\epsilon+\alpha)} = \frac{1}{t^\epsilon} \left(\frac{\log(1+t)}{t}\right)^\alpha$ is a decreasing function on $(0, \infty)$. Since $\Phi_\epsilon^{-1}(\log^\alpha 2) = 1$, we have $\sup_{\epsilon \in (0, \epsilon)} \Phi_\epsilon^{-1}(1) \leq 2^{-\alpha}$. A standard calculus shows that if ω_σ is given by $\omega_\sigma(\epsilon) := \epsilon^{\theta/(1+\epsilon-\epsilon)}$ for all $\epsilon \in (0, \epsilon)$, then for any $\delta \in (0, \epsilon)$,

$$\sup_{\epsilon \in [\delta, \epsilon]} \omega_\sigma(\epsilon) = (\epsilon)^\theta$$

Thus, we conclude that the assumptions of Theorem 4.7 are satisfied with $\epsilon = \alpha$ and thus the required statement follows.

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