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Research Paper

Application on Variational Estimates for Discrete Operators Modeled on Multi-Dimensional Polynomial Subsets of Primes

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Abstract
We follow the pioneer author Bartosz Trojan [27] on the perfect extensions of Birkhoff's and Cotlar's ergodic theorems to multi-dimensional polynomial subsets of prime numbers $\mathbb{P}^k$. We deduce and show an application on them from $\ell^{1+\epsilon}(\mathbb{Z}^d)$-boundedness of $(2+\epsilon)$-variational seminorms for the corresponding discrete operators of Radon type, for $\epsilon > 0$.

1. Introduction

For $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with d_0 invertible commuting and measure preserving transformations $T_1, \dots, T_{d_0}: X \to X$. Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}): \mathbb{R}^k \to \mathbb{R}^{d_0}$ denote a polynomial mapping such that each $\mathcal{P}_j$ is a polynomial on $\mathbb{R}^k$ having integer coefficients without a constant term. Let B be an open bounded convex subset in $\mathbb{R}^k$ containing the origin such that for some $\iota > 0$ and all $N \in \mathbb{N}$,

$$[-\iota N, \iota N]^k \subseteq B_N \subseteq [-N, N]^k, \tag{1.1}$$

where for $\lambda > 0$, we have set

$$B_\lambda = \{x \in \mathbb{R}^k : \lambda^{-1} x \in B\}.$$

Now we consider the following averages

$$\mathcal{A}_N^p f_m(x) = \frac{1}{\pi_B(N)} \sum_{n \in \mathbb{N}^{k'}} \sum_{(1+\epsilon) \in \mathbb{P}^{k''}} \sum_m f_m\left(T_1^{\mathcal{P}_1(n,1+\epsilon)} \dots T_{d_0}^{\mathcal{P}_{d_0}(n,1+\epsilon)} x\right) \mathbb{1}_{B_N}(n, 1+\epsilon),$$

where $k = k' + k''$, P denotes the set of prime numbers, and

$$\pi_B(N) = \sum_{n \in \mathbb{N}^{k'}} \sum_{(1+\epsilon) \in \mathbb{P}^{k''}} \mathbb{1}_{B_N}(n, 1+\epsilon).$$

We establish the following theorem (see [27]).

Theorem A. Assume that $0 < \epsilon < \infty$. For every $f_m \in L^{1+\epsilon}(X, \mu)$ there exists $f_m^* \in L^{1+\epsilon}(X, \mu)$ such that

$$\lim_{N \to \infty} \sum_m \mathcal{A}_N^p f_m(x) = f_m^*(x),$$

for μ-almost all $x \in X$.

Sums over prime numbers are irregular, we them work with weighted averaging operators,

$$\mathrm{M}_N^p f_m(x) = \frac{1}{\vartheta_B(N)} \sum_{n \in \mathbb{N}^{k'}} \sum_{(1+\epsilon) \in \mathbb{P}^{k''}} \sum_m f_m\left(T_1^{\mathcal{P}_1(n,1+\epsilon)} \dots T_{d_0}^{\mathcal{P}_{d_0}(n,1+\epsilon)} x\right) \mathbb{1}_{B_N}(n, 1+\epsilon) \left(\prod_{j=1}^{k''} \log p_j\right),$$

where

$$\vartheta_B(N) = \sum_{n\in\mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathbb{P}^{k''}} \mathbb{1}_{B_N}(n, 1+\epsilon)\left(\prod_{j=1}^{k''} \log p_j\right).$$

Then the pointwise convergence of $(\mathcal{A}_N f_m: N \in \mathbb{N})$ can be deduced from the properties of $(\mathcal{H}_N f_m: N \in \mathbb{N})$, see Proposition 2.1 for details.

Hence we also study pointwise convergence of truncated discrete singular operators. So, for $K_m \in C^1(\mathbb{R}^k \setminus \{0\})$ be a Calderón-Zygmund kernel satisfying the differential inequality

$$|x|^k|K_m(x)| + |x|^{k+1}|\nabla K_m(x)| \le 1, \tag{1.2}$$

for all $x \in \mathbb{R}^k$ with $|x| \ge 1$, and the cancellation condition

$$\int_{B_\lambda \setminus B_{\lambda'}} \sum_m K_m(x)\mathrm{d}x = 0, \tag{1.3}$$

for every $0 < \lambda' \le \lambda$. Then the truncated discrete singular operator $\mathcal{H}_N^{\mathcal{P}}$ is defined as

$$\mathcal{H}_N^{\mathcal{P}} f_m(x) = \sum_{n\in\mathbb{Z}^{k'}} \sum_{(1+\epsilon)\in(\pm\mathcal{P})^{k''}} \sum_m f_m\left(T_1^{\mathcal{P}_1(n,1+\epsilon)} \cdots T_{d_0}^{\mathcal{P}_{d_0}(n,1+\epsilon)} x\right)$$

$$K_m(n, 1+\epsilon)\mathbb{1}_{B_N}(n, 1+\epsilon)\left(\prod_{j=1}^{k''} \log|p_j|\right).$$

The logarithmic weights in $M_N^{\mathcal{P}}$ and $\mathcal{H}_N^{\mathcal{P}}$ correspond to the density of prime numbers. [27] prove the following theorem, which may be thought as an extension of Cotlar's ergodic theorem, see [4].

Theorem B. Assume that $0 < \epsilon < \infty$. For every $f_m \in L^{1+\epsilon}(X, \mu)$ there exists $f_{\mathrm{m}}^* \in L^{1+\epsilon}(X, \mu)$ such that

$$\lim_{N\to\infty} \sum_m \mathcal{H}_N^{\mathcal{P}} f_m(x) = f_m^*(x),$$

for μ-almost all $x \in X$.

The classical approach to the pointwise convergence in $L^{1+\epsilon}(X, \mu)$ proceeds in two steps. Namely, one needs to show $L^{1+\epsilon}(X, \mu)$ boundedness of the corresponding maximal function reducing the problem to showing the convergence on some dense class of $L^{1+\epsilon}(X, \mu)$ functions. However, finding such a class may be a difficult task.

This is the case of one dimensional averages along $(n^2: n \in \mathbb{N})$ studied by [2]. To overcome this issue Bourgain introduced the oscillation seminorm defined for a given lacunary sequence $(N_j: j \in \mathbb{N})$ and a sequence of complex numbers $(a_n^{\mathrm{m}}: n \in \mathbb{N})$ as

$$O_J(a_n^{\mathrm{m}}: n \in \mathbb{N}) = \left(\sum_{j=1}^{J} \sum_m \sup_{N_j \le n \le N_{j+1}} \left|a_n^{\mathrm{m}} - a_{N_j}^m\right|^2\right)^{1/2}.$$

Then the pointwise convergence of $(\mathcal{A}_N f_m: N \in \mathbb{N})$ is reduced to showing that

$$\left\|O_J(\mathcal{A}_N f_m: N \in \mathbb{N})\right\|_{L^2} = o(J^{1/2}),$$

while J tends to infinity. In place of the oscillation seminorm, we investigate $(2+\epsilon)$-variational seminorm. We recall that $(2+\epsilon)$-variational seminorm of a sequence $(a_n^{\mathrm{m}}: n \in \mathbb{N})$ is defined by

$$V_{2+\epsilon}(a_n^{\mathrm{m}}: n \in \mathbb{N}) = \sup_{k_0 < k_2 < \cdots < k_j} \left(\sum_{j=1}^{J} \sum_m \left|a_{k_j}^{\mathrm{m}} - a_{k_j - 1}^m\right|^{2+\epsilon}\right)^{1/2+\epsilon}.$$

In fact, $(2+\epsilon)$-variational seminorm controls O_J as well as the maximal function. Indeed, for any $\epsilon \ge 0$, by Hölder's inequality we have

$$O_I(a_n^{\mathrm{m}}: n \in \mathbb{N}) \le J^{\frac{1}{2} - \frac{1}{2+\epsilon}} V_{2+\epsilon}(a_n^{\mathrm{m}}: n \in \mathbb{N}).$$

Moreover, for any $n_0 \in \mathbb{N}$,

$$\sup_{n\in\mathbb{N}}|a_n^{\mathrm{m}}| \leq |a_{n_0}^m| + V_{2+\epsilon}(a_n^{\mathrm{m}}: n \in \mathbb{N}).$$

So, the main motivation to study $L^{1+\epsilon}(X,\mu)$ boundedness of $(2+\epsilon)$-variational seminorm is the following observation: if $V_{2+\epsilon}(a_n^{\mathrm{m}}: n \in \mathbb{N}) < \infty$ for any $\epsilon \geq 0$ then the sequence $(a_n^{\mathrm{m}}: n \in \mathbb{N})$ converges. Hence, we can deduce Theorem A and Theorem B from the following result (see [27]).

Theorem C. For every $0 < \epsilon < \infty$ there is $C_{1+\epsilon} > 0$ such that for all $0 < \epsilon < \infty$ and all $f_m \in L^{1+\epsilon}(X,\mu)$,

$$\left\| \sum_m V_{2+\epsilon}\left(M_N^p f_m: N \in \mathbb{N}\right) \right\|_{L^{1+\epsilon}} \leq C_{1+\epsilon} \frac{2+\epsilon}{\epsilon} \sum_m \| f_m \|_{L^{1+\epsilon}}, \tag{1.4}$$

and

$$\left\| \sum_m V_{2+\epsilon}\left(\mathcal{H}_N^p f_m: N \in \mathbb{N}\right) \right\|_{L^{1+\epsilon}} \leq C_{1+\epsilon} \frac{2+\epsilon}{\epsilon} \sum_m \| f_m \|_{L^{1+\epsilon}} . \tag{1.5}$$

The constant $C_{1+\epsilon}$ is independent of the coefficients of the polynomial mapping $\mathcal{P}$. The variational estimates for discrete averaging operators have been the subject of many, see [8,10,11,13,15,16,26]. [10], studied the case $d_0 = k = k' = 1$ and has obtained the inequality (1.4) for $0 < \epsilon < \infty$ and $(2+\epsilon) > \max\left\{1+\epsilon, \frac{1+\epsilon}{\epsilon}\right\}$. And, [26] obtained (1.4) for all $0 < \epsilon < \infty$ but for $(1+\epsilon)$ in some vicinity of 2. Only recently in [11] the variational estimates have been established in the full range of parameters, that is $0 < \epsilon < \infty$, covering the case $k'' = 0$.

[26], has proved (1.4) also for the averaging operators modeled on prime numbers, that is when $d_0 = k = k'' = 1$ with a polynomial $P_m(n) = n$. It is worth mentioning that the variational estimates for discrete operators are based on a priori estimates for their continuous counterparts developed in [9], see also [11, Appendix].

For the variational estimates for discrete singular operators see [3,11,13,16]. [16], obtained the inequality (1.5), for the truncated Hilbert transform modeled on prime numbers, which corresponds to $d_0 = k = k'' = 1$ and a polynomial $P_m(n) = n$. In fact, discrete singular operators of Radon type required a new approach. An important milestone has been improve by [7].

The complete development of the discrete singular operators of Radon type has been obtained in [11].

Concerning pointwise ergodic theorems over prime numbers, there are some results using oscillation seminorms. [1], has shown pointwise convergence for the averages along prime numbers for functions from $L^2(X,\mu)$. Then his result was extended to all $L^{1+\epsilon}(X,\mu), \epsilon > 0$, by [24], see also [2, Section 9]. Not long afterwards, [18] has proved Theorem A for $L^2(X,\mu)$, $d_0 = k = k'' = 1$, and any integer-valued polynomial. Nair also studied ergodic averages for functions in $L^{1+\epsilon}(X,\mu)$ for $\epsilon \neq 1$, however, [19, Lemma 14] contains an error. In fact, the estimates on the multipliers W_N are insufficient to show that the sum considered at the end of the proof has bounds independent of $|\alpha - a^m/b|$. Lastly, the extension of Cotlar's ergodic theorem to prime numbers has been established in [14], see [16] and also [27].

In view of the Calderón transference principle, while proving Theorem C, we may work with the model dynamical system, namely, Z^{d_0} with the counting measure and the shift operators. We denote by M_N^p and H_N^p, the corresponding operators, namely,

$$M_N^p f_m(x) = \frac{1}{\vartheta_B(N)} \sum_{n\in\mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathbb{P}^{k''}} \sum_m f_m\left(x - \mathcal{P}(n,1+\epsilon)\right)\mathbb{1}_{B_N}(n,1+\epsilon)\left(\prod_{j=1}^{k''} \log p_j\right), \tag{1.6}$$

and

$$H_N^p f_m(x) = \sum_{n\in\mathbb{Z}^{k'}} \sum_{(1+\epsilon)\in(\pm\mathbb{P})^{k''}} \sum_m f_m\left(x - \mathcal{P}(n,1+\epsilon)\right)K_m(n,1+\epsilon)\mathbb{1}_{B_N}(n,1+\epsilon)\left(\prod_{j=1}^{k''} \log|p_j|\right). \tag{1.7}$$

We now give some details about the method of the proof of Theorem C for the model dynamical system. To simplify the exposition we restrict attention to the averaging operators. We denote by $(\mathrm{m}_0)_N$ the discrete Fourier multiplier corresponding to M_N^p. To deal with $(2+\epsilon)$-variational estimates we apply the method recently used [13], see also [26]. Namely, given $\rho \in (0,1)$ we consider the set $\mathcal{D}_\rho = \{N_n: n \in \mathbb{N}\}$, where $N_n = \lfloor 2^{n^\rho} \rfloor$.

Then in view of (5.6) we can split the $(2+\epsilon)$-variation into two parts (see [27]): long variations and short variations, and study them separately. For each $0<\epsilon<\infty$ we can choose ρ so that the estimate for $\ell^{1+\epsilon}$-norm of short variations is straightforward. Next, to control long variations we adopt the partition of unity constructed in [11], that is

$$1=\sum_{s=0}^{n-1}\Xi_{n,s}^{\beta}+\left(1-\sum_{s=0}^{n-1}\Xi_{n,s}^{\beta}\right),$$

for some parameter $\beta\in\mathbb{N}_0$. Each projector $\Xi_{n,s}^{\beta}$ is supported by a finite union of disjoint cubes centered at rational points belonging to R_s^{β}. In this way, we distinguish the part of the multiplier where we can identify the asymptotic from the highly oscillating piece. The oscillating part is controlled by a multi-dimensional version of Weyl-Vinogradov's inequality with a logarithmic loss together with $\ell^{1+\epsilon}(Z^d)$ estimates for multipliers of Ionescu-Wainger type. By the triangle inequality, to control the first part it is enough to show

$$\left\|\sum_m V_{2+\epsilon}\left(\mathcal{F}^{-1}\left((\mathrm{m}_0)_{N_n}\Xi_{n,s}^{\beta}\hat{f}_m\right):n>s\right)\right\|_{\ell^{1+\epsilon}}\leq C_{1+\epsilon}(s+1)^{-2}\sum_m \|f_m\|_{\ell^{1+\epsilon}}. \quad (1.8)$$

First, by the circle method of Hardy and Littlewood, we find the asymptotic of the multiplier $(\mathrm{m}_0)_{N_n}$. Here we encounter the main difference from [11]. Namely, for ξ^m sufficiently close to the rational point $a^m/1+\epsilon$ we have

$$(\mathrm{m}_0)_{N_n}(\xi^m)=G\left(\frac{a^m}{1+\epsilon}\right)\Phi_{N_n}\left(\xi^m-\frac{a^m}{1+\epsilon}\right)+O\left(\exp\left(-(1+2\epsilon)\sqrt{\log N_n}\right)\right), \quad (1.9)$$

provided that $0\leq\epsilon\leq(\log N_n)^{\beta'}$, where $G(a^m/1+\epsilon)$ is the Gaussian sum and Φ_{N_n} is an integral version of $(\mathrm{m}_0)_{N_n}$.

The limitation on the size of the denominator is a consequence of the fact that for a larger $(1+\epsilon)$ the Siegel-Walfisz theorem has an additional term due to the possible exceptional zero of the exceptional quadratic character.
The second issue is the slower decay of the error term in (1.9). In particular, the later has its impact on the size of the cubes in the partition of unity. Both facts made the analysis of the approximating multipliers $v_{N_{m_0^n}}^s$ harder. To overcome this we directly work with $(m_0)_N$. Moreover, we get completely unified approach to the variational estimates for the averaging operators and the truncated discrete singular operators.
Going back to the sketch of the proof (see [27]), in order to show (1.8), we divide the variation into two parts: $s<n\leq 2^{\kappa_x}$ and $2^{\kappa_x}<n$, where $\kappa_s\simeq(s+1)^{\rho/10}$. For large scales $2^{\kappa_s}<n$, we transfer a priori estimates on $L^{1+\epsilon}$-norm for $(2+\epsilon)$-variation of the related continuous multipliers. Since the Gaussian sums satisfies $|G(a^m/1+\epsilon)|\lesssim(1+\epsilon)^{-\delta}$ for some $\delta>0$, we gain a decay $(s+1)^{-\delta\beta\rho}$ on ℓ^2. Consequently, by interpolation the $\ell^{1+\epsilon}$ norm of $(2+\epsilon)$-variation for large scales is bounded by $(s+1)^{-2}$ provided that β is sufficiently large. In the case of small scales $s<n\leq 2^{\kappa_s}$, the estimate on ℓ^2 is obtained with a help of the numerical inequality (2.3). We again show that ℓ^2 norm is bounded by $(s+1)^{-\delta\beta\rho+1}$. Because of the weaker asymptotic (1.9), to obtain $\ell^{1+\epsilon}$ bounds for $(2+\epsilon)$-variations over small scales required a new approach. We further divide the index set into dyadic blocks, then on each block we construct a good approximation to the multiplier giving bounds on $\ell^{1+\epsilon}$ norm independent of the block. At the cost of additional factor of κ_s^2, we control $\ell^{1+\epsilon}$ norm of $(2+\epsilon)$-variation. Again, by interpolation combined with a choice of β large enough we can make the $\ell^{1+\epsilon}$ norm bounded by $(s+1)^{-2}$.
We collect basic properties of the variational seminorm. We show how to deduce Theorem A from $(2+\epsilon)$-variational estimates (1.4) and (1.5). Then we present the lifting procedure, which allows us to replace any polynomial mapping $\mathcal{P}$ by a canonical one Q_m. We describe multipliers of Ionescu-Wainger type whose $\ell^{1+\epsilon}$ norm estimates are essential to our argument. We show a multi-dimensional version of Weyl-Vinogradov's inequality with a logarithmic loss. Moreover, we prove the estimate on the Gaussian sums of a mixed type. We devoted to study the asymptotic behavior of multipliers M_N and H_N, respectively. Finally, to get completely unified approach to the variational estimates for the averaging operators and truncated singular operators, at the beginning, we list the properties shared by them which are sufficient to prove Theorem $(1+\epsilon)$. We show the estimates on long and short variations.

We write $A \lesssim B (A \gtrsim B)$ if there is an absolute constant $\epsilon \geq 0$ such that $A \leq (1+\epsilon)B (A \geq (1+\epsilon)B)$. Moreover, $(1+\epsilon)$ stand for a large positive constant whose value may vary from occurrence to occurrence. If $A \lesssim B$ and $A \gtrsim B$ hold simultaneously then we write $A \simeq B$. Lastly, we write $A \lesssim_\delta B (A \gtrsim_\delta B)$ to indicate that the constant $(1+\epsilon)$ depends on some $\delta > 0$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a vector $x \in \mathbb{R}^d$, we set $|x|_\infty = \max\{|x_j|: 1 \leq j \leq d\}$. Given a subset $A \subseteq \mathbb{Z}$ and $x \in \mathbb{R}$, we set $A_x = A \cap [0, x]$.

2. Preliminaries

2.1. Variational norm. Let $0 \leq \epsilon < \infty$. For a sequence $(a_j^{\mathrm{m}}: j \in A), A \subseteq Z$, we define $(2+\epsilon)$-variational seminorm by

$$V_{2+\epsilon}(a_j^{\mathrm{m}}: j \in A) = \sup_{\substack{k_0 < \cdots < k_j \\ k_j \in A}} \left(\sum_{j=1}^{J} \sum_m |a_{kj}^{\mathrm{m}} - a_{kj-1}^{m}|^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}}.$$

The function $(2+\epsilon) \mapsto V_{2+\epsilon}(a_j^{\mathrm{m}}: j \in A)$ is non-decreasing, thus

$$V_{2+\epsilon}(a_j^{\mathrm{m}}: j \in A) \leq V_1(a_j^{\mathrm{m}}: j \in A),$$

and by Minkowski's inequality

$$V_{2+\epsilon}(a_j^{\mathrm{m}}: j \in A) \leq 2 \left(\sum_{j \in A} \sum_m |a_j^m|^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}}.$$

Moreover, for any $j_0 \in A$,

$$\sup_{j \in A} |a_j^m| \leq V_{2+\epsilon}(a_j^{\mathrm{m}}: j \in A) + |a_{j_0}^m|, \tag{2.1}$$

Finally, for any increasing sequence $((u_m)_k: 0 \leq k \leq K_m)$, we have

$$V_{2+\epsilon}(a_j^{\mathrm{m}}: (u_m)_0 \leq j \leq (u_m)_{K_m}) \leq K_m^{\frac{1+\epsilon}{2+\epsilon}} \left(\sum_{k=1}^{K_m} \sum_m V_{2+\epsilon}(a_j^{\mathrm{m}}: (u_m)_{k-1} \leq j \leq (u_m)_k)^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}}. \tag{2.2}$$

The following is essential in studying variational seminorms.

Lemma 1. [15, Lemma 1] If $\epsilon \geq 0$ then for any sequence $(a_j^{\mathrm{m}}: 0 \leq j \leq 2^s)$ of complex numbers

$$V_{2+\epsilon}(a_j^{\mathrm{m}}: 0 \leq j \leq 2^s) \leq \sqrt{2} \sum_{i=0}^{s} \left(\sum_{j=0}^{2^{s-i}-1} \sum_m |a_{(j+1)2^i}^{\mathrm{m}} - a_{j2^i}^{m}|^2 \right)^{\frac{1}{2}}. \tag{2.3}$$

2.2. Pointwise ergodic theorems. We show how to deduce the pointwise ergodic theorem (Theorem A) from a priori $(2+\epsilon)$-variational estimates for $\mathcal{M}_N^p$.

Proposition 2.1 (see [27]). Let $0 < \epsilon < \infty$. Suppose that there is $\epsilon \geq 0$ such that for all $f_m \in L^{1+\epsilon}(X, \mu)$,

$$\left\| \sum_m V_{2+\epsilon}(M_N^p f_m: N \in \mathbb{N}) \right\|_{\ell^{1+\epsilon}} \leq (1+\epsilon) \sum_m \| f_m \|_{\ell^{1+\epsilon}}. \tag{2.4}$$

Then there is $\epsilon \geq 0$ such that for all $f_m \in L^{1+\epsilon}(X, \mu)$,

$$\left\| \sup_{N \in \mathbb{N}} \sum_m |\mathcal{A}_N f_m| \right\|_{\ell^{1+\epsilon}} \leq (1+\epsilon) \sum_m \| f_m \|_{\ell^{1+\epsilon}},$$

and the averages $(\mathcal{A}_N f_m(x): N \in \mathbb{N})$ converges for μ-almost all $x \in X$.

Proof. Let us fix $N \in \mathbb{N}$. For each $m_0 \in \{1, \ldots, N\}$ and $s \in \{1, \ldots, k''\}$, we set

$$S_{N,m_0}^{(s)} f_m(x) = \sum_{n \in \mathbb{N}^{k'}} \sum_{\substack{(1+\epsilon) \in \mathbb{P}^{k''} \\ P_s \leq m_0}} \sum_m f_m \left(T_1^{\mathcal{P}_1(n,1+\epsilon)} \cdots T_{d_0}^{\mathcal{P}_{d_0}(n,1+\epsilon)} x \right) \mathbb{1}_{B_N}(n, 1+\epsilon) \left(\prod_{j=s+1}^{k^N} \log p_j \right),$$

and $S_{N,N}^{(0)} f_m = \vartheta_B(N). M_N^{\mathcal{P}} f_m$. For $0 \leq s < k''$, by the partial summation we obtain

$$\begin{aligned} S_{N,N}^{(s)} f_m &= \sum_{m=2}^{N} \sum_m \left(S_{N,m_0}^{(s+1)} f_m - S_{N,m_0-1}^{(s+1)} f_m \right) \log m_0 \\ &= \sum_m (\log N) S_{N,N}^{(s+1)} f_m + \sum_{m_0=2}^{N-1} \sum_m (\log m_0 - \log(m_0+1)) S_{N,m_0}^{(s+1)} f_m. \end{aligned}$$

Hence,

$$\left\|\sum_m S_{N,N}^{(s)} f_m - \sum_m (\log N) S_{N,N}^{(s+1)} f_m\right\|_{L^{1+\epsilon}} \le \sum_{m_0=2}^{N-1} \sum_m \|S_{N,m_0}^{(s+1)} f_m\|_{L^{1+\epsilon}} m_0^{-1}$$

$$\lesssim N^{k-1} (\log N)^{-s} \sum_{m_0=2}^{N-1} \sum_m \| f_m \|_{L^{1+\epsilon}} (\log m_0)^{-1} \quad (2.5)$$

$$\leqslant N^k (\log N)^{-(s+1)} \sum_m \| f_m \|_{L^{1+\epsilon}},$$

where we have used the trivial estimate

$$\left\|\sum_m S_{N,m_0}^{(s+1)} f_m\right\|_{L^{1+\epsilon}} \leqslant N^{k-1} (\log N)^{-s} m_0 (\log m_0)^{-1} \sum_m \| f_m \|_{L^{1+\epsilon}},$$

which is a consequence of (1.1) and the prime number theorem. Observe that

$$S_{N,N}^{(k^n)} f_m = \pi_B(N), A_N^{\mathcal{P}} f_m,$$

thus by repeated application of (2.5), we arrive at the conclusion that

$$\left\|\sum_m \vartheta_B(N) M_N^{p} f_m - \sum_m (\log N)^{k''} \pi_B(N) A_N^{p} f_m\right\|_{L^{1+\epsilon}}$$

$$\leqslant \vartheta_B(N) (\log N)^{-1} \sum_m \| f_m \|_{L^{1+\epsilon}}, \quad (2.6)$$

because the prime number theorem implies that $\vartheta_B(N) \simeq N^k$. In particular, by taking $f_m = \mathbb{1}_X$ and $\epsilon = \infty$ in (2.6) we get

$$\pi_B(N) = \vartheta_B(N) (\log N)^{-k\prime} \big(1 + O((\log N)^{-1})\big).$$

Hence, for any $0 \le \epsilon \le \infty$ and $f_m \in L^{1+\epsilon}(X, \mu)$,

$$\left\|\sum_m M_N^{\mathcal{P}} f_m - \sum_m \mathcal{A}_N^{\mathcal{P}} f_m\right\|_{L^{1+\epsilon}} \lesssim (\log N)^{-1} \sum_m \| f_m \|_{L^{1+\epsilon}}. \quad (2.7)$$

Next, if $\epsilon > 0$ then we can write

$$\left\|\sum_m \sup_{n\in\mathbb{N}} | \mathcal{A}_{2^n}^{\mathcal{P}} f_m|\right\|_{L^{1+\epsilon}} \le \left\|\sum_m \sup_{n\in\mathbb{N}} |M_{2^n}^{\mathcal{P}} f_m|_{L^{1+\epsilon}} + \sum_m \sup_{n\in\mathbb{N}} |M_{2^n}^{\mathcal{P}} f_m - \mathcal{A}_{2^n}^{\mathcal{P}} f_m|\right\|_{L^{1+\epsilon}}$$

$$\le \sum_m \left\|\sup_{n\in\mathbb{N}} |M_{2^n}^{\mathcal{P}} f_m|_{L^{1+\epsilon}} + \left(\sum_{n\in\mathbb{N}} n^{-(1+\epsilon)}\right)^{\frac{1}{1+\epsilon}}\right\| \sum_m \|f_m\|_{L^{1+\epsilon}}.$$

In view of (2.1), a priori estimate (2.4) entails that

$$\left\|\sum_m \sup_{N\in\mathbb{N}} | \mathcal{A}_{2^n}^{\mathcal{P}} f_m|\right\|_{L^{1+\epsilon}} \lesssim \sum_m \| f_m \|_{L^{1+\epsilon}}.$$

Hence, while proving μ-almost everywhere convergence of the averages $(\mathcal{A}_N f_m : N \in \mathbb{N})$ for $f_m \in L^{1+\epsilon}(X, \mu)$, we may assume that the function f_m is bounded. By (2.7), for $\epsilon = \infty$, we can write

$$\left|\sum_m \mathcal{M}_N^{p} f_m(x) - \sum_m \mathcal{A}_N^{p} f_m(x)\right| \le \sum_m \|\mathcal{M}_N^{p} f_m - \mathcal{A}_N^{p} f_m\|_{L^\infty} \lesssim (\log N)^{-1} \sum_m \| f_m \|_{L^\infty}.$$

Therefore, the convergence of $\big(M_N^{p} f_m(x) : N \in \mathbb{N}\big)$ implies the convergence of $\big(\sum_m \mathcal{A}_N^{p} f_m(x) : N \in \mathbb{N}\big)$ to the same limit.

Thanks to the Calderón's transference principle we can restrict attention to the model dynamical system, that is, Z^{d_0} with the counting measure and the shift operator. Hence, it suffices to study the operators (1.6) and (1.7) on $\ell^{1+\epsilon}(Z^{d_0})$.

2.3. Lifting lemma. For the polynomial mapping $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_{d_0})$, we define

$$\deg \mathcal{P} = \max\{\deg \mathcal{P}_i : 1 \le j \le d_0\}.$$

We use the set

$$\Gamma = \{\gamma \in Z^k \setminus \{0\}: 0 \le \gamma_j \le \deg \mathcal{P}, \text{ for each } j = 1, \dots, k\}$$

equipped with the lexicographic order. Then each $\mathcal{P}_j$ can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma\in\Gamma} \sum_m c_{j,\gamma}^m x^\gamma.$$

for some $c_{j,\gamma}^m \in \mathbb{Z}$. The cardinality of the set Γ is denoted by d. We identify $\mathbb{R}^d$ with $\mathbb{R}^\Gamma$. Let A be a diagonal $d \times d$ matrix such that for all $\gamma \in \Gamma$ and $v \in \mathbb{R}^d$,

$$(Av)_\gamma = |\gamma| v_\gamma. \tag{2.8}$$

For $t > 0$, we set

$$t^A v = (t^{|\gamma|} v_\gamma : \gamma \in \Gamma).$$

Finally, we introduce the canonical polynomial mapping,

$$Q_m = ((Q_m)_\gamma : \gamma \in \Gamma): \mathbb{R}^k \to \mathbb{R}^d,$$

by setting $(Q_m)_\gamma(x) = x^\gamma$. Now, if we define $L: \mathbb{R}^d \to \mathbb{R}^{d_0}$ to be the linear transformation such that for $v \in \mathbb{R}^d$,

$$(Lv)_j = \sum_{\gamma\in\Gamma} \sum_m c_{j,\gamma}^m v_\gamma,$$

then $LQ_m = \mathcal{P}$. The following lemma allows us to reduce the problems to studying the canonical polynomial mappings (see [27]).

Lemma 2. [12, Lemma 2.1] Let $R_N^{\mathcal{P}}$ be any of the operators $M_N^{\mathcal{P}}$ or $H_N^{\mathcal{P}}$. Suppose that for some $0 < \epsilon < \infty$,

$$\left\| \sum_m V_{2+\epsilon}(R_N^{Q_m} f_m : N \in \mathbb{N}) \right\|_{\ell^{1+\epsilon}(\mathbb{Z}^d)} \le C_{1+\epsilon,2+\epsilon} \sum_m \| f_m \|_{\ell^{1+\epsilon}(\mathbb{Z}^d)},$$

then

$$\left\| \sum_m V_{2+\epsilon}(R_N^{p} f_m : N \in \mathbb{N}) \right\|_{\ell^{1+\epsilon}(\mathbb{Z}^{d_0})} \le C_{1+\epsilon,2+\epsilon} \sum_m \| f_m \|_{\ell^{1+\epsilon}(\mathbb{Z}^{d_0})}.$$

In the rest of the article by M_N and H_N we denote the averaging and the truncated discrete singular operator for the canonical polynomial mapping Q_m, that is $M_N = M_N^{Q_m}$ and $H_N = H_N^{Q_m}$.

2.4. Ionescu-Wainger type multipliers. Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^d$, that is for any $f_m \in L^1(\mathbb{R}^d)$,

$$\mathcal{F} f_m(\xi^m) = \int_{\mathbb{R}^d} \sum_m f_m(x) e^{2\pi i \xi^m \cdot x} dx.$$

If $f_m \in \ell^1(Z^d)$, then we set

$$\hat{f_m}(\xi^m) = \sum_{x\in\mathbb{Z}^d} \sum_m f_m(x) e^{2\pi i \xi^m \cdot x}.$$

To simplify the notation, by $\mathcal{F}^{-1}$ we denote the inverse Fourier transform on $\mathbb{R}^d$ as well as the inverse Fourier transform on the d-dimensional torus identified with $(0,1]^d$. We also fix $\eta: \mathbb{R}^d \to \mathbb{R}$, a smooth function such that $0 \le \eta \le 1$, and

$$\eta(x) = \begin{cases} 1 & \text{if } |x|_\infty \le \dfrac{1}{32d}, \\ 0 & \text{if } |x|_\infty \ge \dfrac{1}{16d}. \end{cases}$$

We additionally assume that η is a convolution of two non-negative smooth functions with supports contained inside $\left[-\frac{1}{8d}, \frac{1}{8d}\right]^d$.

Next, we recall necessary notation to define auxiliary multipliers of Ionescu-Wainger type. For details see [12]. The following construction depends on a parameter $\beta \in \mathbb{N}$.

For $n \in \mathbb{N}$, we set $n_0 = \lfloor n^{1/20} \rfloor$ and $(Q_m)_0 = (n_0!)^D$ where $D = 20\beta + 1$. We define

$$\Pi = \bigcup_{k=1}^{D} \Pi_k,$$

wherein for $k \in \{1, \dots, D\}$ we have set

$$\Pi_k = \{p_1^{\gamma_1} \cdots p_k^{\gamma_k} : \gamma_j \in \mathbb{N}_D \text{ and } p_j \in \mathbb{P} \cap (n_0, n^\beta] \text{ are distint for all } 1 \le j \le k\}.$$

Let

$$(P_m)_n = \{Q_m \cdot w: Q_m \mid (Q_m)_0 \text{ and } w \in \Pi \cup \{1\}\}.$$

Notice that $\mathbb{N}_{n^\beta} \subseteq (P_m)_n \subseteq \mathbb{N}_{e^{n^{1/10}}}$. For $(1+\epsilon) \in \mathbb{N}$, we define

$$A_{1+\epsilon} = \{a^m \in \mathbb{N}_{1+\epsilon}: (a^m, 1+\epsilon) = 1\},$$

and

$$\mathbf{A}_{1+\epsilon} = \{a^m \in \mathbb{N}_{1+\epsilon}^k: \gcd(1+\epsilon, a_1^m, \dots, a_k^m) = 1\}.$$

Lastly, we set

$$\mathcal{U}_n^\beta = \{a^m/1+\epsilon: a^m \in \mathbf{A}_{1+\epsilon} \text{ and } (1+\epsilon) \in (P_m)_n\}. \tag{2.9}$$

Given $(\Theta_j: j \in \mathbb{Z})$ a sequence of multipliers on $\mathbb{R}^d$ such that for each $0 < \epsilon < \infty$ there is $A_{2+\epsilon} > 0$ such that for all $f_m \in L^2(\mathbb{R}^d) \cap L^{2+\epsilon}(\mathbb{R}^d)$,

$$\left\| \left(\sum_{j \in Z} \sum_m \left| \mathcal{F}^{-1}(\Theta_j \mathcal{F} f_m) \right|^2 \right)^{1/2} \right\|_{L^{2+\epsilon}} \le A_{2+\epsilon} \sum_m \| f_m \|_{L^{2+\epsilon}},$$

its discrete counterpart is given by the formula

$$\Theta_j^\beta(\xi^m) = \sum_{a^m/1+\epsilon \in U_n^\beta} \eta(\mathcal{E}_n^{-1}(\xi^m - a^m/1+\epsilon))\Theta_j(\xi^m - a^m/1+\epsilon),$$

where $\mathcal{E}_n$ being a diagonal $d \times d$ matrix with positive entries $(\epsilon_{n,\gamma}: \gamma \in \Gamma)$ such that $\epsilon_{n,\gamma} \le \exp(-n^{1/5})$. Then by [13, Theorem 2.1], for each $0 < \epsilon < \infty$ and any finitely supported function $f_m: \mathbb{Z}^d \to \mathbb{C}$,

$$\left\| \left(\sum_{j \in Z} \sum_m \left| \mathcal{F}^{-1}\left(\Theta_j^\beta \hat{f}_m\right) \right|^2 \right)^{1/2} \right\|_{\ell^{1+\epsilon}} \lesssim_{\beta, 1+\epsilon} \log(n+1) A_{2(2+\epsilon)} \sum_m \| f_m \|_{\ell^{1+\epsilon}}, \tag{2.10}$$

where $(2+\epsilon) = \max\left\{\left\lceil \frac{1+\epsilon}{2} \right\rceil, \left\lceil \frac{1+\epsilon}{2\epsilon} \right\rceil\right\}$. The scalar-valued version of (2.10) was proved in [7], see also [12]. The vector-valued extension was recently observed in [13]. Essentially its proof follows the same line as scalar valued except that in place of Marcinkiewicz-Zygmund inequality one uses Kahane's vector-valued extension of Khinchine's inequality, see [13, Theorem 2.1] for details.

3. Trigonometric Sums

3.1. Weyl-Vinogradov sum. We say that a subset of integers $\mathcal{A}$ is polynomially regular, if for all $\alpha, \alpha_1 > 0$, there are $\beta_0 > 0$ and a constant $\epsilon \ge 0$ so that for any integer $1 \le Q_m \le (\log N)^{\alpha_1}, \beta > \beta_0$ and any polynomial P_m of a form

$$P_m(x) = \frac{1+\epsilon}{1+2\epsilon} x^d + \cdots + \xi_1^m x,$$

for some coprime integers $(1+\epsilon)$ and $(1+2\epsilon)$, such that $\epsilon \ge -1$, and

$$(\log N)^\beta \le 1+2\epsilon \le N^d (\log N)^{-\beta},$$

we have

$$\left| \sum_{\substack{n \in A \\ n = 2+\epsilon \bmod Q_m}} \sum_m e^{2\pi i P_m(n)} \mathbb{1}_{[-N,N]}(n) \right| \le (1+\epsilon) \sum_m Q_m^{-1} N (\log N)^{-\alpha}, \tag{3.1}$$

for all $(2+\epsilon) \in \{1, \dots, Q_m\}$ and $N \in \mathbb{N}$.

We check that Z is polynomially regular. We write

$$\sum_{\substack{n \in Z \\ n = (2+\epsilon) \bmod Q_m}} \sum_m e^{2\pi i P_m(n)} \mathbb{1}_{[1,N]}(n) = \sum_{m_0=1}^{\left\lfloor \frac{N}{Q_m} \right\rfloor} \sum_m e^{2\pi i \hat{P}_m(m_0)} + O(Q_m), \tag{3.2}$$

where

$\hat{P}_m(m_0) = P_m(Q_m m_0 + 2 + \epsilon) = \frac{1+\epsilon}{1+2\epsilon} Q_m^d m_0^d$ + lower powers of m.

Set $M_m = \lfloor N/Q_m \rfloor$ and $(1+\epsilon)'/(1+2\epsilon)' = Q_m^d (1+\epsilon)/(1+2\epsilon)$ with $((1+\epsilon)', (1+2\epsilon)') = 1$. Then

$$(\log M_m)^{\beta - d\alpha_1} \le (1+2\epsilon) Q_m^{-d} \le (1+2\epsilon)' \le (1+2\epsilon) \le M_m^d (\log M_m)^{-\beta + d\alpha_1},$$

and hence, by Weyl estimates with logarithmic loss (see $e.g.$ [25, Remark after Theorem 1.5]),

$$\left| \sum_{m_0=1}^{M_m} \sum_m e^{2\pi i \hat{P_m}(m_0)} \right| \le (1+\epsilon) \sum_m M_m (\log M_m) \left(\frac{1}{(1+2\epsilon)'} + \frac{1}{M_m} + \frac{(1+2\epsilon)'}{M_m^d} \right)^{\frac{1}{2d^2 - 2d + 1}}.$$

Therefore, for $\beta > \beta_0 = (1+\alpha)(2d^2 - 2d + 1) + d\alpha_1$, by (3.2), we conclude that

$$\left| \sum_{\substack{n\in Z \\ n=(2+\epsilon)=m(1+\epsilon)dQ_m}} \sum_m e^{2\pi i P_m(n)} \mathbb{1}_{[1,N]}(n) \right| \le (1+\epsilon) \sum_m Q_m^{-1} N (\log N)^{-\alpha},$$

proving the claim. Another example of polynomially regular sets is the set of prime numbers. This is a consequence of [6, Theorem 10].

Our aim is to understand exponential sums over Cartesian products of polynomially regular sets. We fix a function $\phi_m: \mathbb{R}^k \to \mathbb{C}$ satisfying

$$|\phi_m(x)| \le 1+\epsilon,\ |\nabla\phi_m(x)| \le (1+|x|)^{-1}. \tag{3.3}$$

The main result is the following theorem.

Theorem 1 (see [27]). Let $\mathcal{A}_1, \dots \mathcal{A}_k$ be polynomially regular subsets of $\mathbb{Z}$. For all $\alpha > 0$ there are $\beta_0 > 0$ and a constant $\epsilon \ge 0$ so that for all $\beta > \beta_0$ and any polynomial P_m of a form

$$P_m(x) = \sum_{0<|y|\le d} \sum_m \xi_\gamma^m x^\gamma,$$

wherein for some $0 < |\gamma_0| \le d$,

$$\left| \xi_{\gamma_0}^m - \frac{1+\epsilon}{1+2\epsilon} \right| \le \frac{1}{(1+2\epsilon)^2},$$

for some coprime integers $(1+\epsilon)$ and $(1+2\epsilon)$ such that $\epsilon \ge -1$, and

$$(\log N)^\beta \le 1+2\epsilon \le N^{|\gamma_0|} (\log N)^{-\beta},$$

we have

$$\sup_{\substack{\Omega\subseteq[-N,N]^k \\ \Omega \text{ convex}}} \left| \sum_{n\in\mathcal{A}_1\times\dots\times\mathcal{A}_k} \sum_m e^{2\pi i P_m(n)} \mathbb{1}_\Omega(n) \phi_m(n) \right| \le (1+\epsilon) N^k (\log N)^{-\alpha}.$$

The constant $(1+\epsilon)$ depends on α, d and a constant in (3.3).

Proof. Let us first assume that $\phi_m \equiv 1$. The proof consists of three steps.
Step 1. We consider the case when $k = 1$ and $|\gamma_0| = d$. Take $\alpha > 0$ and $\alpha_1 > 0$, and let $\beta > \beta_0 = 3\beta_1 + 3d\alpha$, where β_1 is the value of β_0 determined by $\mathcal{A}_1$ for α and α_1. Suppose that $(1+\epsilon)$ and $(1+2\epsilon)$ are coprime integers such that $\epsilon \ge -1$, and

$$\left| \xi_d^m - \frac{1+\epsilon}{1+2\epsilon} \right| \le \frac{1}{(1+2\epsilon)^2},$$

with $(\log N)^\beta \le 1+2\epsilon \le N^d (\log N)^{-\beta}$. By Dirichlet's principle, there are coprime integers $(1+\epsilon)'$ and $(1+2\epsilon)'$ such that $1 \le (1+\epsilon)' \le (1+2\epsilon)' \le N^d (\log N)^{-\frac{1}{3}\beta}$, and

$$\left| \xi_d^m - \frac{(1+\epsilon)'}{(1+2\epsilon)'} \right| \le \frac{1}{(1+2\epsilon)'} N^{-d} (\log N)^{\frac{1}{3}\beta}.$$

If $(1+\epsilon)'/(1+2\epsilon)' \ne (1+\epsilon)/(1+2\epsilon)$ then

$$\frac{1}{(1+2\epsilon)(1+2\epsilon)'} \le \left| \frac{1+\epsilon}{1+2\epsilon} - \frac{(1+\epsilon)'}{(1+2\epsilon)'} \right| \le \left| \xi_d^m - \frac{1+\epsilon}{1+2\epsilon} \right| + \left| \xi_d^m - \frac{(1+\epsilon)'}{(1+2\epsilon)'} \right| \le \frac{1}{(1+\epsilon)^2} + N^{-d} (\log N)^{\frac{1}{3}\beta}.$$

Hence, we obtain

$$\frac{1}{(1+2\epsilon)'} \le \frac{1}{1+2\epsilon} + (1+2\epsilon) N^{-d} (\log N)^{\frac{1}{3}\beta} \le (\log N)^{-\beta} + (\log N)^{-\frac{2}{3}\beta}.$$

Thus

$$(\log N)^{\frac{1}{3}\beta} \le (1+2\epsilon)' \le N^d (\log N)^{-\frac{1}{3}\beta}.$$

Observe that the last estimate is also valid if $(1+2\epsilon)' = 1+2\epsilon$. Let Q_m be an integer such that $1 \le Q_m \le (\log N)^{\alpha_1}$. Given $(2+\epsilon) \in \{1, \dots, Q_m\}$, we set

$$S_{2+\epsilon,N} = \sum_{\substack{n\in A_1 \\ n=(2+\epsilon) \bmod Q_m}} \sum_m e^{2\pi i P_m(n)} \mathbb{1}_{[1,N]}(n), \text{ and } \tilde{S}_{2+\epsilon,N} = \sum_{\substack{n\in A_1 \\ n=(2+\epsilon) \bmod Q_m}} \sum_m e^{2\pi i \tilde{P_m}(n)} \mathbb{1}_{[1,N]}(n),$$

where

$$\tilde{P_m}(x) = \frac{(1+\epsilon)'}{(1+2\epsilon)'} x^d + \dots + \xi_1^m x.$$

We first show that

$$\sup_{1\le N'\le N}\left|\tilde{S}_{2+\epsilon,N'}\right| \le (1+\epsilon)\sum_{m} Q_m^{-1}N(\log N)^{-\alpha}, \tag{3.4}$$

for all $\beta > \beta_0$. If $1 \le N' \le N(\log N)^{-\alpha}$, then there is nothing to be proven. For $N(\log N)^{-\alpha} \le N' \le N$, we have

$$(\log N')^{\frac{1}{3}\beta} \le (1+2\epsilon)' \le N^d(\log N)^{-\frac{1}{3}\beta} \le (N')^d(\log N)^{d\alpha-\frac{1}{3}\beta} \le (N')^d(\log N')^{d\alpha-\frac{1}{3}\beta},$$

thus, by (3.1), we obtain

$$\left|\tilde{S}_{2+\epsilon,N'}\right| \le (1+\epsilon)\sum_{m} Q_m^{-1}N'(\log N')^{-\alpha} \le (1+\epsilon)'\sum_{m} Q_m^{-1}N(\log N)^{-u_m},$$

proving (3.4). We now set $\theta = \xi_d^{\mathrm{m}} - (1+\epsilon)'/(1+2\epsilon)'$ and apply the partial summation to get

$$\begin{aligned}\left|S_{2+\epsilon,N'}\right| &= \left|\sum_{n=1}^{N'} \left(\tilde{S}_{2+\epsilon,n} - \tilde{S}_{2+\epsilon,n-1}\right)e^{2\pi i\theta n^d}\right| \\ &\le \left|\tilde{S}_{2+\epsilon,N'}\right| + \left|\tilde{S}_{2+\epsilon,0}\right| + \sum_{n=1}^{N'-1}\left|\tilde{S}_{2+\epsilon,n}\right|\left|e^{2\pi i\theta n^d} - e^{2\pi i\theta(n+1)^d}\right|.\end{aligned}$$

Since

$$|\theta| \le \frac{1}{(1+2\epsilon)'}N^{-d}(\log N)^{\frac{1}{3}\beta} \lessdot N^{-d},$$

by (3.4), we obtain

$$\begin{aligned}\sup_{1\le N'\le N}\left|S_{2+\epsilon,N'}\right| &\lesssim \sum_{m} Q_m^{-1}N(\log N)^{-\alpha}\sum_{n=1}^{N-1}|\theta|n^{d-1} \\ &\lesssim \sum_{m} Q_m^{-1}N(\log N)^{-\alpha},\end{aligned}$$

which finishes the proof of Step 1.

Step 2. We next consider $k \ge 2$ and $\gamma_0 \ne (0,\dots,0,\ell,0,\dots,0)$ for any $\ell \le d$. Without loss of generality we may assume that $\gamma_0(1) \ge 1$. By the triangle inequality followed by Cauchy-Schwarz inequality we get

$$\begin{aligned}\left|\sum_{n\in\mathcal{A}_1\times\dots\times\mathcal{A}_k}\sum_{m} e^{2\pi iP_m(n)}\mathbb{1}_\Omega(n)\right| &\le \sum_{\tilde{n}\in\mathcal{A}_2\times\dots\times\mathcal{A}_k}\left|\sum_{n_1\in A_1}\sum_{m} e^{2\pi iP_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right| \\ &\lesssim N^{\frac{k-1}{2}}\left(\sum_{\tilde{n}\in\mathcal{A}_2\times\dots\times\mathcal{A}_k}\left|\sum_{n_1\in\mathcal{A}_1}\sum_{m} e^{2\pi iP_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5}$$

Next, we have

$$\begin{aligned}\sum_{\tilde{n}\in\mathcal{A}_2\times\dots\times\mathcal{A}_k}\left|\sum_{n_1\in\mathcal{A}_1}\sum_{m} e^{2\pi iP_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right|^2 &\le \sum_{\tilde{n}\in\mathbb{Z}^{k-1}}\left|\sum_{n_1\in\mathcal{A}_1}\sum_{m} e^{2\pi iP_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right|^2 \\ &\le \sum_{n_1,n_1'\in\mathcal{A}_1}\left|\sum_{\tilde{n}\in\mathbb{Z}^{k-1}}\sum_{m} e^{2\pi i\left(P_m(n_1;\tilde{n})-P_m(n_1';\tilde{n})\right)}\mathbb{1}_\Omega(n_1,\tilde{n})\mathbb{1}_\Omega(n_1',\tilde{n})\right| \\ &\le \sum_{n_1,n_1'\in\mathbb{Z}}\left|\sum_{\tilde{n}\in Z^{k-1}}\sum_{m} e^{2\pi i\left(P_m(n_1;\tilde{n})-P_m(n_1';\tilde{n})\right)}\mathbb{1}_\Omega(n_1,\tilde{n})\mathbb{1}_\Omega(n_1',\tilde{n})\right|,\end{aligned} \tag{3.6}$$

which, by another application of Cauchy-Schwarz inequality, is bounded by

$$N\left(\sum_{n_1,n_1'\in\mathbb{Z}}\left|\sum_{n\in Z^{k-1}}\sum_{m} e^{2\pi i\left(P_m(n_1,\tilde{n})-P_m(n_1';\tilde{n})\right)}\mathbb{1}_\Omega(n_1,\tilde{n})\mathbb{1}_\Omega(n_1',\tilde{n})\right|^2\right)^{\frac{1}{2}}.$$

Finally,

$$\sum_{n_1,n_1'\in\mathbb{Z}}\left|\sum_{n\in Z^{k-1}}\sum_{m} e^{2\pi i\left(P_m(n_1,\tilde{n})-P_m(n_1';\tilde{n})\right)}\mathbb{1}_\Omega(n_1,\tilde{n})\mathbb{1}_\Omega(n_1',\tilde{n})\right|^2$$

$$=\sum_{n_1,n_1'\in\mathbb{Z}}\sum_{n\in Z^{k-1}}\sum_{m} e^{2\pi i\left(P_m(n_1,\tilde{n})-P_m(n_1';\tilde{n})\right)}\mathbb{1}_\Omega(n_1,\tilde{n})\mathbb{1}_\Omega(n_1',\tilde{n}) \tag{3.7}$$

where

$$\Theta=\{(x_1,\tilde{x},x_1',\tilde{x}')\in\Omega\times\Omega:(x_1,\tilde{x}'),(x_1',\tilde{x})\in\Omega\},$$

and

$$Q_m(x_1,\tilde{x},x_1',\tilde{x}')=P_m(x_1;\tilde{x})-P_m(x_1';\tilde{x})-P_m(x_1;\tilde{x}')+P_m(x_1';\tilde{x}').$$

Notice that the set Θ is a convex subset of a cube $[-N,N]^{2k}$. Moreover, the polynomial $Q_m(x,x')$ has degree at least $|\gamma_0|$ having a coefficient $\xi_{\gamma_0}^m$ in front of the monomial x^{γ_0}. Therefore, by [12, Theorem 3.1], there are $\beta_0>0$ and $\epsilon\geq 0$ such that

$$\left|\sum_{n_1,n_1'\in Z}\sum_{\tilde{n},\tilde{n}'\in Z^{k-1}}\sum_{m} e^{2\pi i Q_m(n_1,\tilde{n},n_1',\tilde{n}')}\mathbb{1}_\Theta(n_1,\tilde{n},n_1',\tilde{n}')\right|\leq(1+\epsilon)N^{2k}(\log N)^{-4\alpha},$$

provided that $\beta>\beta_0$. Hence, by (3.5), (3.6) and (3.7) we obtain

$$\left|\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k}\sum_{m} e^{2\pi i P_m(n)}\mathbb{1}_\Omega(n)\right|\leq N^k(\log N)^{-\alpha}.$$

Step 3. Suppose that $k\geq 1$ and $\gamma_0=(0,\ldots,0,\ell,0,\ldots,0)$ for $1\leq\ell\leq d$. Without loss of generality we may assume that $\gamma_0=(\ell,\ldots,0)$. The proof is by a backward induction over $\ell\in\{1,\ldots,d\}$. We write

$$\left|\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k}\sum_{m} e^{2\pi i P_m(n)}\mathbb{1}_\Omega(n)\right|\leq\sum_{\tilde{n}\in\mathcal{A}_2\times\ldots\times\mathcal{A}_k}\left|\sum_{n_1\in\mathcal{A}_1}\sum_{m} e^{2\pi i P_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right|. \tag{3.8}$$

If $\ell=d$ the conclusion follows by Step 1. Suppose that $\ell<d$. In view of Step 2 and the inductive hypothesis, the estimate holds for any $|\gamma_0|=j,\ell<j\leq d$. Let β_1 be the largest value of β_0 among those that were determined in Step 2 and resulting from the inductive hypothesis. By Dirichlet's principle, for each $\ell<|\gamma|\leq d$, we select coprime integers $(1+\epsilon)_\gamma$, such that $1\leq(1+\epsilon)_\gamma\leq(1+2\epsilon)_\gamma\leq N^{|\gamma|}(\log N)^{-\beta_1}$, satisfying

$$\left|\xi_\gamma^{\mathrm{m}}-\frac{(1+\epsilon)_\gamma}{(1+2\epsilon)_\gamma}\right|\leq\frac{1}{(1+2\epsilon)_\gamma}N^{-|\gamma|}(\log N)^{\beta_1}.$$

If for some $\gamma\in\Gamma,\ell<|\gamma|\leq d$ we have $(\log N)^{\beta_1}\leq(1+2\epsilon)_\gamma$, then the conclusion follows by the inductive hypothesis or Step 2. Otherwise, we set $\theta_\gamma=\xi_\gamma^{\mathrm{m}}-(1+\epsilon)_\gamma/(1+2\epsilon)_\gamma$ and $Q_m=\operatorname{lcm}\{(1+2\epsilon)_\gamma:\ell<|\gamma|\leq d\}$. We have

$$|\theta_\gamma|\leq\frac{1}{(1+2\epsilon)_\gamma}N^{-|\gamma|}(\log N)^{\beta_1}, \tag{3.9}$$

and

$$Q_m\leq(\log N)^{\alpha_1},$$

where $\alpha_1=\beta_1\cdot\#\{\gamma\in\mathbb{N}_0^k:\ell<|\gamma|\leq d\}$. We have

$$\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k}\left|\sum_{n_1\in\mathcal{A}_1}\sum_{m} e^{2\pi i P_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right|\leq\sum_{\tilde{n}\in Z^{k-1}}\left|\sum_{n_1\in\mathcal{A}_1}\sum_{m} e^{2\pi i P_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right|$$

$$\leq\sum_{(r_1,r')\in N_{Q_m}^k}\sum_{m}\sum_{\substack{\tilde{n}\in Z^{k-1}\\ \tilde{n}\equiv\tilde{r}\bmod Q_m}}\left|\sum_{\substack{n_1\in\mathcal{A}_1\\ \tilde{n}\equiv\tilde{r}\bmod Q_m}} e^{2\pi i P_m(n_1;\tilde{n})}\mathbb{1}_\Omega(n_1,\tilde{n})\right| \tag{3.10}$$

Setting

$$(P_m)_0(x)=\sum_{0<|\gamma|\leq\ell}\sum_{m}\xi_\gamma^m x^\gamma,$$

we can write

$$P_m(Q_m m_0+2+\epsilon) \equiv \sum_{\ell<|\gamma|\le d}\sum_m \xi_\gamma^{m}(Q_m m_0+2+\epsilon)^\gamma+\sum_m (P_m)_0(Q_m m_0+2+\epsilon)\ (\mathrm{mod}1)$$
$$\equiv \sum_{\ell<|\gamma|\le d}\sum_m \frac{(1+\epsilon)_\gamma}{(1+2\epsilon)_\gamma}(2+\epsilon)^\gamma+\sum_{\ell<|\gamma|\le d}\sum_m \theta_\gamma(Q_m m_0+2+\epsilon)^\gamma+\sum_m (P_m)_0(Q_m m_0+2+\epsilon)\ (\mathrm{mod}1).$$

Thus

$$\sum_{(r_1,\tilde r)\in N_{Q_m}^k}\sum_m \sum_{\substack{\tilde n\in Z^{k-1}\\ \tilde n\equiv\tilde r \bmod Q_m}}\left|\sum_{\substack{n_1\in\mathcal{A}_1\\ n_1\equiv r_1 \bmod Q_m}} e^{2\pi i P_m(n_1;\tilde n)}\mathbb{1}_\Omega(n_1,\tilde n)\right|$$
$$=\sum_{(r_1,\tilde r)\in N_{Q_m}^k}\sum_m \sum_{\substack{\tilde n\in Z^{k-1}\\ \tilde n\equiv\tilde r \bmod Q_m}}\left|\sum_{n_1\in Z} A_{n_1;\tilde n}(S_{n_1;\tilde n}^{(2+\epsilon)}-S_{n_1-1;\tilde n}^{(2+\epsilon)}\right| \qquad (3.11)$$

where

$$A_{n_1,\tilde n}=e^{2\pi i\Sigma_{\ell<|\gamma|\le d}\theta_\gamma(n_1,\tilde n)^\gamma},$$

and

$$S_{n_1,\tilde n}^{(2+\epsilon)}=\sum_{\substack{n_1'\le n_1\\ n_1'=r_1 \bmod Q_m}}\sum_m e^{2\pi i(P_m)_0(n_1';\tilde n)}\mathbb{1}_\Omega(n_1',\tilde n)\mathbb{1}_{\mathcal{A}_1}(n_1').$$

To estimate the inner sum on the right-hand side of (3.11), we apply the partial summation. Setting

$$((M_m)_0,(M_m)_0+1,\dots,(M_m)_1)=\{n_1\in\mathbb{Z}:(n_1,\tilde n)\in\Omega\},$$

we can write

$$\left|\sum_{n_1=(M_m)_0}^{(M_m)_1} A_{n_1,\tilde n}\left(S_{n_1,\tilde n}^{(2+\epsilon)}-S_{n_1-1,\tilde n}^{(2+\epsilon)}\right)\right|\le\left|S_{(M_m)_1,\tilde n}^{(2+\epsilon)}\right|+\sum_{n_1=(M_m)_0}^{(M_m)_1-1}\left|S_{n_1,\tilde n}^{(2+\epsilon)}\right|\cdot\left|A_{n_1,\tilde n}-A_{n_1+1,\tilde n}\right|.$$

By(3.9), for $(n_1,\tilde n)\in\Omega$ we have

$$\left|A_{n_1,\tilde n}-A_{n_1+1,\tilde n}\right|\lesssim\sum_{\ell<|\gamma|\le d}|\theta_\gamma|N^{|\gamma|-1}\lesssim N^{-1}(\log N)^{\beta_1}.$$

Recall that $\gamma_0=(\ell,0,\dots,0)$ and

$$\left|\xi_{\gamma_0}^{m}-\frac{1+\epsilon}{1+2\epsilon}\right|\le\frac{1}{(1+2\epsilon)^2},$$

thus, by Step 1 applied to $S_{n_1,\tilde n}^{(2+\epsilon)}$ we obtain

$$\sup_{(M_m)_0\le n_1\le(M_m)_1}\left|S_{n_1,\tilde n}^{(2+\epsilon)}\right|\lesssim\sum_m NQ_m^{-1}(\log N)^{-\alpha-\beta_1},$$

whenever $\beta>\beta_2$, where β_2 is the value of β_0 determined in Step 1 for $\alpha+\beta_1$ and α_1. Hence,

$$\left|\sum_{n_1=(M_m)_0}^{(M_m)_1} A_{n_1,\tilde n}\left(S_{n_1,\tilde n}^{(2+\epsilon)}-S_{n_1-1,\tilde n}^{(2+\epsilon)}\right)\right|\lesssim\sum_m Q_m^{-1}N(\log N)^{-\alpha}.$$

Consequently, by (3.8), (3.10) and (3.11) we get

$$\left|\sum_{n\in\mathcal{F}_1\times\ldots\times\mathcal{A}_k}\sum_m e^{2\pi i P_m(n)}\mathbb{1}_\Omega(n)\right|\le N^k(\log N)^{-\alpha},$$

provided that $\beta>\beta_0=\max\{\beta_1,\beta_2\}$.

Finally, we deal with a general ϕ_m. Given α, let β_0 be such that

$$\sup_{\substack{\Omega\subseteq[-N.N]^k\\ \Omega\text{ convex}}}\left|\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k}\sum_m e^{2\pi i P_m(n)}\mathbb{1}_\Omega(n)\right|\le(1+\epsilon)N^k(\log N)^{-(k+1)\alpha-k}. \qquad (3.12)$$

We divide the cube $[-N,N]^k$ into J closed cubes $((Q_m)_j:1\le j\le J)$ with sides parallel to the axes and having side lengths $O(N(\log N)^{-\alpha-1})$. Thus

$$J=O\left((\log N)^{k(\alpha+1)}\right). \qquad (3.13)$$

By $(Q_m)_j^{\circ}$ we denote the interior of $(Q_m)_j$. We assume that $(Q_m)_j^{\circ}$ are disjoint with the axes. Let n_j be the vertex of $(Q_m)_i$ at the largest distance to the origin. Then by the mean value theorem and (3.3), we have

$$\left|\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k}\sum_{m} e^{2\pi i P_m(n)}\mathbb{1}_{Q_j^\circ\cap\Omega}(n)\left(\phi_m(n)-\phi_m(n_j)\right)\right| \lesssim \sum_{n\in(Q_m)_j}\sum_{m}\sup_{t\in[0.1]}\left|\nabla\phi_m(tn+(1-t)n_j)\right|\cdot|n-n_j|$$
$$\lesssim N(\log N)^{-\alpha-1}\sum_{n\in(Q_m)_j}\sum_{m}(1+|n|)^{-1},$$

thus

$$\left|\sum_{j=1}^{J}\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k}\sum_{m} e^{2\pi i P_m(n)}\mathbb{1}_{(Q_m)_j^\circ\cap\Omega}(n)\left(\phi_m(n)-\phi_m(n_j)\right)\right| \lesssim N^k(\log N)^{-\alpha}. \tag{3.14}$$

On the other hand, in view of (3.12), we get

$$\left|\sum_{m}\phi_m(n_j)\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k} e^{2\pi i P_m(n)}\mathbb{1}_{(Q_m)_j^\circ\cap\Omega}(n)\right| \lesssim N^k(\log N)^{-(k+1)\alpha-k},$$

hence, by (3.13),

$$\left|\sum_{j=1}^{J}\sum_{m}\phi_m(n_j)\sum_{n\in\mathcal{A}_1\times\ldots\times\mathcal{A}_k} e^{2\pi i P_m(n)}\mathbb{1}_{(Q_m)_j^\circ\cap\Omega}(n)\right| \lesssim N^k(\log N)^{-\alpha},$$

which together with (3.14) completes the proof.

We next apply Theorem 1 to get the following variant of Weyl-Vinogradov's inequality.

Theorem 2 (see [27]). Let $\xi^m\in\mathbb{T}^d$. Assume that there is a multi-index $\gamma_0\in\Gamma$, such that

$$\left|\xi^m_{\gamma_0}-\frac{1+\epsilon}{1+2\epsilon}\right|\le\frac{1}{(1+2\epsilon)^2},$$

for some coprime integers $(1+\epsilon)$ and $(1+2\epsilon)$ such that $\epsilon\ge 0$. Then for all $\alpha>0$, there is $\beta_\alpha>0$, so that for any $\beta>\beta_\alpha$, if

$$(\log N)^\beta\le 1+2\epsilon\le N^{|\gamma_0|}(\log N)^{-\beta},$$

then

$$\sup_{\substack{\Omega\subseteq[-N,N]^k\\ \Omega\text{ convex}}}\left|\sum_{n\in\mathbb{N}^{k'}}\sum_{(1+\epsilon)\in\mathbb{P}^{k''}}\sum_{m} e^{2\pi i\xi^m\cdot Q_m(n,1+\epsilon)}\mathbb{1}_\Omega(n,1+\epsilon)\phi_m(n,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)\right|\le(1+\epsilon)N^k(\log N)^{-\alpha}.$$

The constant $(1+\epsilon)$ depends on α, d and a constant in (3.3).

Proof. We claim that the following holds true.

Claim 1. For all $\alpha>0$, there is $\beta_\alpha>0$, such that for all $\beta>\beta_\alpha, N\in\mathbb{N}$, and $(2+\epsilon)\in\{0,\ldots,k''\}$, if there is a multi-index $\gamma_0\in\Gamma$, such that

$$\left|\xi^m_{\gamma_0}-\frac{1+\epsilon}{1+2\epsilon}\right|\le\frac{1}{(1+2\epsilon)^2},$$

for some coprime integers $(1+\epsilon)$ and $(1+2\epsilon)$, such that $\epsilon\ge 0$, and

$$(\log N)^\beta\le 1+\epsilon\le N^{|\gamma_0|}(\log N)^{-\beta},$$

then

$$\sup_{\substack{\Omega\subseteq[-N,N]^k\\ \Omega\text{ convex}}}\left|\sum_{\Omega\in\mathbb{N},\mathbb{N}^{k'}}\sum_{(1+\epsilon)\in\mathbb{P}^{k''}}\sum_{m} e^{2\pi i\xi^m\cdot Q_m(n,1+\epsilon)}\mathbb{1}_\Omega(n,1+\epsilon)\phi_m(n,1+\epsilon)\left(\prod_{j=3+\epsilon}^{k''}\log p_j\right)\right|$$
$$\le(1+\epsilon)N^k(\log N)^{-\alpha+k''-(2+\epsilon)}.$$

The proof is by a backward induction over $(2+\epsilon)$. For $(2+\epsilon)=k''$ the assertion follows by Theorem 1 . For $(2+\epsilon)\in\{1,\ldots,k''\}, N\in\mathbb{N}$ and $m_0\in\{1,\ldots,N\}$, we set

$$S_{N,m_0}^{(2+\epsilon)}(\xi^m)=\sum_{n\in\mathbb{N}^{k'}}\sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''}\\ p_{2+\epsilon}\le m_0}}\sum_{m} e^{2\pi i\xi^m\cdot Q_m(n,1+\epsilon)}\mathbb{1}_\Omega(n,1+\epsilon)\phi_m(n,1+\epsilon)\left(\prod_{j=3+\epsilon}^{k''}\log p_j\right),$$

and

$$S_{N,N}^{(0)}(\xi^m) = \sum_{n\in\mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathrm{P}^{k''}} \sum_{m} e^{2\pi i\xi^m\cdot Q_m(n,1+\epsilon)}\mathbb{1}_\Omega(n,1+\epsilon)\phi_m(n,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right),$$

where Ω is a convex subset of $[-N,N]^k$. For $0 \le 2+\epsilon < k''$, by the partial summation, we can write

$$\begin{aligned} S_{N,N}^{(2+\epsilon)} &= \sum_{m_0=1}^{N}\left(S_{N,m_0}^{(3+\epsilon)} - S_{N,m_0-1}^{(3+\epsilon)}\right)\log m_0 \\ &= S_{N,N}^{(3+\epsilon)}(\log N) + \sum_{m_0=1}^{N-1} S_{N,m_0}^{(3+\epsilon)}(\log(m_0)-\log(m_0+1)). \end{aligned}$$

Hence, by the inductive hypothesis we get

$$\begin{aligned} \left|S_{N,N}^{(2+\epsilon)}\right| &\le \left|S_{N,N}^{(3+\epsilon)}\right|(\log N) + \sum_{m_0=1}^{N-1}\left|S_{N,m_0}^{(3+\epsilon)}\right| m_0^{-1} \\ &\le (1+\epsilon)' N^k(\log N)^{-\alpha+k''-(2+\epsilon)}, \end{aligned}$$

proving the claim. Now, the theorem follows by Claim 1 for $\epsilon = -2$.

3.2. Gaussian sums. Given $(1+2\epsilon)\in\mathbb{N}$ and $(1+\epsilon)\in\mathbf{A}_{1+2\epsilon}$, the Gaussian sum is

$$G\left(\frac{1+\epsilon}{1+2\epsilon}\right) = \frac{1}{(1+2\epsilon)^{k'}}\cdot\frac{1}{\varphi_m(1+2\epsilon)^{k''}}\sum_{x\in\mathbb{N}_{1+2\epsilon}^{k'}}\sum_{y\in A_{1+2\epsilon}^{k''}} e^{2\pi i\left(\frac{1+\epsilon}{1+2\epsilon}\right)\cdot Q_m(x,y)},$$

where φ_m is Euler's totient function, $i.e$ $\varphi_m(1+2\epsilon)$ equals to the number of elements in $A_{1+2\epsilon}$. The following theorem provides a very useful estimate on the Gaussian sums.

Theorem 3 (see [27]). There are $\epsilon \ge 0$ and $\delta > 0$ such that for all $(1+2\epsilon)\in\mathbb{N}$ and $(1+\epsilon)\in\mathbf{A}_{1+2\epsilon}$,

$$|G(1+\epsilon/1+2\epsilon)| \le (1+\epsilon)(1+2\epsilon)^{-\delta}.$$

Proof. Let us recall that for $1+\epsilon, 1+2\epsilon\in\mathbb{N}$, (see $e.g.$ [17, Theorem 4.1])

$$\frac{1}{\varphi_m(1+2\epsilon)}\sum_{x\in A_{1+2\epsilon}} e^{2\pi i(1+\epsilon)x/1+2\epsilon} = \frac{\mu(1+2\epsilon/\gcd(1+\epsilon,1+2\epsilon))}{\varphi_m(1+2\epsilon/\gcd(1+\epsilon,1+2\epsilon))}, \tag{3.15}$$

wherein $\mu(1+2\epsilon)$ is Möbius function defined for $(1+2\epsilon) = p_1^{j_1}\cdots p_{m_0}^{j_{m_0}}, p_j$ are distinct prime numbers, as

$$\mu(1+2\epsilon) = \begin{cases} (-1)^{m_0} & \text{if } j_1 = \cdots = j_{m_0} = 1, \\ 0 & \text{otherwise}. \end{cases}$$

For each $\epsilon > 0$ there is $C_\epsilon > 0$, such that (see $e.g.$ [17, Theorem 2.9])

$$\varphi_m(1+2\epsilon) \ge C_\epsilon(1+2\epsilon)^{1-\epsilon}. \tag{3.16}$$

We start the proof of the theorem by considering $d = 1$. Then

$$G\left(\frac{1+\epsilon}{1+2\epsilon}\right) = \prod_{\gamma=(\gamma',0)\in\Gamma}\left(\frac{1}{1+2\epsilon}\sum_{x=1}^{1+\epsilon} e^{\frac{2\pi i(1+\epsilon)_\gamma x}{1+2\epsilon}}\right)\prod_{\gamma=(0,\gamma'')\in\Gamma}\left(\frac{1}{\varphi_m(1+2\epsilon)}\sum_{x\in A_{1+2\epsilon}} e^{\frac{2\pi i(1+\epsilon)_\gamma x}{1+2\epsilon}}\right).$$

Suppose that $k' \ge 1$. If $G((1+\epsilon)/(1+2\epsilon)) \ne 0$ then $1+2\epsilon \mid (1+\epsilon)_\gamma$ for all $\gamma = (\gamma',0)\in\Gamma$. Since $(1+\epsilon)\in\mathbf{A}_{1+2\epsilon}$, we must have $k'' \ge 1$.

For $\gamma = (0,\gamma'')\in\Gamma$, we set $b_\gamma/(1+2\epsilon)_\gamma = (1+\epsilon)_\gamma/(1+2\epsilon)$, where $\left(b_\gamma,(1+\epsilon)_\gamma\right) = 1$. By (3.15), $G(1+\epsilon/1+2\epsilon) \ne 0$ entails that each $(1+\epsilon)_\gamma$ is square-free. Since for any $(1+\epsilon)$ prime factor $(1+2\epsilon)$ there is $\gamma = (0,\gamma'')\in\Gamma$ such that $(1+\epsilon)\nmid(1+2\epsilon)/(1+2\epsilon)_\gamma$, we conclude that $(1+2\epsilon)$ is square-free. Because $(1+2\epsilon) = \mathrm{lcm}\left((1+\epsilon)_\gamma : \gamma = (0,\gamma'')\in\Gamma\right)$,

$$\left|G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\right| \le \sum_m \prod_{\gamma=(0,\gamma'')\in\Gamma}\frac{1}{\varphi_m\left((1+2\epsilon)_\gamma\right)} \le \frac{1}{\varphi_m(1+2\epsilon)},$$

which together with (3.16) gives

$$\left|G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\right| \le C_\epsilon(1+2\epsilon)^{\epsilon-1}.$$

Next, let us consider the case $d \ge 2$. For a given polynomial P_m on $\mathbb{R}^k$ with integral coefficients we define

$$S(1+2\epsilon,P_m) = \sum_{x\in\mathbb{N}_{1+2\epsilon}^{k'}}\sum_{y\in A_{1+2\epsilon}^{k''}}\sum_m \exp(2\pi i P_m(x,y)/1+2\epsilon).$$

Let

$$P_m(x) = \sum_{0<|\gamma|\le d} (1+\epsilon)_\gamma x^\gamma,$$

where $(1+\epsilon) \in \mathbf{A}_{1+2\epsilon}$. Our aim is to show that there are $\epsilon \ge 0$ and $\delta > 0$ such that for all $(1+2\epsilon) \in \mathbb{N}$ and $(1+\epsilon) \in \mathbf{A}_{1+2\epsilon}$,

$$|S(1+2\epsilon, P_m)| \le (1+\epsilon)(1+2\epsilon)^{k-\delta}. \tag{3.17}$$

First, observe that for $(1+2\epsilon) = q_1 q_2, (q_1, q_2) = 1$, we have

$$S(1+2\epsilon, P_m) = S\big(q_1, q_2^{-1}P_m(q_2\,\cdot)\big)S\big(q_2, q_1^{-1}P_m(q_1\,\cdot)\big).$$

Therefore, if $(1+2\epsilon) = p_1^{j_1} \cdots p_m^{j_{m_0}}$ for some distinct prime numbers p_j, then

$$S(1+2\epsilon, P_m) = \prod_{s=1}^{m_0} S\big(p_s^{j_s}, (P_m)_s\big),$$

where

$$(P_m)_s(x) = \frac{p_s^{j_s}}{1+2\epsilon} P_m\left(\frac{1+2\epsilon}{p_s^{j_s}} x\right).$$

Since $\omega(1+2\epsilon)$, the number of distinct prime factors of $(1+2\epsilon)$, satisfies (see $e.g.$ [17, Theorem 2.10])

$$\omega(1+2\epsilon) \le (1+\epsilon)\frac{\log(1+2\epsilon)}{\log\log(1+2\epsilon)},$$

we have

$$2^{\omega(1+2\epsilon)} \le C'_\epsilon (1+2\epsilon)^\epsilon.$$

Hence, it is enough to proof (3.17) for $(1+2\epsilon) = p^j$ with $(1+\epsilon)$ being a prime number and $j \ge 1$. Since for any arithmetic function F_m, we have

$$\sum_{x\in A_{p^j}} \sum_m F_m(x) = \sum_{x\in \mathbb{N}_{p^j}} \sum_m F_m(x) - \sum_{x\in\mathbb{N}_{p^{j-1}}} \sum_m F_m((1+\epsilon)x),$$

if $j \ge 2$ we write

$$S\big(p^j, P_m\big) = \sum_{\sigma\in\{0,1\}^{k''}} (-1)^\sigma \sum_{(x',x'')\in\Omega^\sigma} \sum_m \exp\big(2\pi i P_m(x', p^\sigma x'')/p^j\big),$$

where for $\sigma \in \{0,1\}^{k''}$, we have set

$$\Omega^\sigma = \mathbb{N}_{p^j}^{k'} \times \mathbb{N}_{p^{j-\sigma_1}} \times \dots \times \mathbb{N}_{p^{j-\sigma_{k''}}}.$$

Fix $\sigma \in \{0,1\}^{k''}$. For each $\gamma \in \Gamma$, we define

$$\frac{b_\gamma}{(1+2\epsilon)_\gamma} = \frac{(1+\epsilon)_\gamma p^\sigma \gamma''}{p^j},$$

where $\big(b_\gamma, (1+2\epsilon)_\gamma\big) = 1$. Let

$$(1+2\epsilon) = \operatorname{lcm}\big((1+2\epsilon)_\gamma : \gamma \in \Gamma, |\gamma| \ge 2\big), \text{ and } Q_m = \operatorname{lcm}\big((1+2\epsilon)_\gamma : \gamma \in \Gamma\big).$$

Observe that

$$\sum_{(x',x'')\in\Omega^\sigma} \sum_m \exp\left(\frac{2\pi i P_m(x', p^\sigma x'')}{p^j}\right) \ne 0 \tag{3.18}$$

entails that $(1+2\epsilon) = Q_m$. To obtain a contradiction, suppose that $(1+2\epsilon) < Q_m$. Let $\gamma_0 \in \Gamma, |\gamma_0| = 1$ be such that $(1+2\epsilon)_{\gamma_0} = Q_m$.

Thus $(1+2\epsilon) \mid p^{j-\sigma_1}$. For any $(2+\epsilon) \in \mathbb{N}_{1+2\epsilon}^k$ we can wright

$$\sum_{\substack{(x',x'')\in\Omega^\sigma \\ x\equiv(2+\epsilon) \bmod (1+2\epsilon)}} \exp\big(2\pi i P_m(x', p^\sigma x'')/p^j\big)$$

$$= \exp\big(2\pi i \tilde{P}_m(x', p^\sigma x'')/p^j\big) \sum_{\substack{x\in\Omega^\sigma \\ x\equiv(2+\epsilon) \bmod (1+2\epsilon)}} \prod_{\substack{\gamma\in\Gamma \\ |\gamma|=1}} \exp\big(2\pi i b_\gamma x^\gamma/(1+2\epsilon)_\gamma\big),$$

where

$$\tilde{P}_m(x) = \sum_{\substack{\gamma\in\Gamma \\ |\nu|\ge 2}} (1+\epsilon)_\gamma x^\gamma.$$

Thus (3.18) implies that $(1+2\epsilon)_{\gamma_0} \mid b_{\gamma_0}(1+2\epsilon)$, which is impossible. Hence, $(1+2\epsilon) = Q_m$.
Now, let $\gamma_0 \in \Gamma, |\gamma_0| \geq 2$, be such that $(1+2\epsilon)_{\gamma_0} = Q_m$. Then

$$(1+2\epsilon)_{\gamma_0} = Q_m = (1+2\epsilon) \leq \prod_{\substack{\gamma\in\mathbb{I}\\ |\gamma|\geq 2}} (1+2\epsilon)_\gamma \leq (1+2\epsilon)_{\gamma_0}^d,$$

and thus

$$Q_{\mathrm{m}}^{\frac{1}{d}} \leq (1+2\epsilon)_{\gamma_0} = Q_m < Q_m^{|\gamma_0|-\frac{1}{d}}. \tag{3.19}$$

Suppose that $Q_m < p^j$. Since $(1+\epsilon) \in \mathbf{A}_{1+2\epsilon}$, we must have $\sigma \neq 0$. Then for $j \leq D = \max\{|\gamma''|: \gamma \in \Gamma\}$, by a trivial estimate we have

$$\left|\sum_{(x',x'')\in\Omega^\sigma} \sum_m \exp\left(2\pi i P_m(x', p^\sigma x'')/p^j\right)\right| \leq p^{kj-|\sigma|} \leq p^{kj(1-\delta_1)},$$

provided $0 < \delta_1 < (kD)^{-1}$. Since $Q_m \geq p^{j-D}$, for $j \geq D+1$ we have

$$Q_{\mathrm{m}}^{\frac{1}{d}} \geq p^{j\epsilon},$$

whenever $0 < \epsilon < (d(D+1))^{-1}$. Hence, by (3.19),

$$p^{j\epsilon} \leq (1+2\epsilon)_{\gamma_0} \leq p^{j(|\gamma_0|-\epsilon)}.$$

Obviously, the last estimate is also valid for $Q_m = p^j$. Since $\Omega^\sigma \subseteq \mathbb{N}_{p^j}^k$, by [21, Proposition 3], there are $\epsilon \geq 0$ and $\delta_2 > 0$ such that

$$\left|\sum_{(x',x'')\in\Omega^\sigma} \sum_m \exp\left(2\pi i P_m(x', p^\sigma x'')/p^j\right)\right| \leq (1+\epsilon)p^{j(k-\delta_2)},$$

which finishes the proof of (3.17) for $(1+2\epsilon) = p^j$, and the theorem follows.

4. Multipliers

We develop some estimates on discrete Fourier multipliers corresponding to operators M_N and H_N [27].

4.1. Averaging operators. For a function $f_m: \mathbb{Z}^d \to \mathbb{C}$ with a finite support we have

$$M_N f_m(x) = \mathcal{F}^{-1}\big((\mathrm{m}_0)_N \hat{f_m}\big)(x),$$

where $(\mathrm{m}_0)_N$ is the discrete Fourier multiplier

$$(\mathrm{m}_0)_N(\xi^m) = \frac{1}{\vartheta_B(N)} \sum_{n\in\mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathcal{P}^{k''}} \sum_m e^{2\pi i \xi^m \cdot Q_m(n,1+\epsilon)} \mathbb{1}_{B_N}(n, 1+\epsilon)\left(\prod_{j=1}^{k''} \log p_j\right),$$

wherein ϑ_B is the Chebyshev function

$$\vartheta_B(N) = \sum_{n\in\mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathbb{P}^{k''}} \mathbb{1}_{B_N}(n, 1+\epsilon)\left(\prod_{j=1}^{k''} \log p_j\right).$$

By (1.1) and the prime number theorem,

$$\vartheta_B(N) \simeq N^k. \tag{4.1}$$

Next, let us define

$$\Phi_N(\xi^m) = \frac{1}{|B|}\int_B \sum_m e^{2\pi i \xi^m \cdot Q_m(Nx)} \mathrm{d}x,$$

where $|B|$ denotes Euclidean measure of B. By a multi-dimensional version of van der Corput's lemma (see [22, Proposition 2.1]) we have

$$|\Phi_N(\xi^m)| \lesssim \min\left\{1, |N^A \xi^m|_\infty^{-\frac{1}{d}}\right\},$$

where A is the matrix defined in (2.8). Moreover,

$$|\Phi_N(\xi^m) - 1| \lesssim min\{1, |N^A\xi^m|_\infty\}. \tag{4.2}$$

Therefore, for $N < N' \leq 2N$, we have

$$|\Phi_N(\xi^m) - \Phi_{N'}(\xi^m)| \lesssim min\left\{|N^A\xi^m|_\infty, |N^A\xi^m|_\infty^{-\frac{1}{d}}\right\}. \tag{4.3}$$

We start with the following proposition.

Proposition 4.1 (see [27]). For each $\beta' > 0$ there are $\epsilon \geq 0$ such that for all $N \in \mathbb{N}$, and $\xi^m \in \mathbb{T}^d$ satisfying

$$\left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right| \leq N^{-|\gamma|}L, \quad \text{for all } \gamma \in \Gamma, \tag{4.4}$$

where $1 \leq 1+2\epsilon \leq (\log N)^{\beta'}$, $(1+\epsilon) \in \mathbf{A}_{1+2\epsilon}$, and $1 \leq L \leq \exp((1+2\epsilon)\sqrt{\log N})$, we have

$$|(\mathrm{m}_0)_N(\xi^m) - G((1+\epsilon)/(1+2\epsilon))\Phi_N(\xi^m - (1+\epsilon)/(1+2\epsilon))| \leq (1+\epsilon)L\exp\big(-(1+2\epsilon)\sqrt{\log N}\big).$$

The constant $(1+2\epsilon)$ is absolute.

Proof. Observe that for a prime number $1+\epsilon$, $(1+\epsilon) \mid (1+2\epsilon)$ if and only if $((1+\epsilon)\mathrm{mod}(1+2\epsilon), 1+2\epsilon) > 1$. Hence, for each $s \in \{1, \dots, k''\}$, we have

$$\left|\sum_{u_m\in\mathbb{N}^{k'}} \sum_{\substack{r''\in\mathbb{N}^{k''} \\ (r''_s, 1+2\epsilon)>1}} \sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''} \\ (1+\epsilon)=r''mod\,(1+2\epsilon)}} \sum_m e^{2\pi i\xi^m . Q_m(u_m, 1+\epsilon)} \mathbb{1}_{B_N}(u_m, 1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)\right|$$
$$\lesssim N^{k-1} \sum_{(1+\epsilon)|(1+2\epsilon)} \log(1+\epsilon) \lesssim N^{k-1}\log(1+2\epsilon).$$

Let $\theta = \xi^m - (1+\epsilon)/(1+2\epsilon)$. Then by (4.4),

$$|\theta_\gamma| \leq N^{-|\gamma|}L, \quad \text{for all } \gamma \in \Gamma. \tag{4.5}$$

Since for $(u_m, 1+\epsilon) \in \mathbb{N}^{k'} \times \mathbb{P}^{k''}$ such that $u_m \equiv r'\mathrm{mod}(1+2\epsilon)$, and $(1+\epsilon) \equiv r''\mathrm{mod}(1+2\epsilon)$,

$$\begin{aligned} \xi_\gamma^m u_m^{\gamma'}(1+\epsilon)^{\gamma''} \quad &\equiv \frac{(1+\epsilon)_\gamma}{1+2\epsilon} u_m^{\gamma'} p^{\gamma''} + \theta_\gamma n^{\gamma'}(1+\epsilon)^{\gamma''} \ (\mathrm{mod}1) \\ &\equiv \frac{(1+\epsilon)_\gamma}{1+2\epsilon}(r')^{\gamma'}(r'')^{\gamma''} + \theta_\gamma u_m^{\gamma'}(1+\epsilon)^{\gamma''} \ (\mathrm{mod}1), \end{aligned}$$

we have

$$\sum_{u_m\in\mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathbb{P}^{k''}} \sum_m e^{2\pi i\xi^m . Q_m(u_m, 1+\epsilon)} \mathbb{I}_{B_N}(u_m, 1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)$$

$$= \sum_{r'\in\mathbb{N}^{k'}_{1+2\epsilon}} \sum_{r''\in A^{k''}_{1+2\epsilon}} \sum_m e^{2\pi i((1+\epsilon)/(1+2\epsilon)).Q_m(r',r'')} \sum_{\substack{u_m\in\mathbb{N}^{k'}\\ u_m=r' \bmod (1+2\epsilon)}} \sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''}\\ (1+\epsilon)=r''\bmod (1+2\epsilon)}} \sum_m e^{2\pi i\xi^m.Q_m(u_m,1+\epsilon)} \mathbb{1}_{B_N}(u_m,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right) + O(N^{k-1}\log\log N). \quad (4.6)$$

Let us fix $u_m \in \mathbb{N}^{k'}$, $\tilde{p}\in\mathbb{P}^{k''-1}$ and $r_1''\in A_{1+2\epsilon}$. Then

$$\{v\in\mathbb{N}: (u_m, v, \tilde{p})\in B_N\} = (V_0+1,\dots,V_1),$$

for some $0\le V_0\le V_1\le N$. Let $\tilde{V}_0=\max\{N^{1/2},V_0\}$ and $\tilde{V}_1=\max\{N^{1/2},V_1\}$. We have

$$\sum_{\substack{p_1\in P_{V_1}\backslash P_{V_0}\\ p_1=r_1'' \bmod (1+2\epsilon)}} \sum_m e^{2\pi i\theta.Q_m(u_m,p_1,\tilde{p})}\log p_1 = \sum_{\substack{p_1\in P_{\tilde{V}_1}\backslash P_{\tilde{V}_0}\\ p_1=r_1'' \bmod (1+2\epsilon)}} \sum_m e^{2\pi i\theta.Q_m(u_m,p_1,\tilde{p})}\log p_1 + O(N^{1/2}).$$

By the partial summation we obtain

$$\sum_{\substack{p_1\in P_{V_1}\backslash P_{V_0}\\ p_1=r_1'' \bmod (1+2\epsilon)}} \sum_m e^{2\pi i\theta.Q_m(u_m,p_1,\tilde{p})}\log p_1 = \sum_{\substack{p_1\in P_{\tilde{V}_1}\backslash P_{\tilde{V}_0}\\ p_1=r_1'' \bmod (1+2\epsilon)}} \sum_m e^{2\pi i\theta.Q_m(u_m,v_1,\tilde{p})}\mathbb{1}_{P(v_1)}\log v_1$$
$$= \sum_m \vartheta(\tilde{V}_1; 1+2\epsilon, r_1'')e^{2\pi i\theta.Q_m(u_m,\tilde{V}_1,\tilde{p})} - \sum_m \vartheta(\tilde{V}_0; 1+2\epsilon, r_1'')e^{2\pi i\theta.Q_m(u_m,\tilde{V}_0,\tilde{p})}$$
$$-\int_{\tilde{V}_0}^{\tilde{V}_1}\sum_m \vartheta(t;1+2\epsilon,2+\epsilon)\frac{d}{dt}\left(e^{2\pi i\theta.Q_m(u_m,t,\tilde{p})}\right)dt, \quad (4.7)$$

where for $x\ge 2$, we have set

$$\vartheta(x;1+2\epsilon,2+\epsilon) = \sum_{\substack{(1+\epsilon)\in\mathbb{P}_x\\ (1+\epsilon)=(2+\epsilon)\bmod(1+2\epsilon)}} \log(1+\epsilon).$$

Analogously, we can write

$$\sum_{V_0<v_1\le V_1}\sum_m e^{2\pi i\theta\cdot Q_m(u_m,v_1,\tilde{p})} = \sum_m \tilde{V}_1 e^{2\pi i\theta\cdot Q_m(u_m,\tilde{V}_1,\tilde{p})} - \sum_m \tilde{V}_0 e^{2\pi i\theta\cdot Q_m(u_m,\tilde{V}_0,\tilde{p})}$$
$$-\int_{\tilde{V}_0}^{\tilde{V}_1}\sum_m t\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2\pi i\theta\cdot Q_m(u_m,t,\tilde{p})}\right)\mathrm{d}t + O\left(N^{\frac{1}{2}}\right). \quad (4.8)$$

Furthermore, in view of the Siegel-Walfisz theorem ([20,23], see also [17, Corollary 11.21]), there are $\epsilon\ge 0$ such that for all $x\ge 2$, $(2+\epsilon,1+2\epsilon)=1$ and $1\le 1+2\epsilon\le(\log x)^{2\beta'}$,

$$\left|\vartheta(x;1+2\epsilon,2+\epsilon) - \frac{x}{\varphi_m(1+2\epsilon)}\right| \le (1+\epsilon)x\exp\left(-(1+2\epsilon)\sqrt{\log x}\right). \quad (4.9)$$

Hence, by (4.7), (4.8) and (4.5), we obtain

$$\left|\sum_{\substack{p_1\in P_{V_1}\backslash P_{V_0}\\ p_1=r_1'' \bmod (1+2\epsilon)}} \sum_m e^{2\pi i\theta.Q_m(u_m,p_1,\tilde{p})}\log p_1 - \frac{1}{\varphi_m(1+2\epsilon)}\sum_{V_0<v_1\le V_1}\sum_m e^{2\pi i\theta\cdot Q_m(u_m,v_1,\tilde{p})}\right|$$
$$\lesssim N^{\frac{1}{2}} + \sum_m\left|\vartheta(\tilde{V}_1;1+2\epsilon,r_1'') - \frac{\tilde{V}_1}{\varphi_m(1+2\epsilon)}\right| + \sum_m\left|\vartheta(\tilde{V}_0;1+2\epsilon,r_1'') - \frac{\tilde{V}_0}{\varphi_m(1+2\epsilon)}\right|$$
$$+\left(\sum_{\gamma\in\Gamma}|\theta_\gamma|N^{|\gamma|-1}\right)\int_{\tilde{V}_0}^{\tilde{V}_1}\sum_m\left|\vartheta(t;1+2\epsilon,r_1'') - \frac{t}{\varphi_m(1+2\epsilon)}\right|dt$$
$$\lesssim N\exp\left(-(1+2\epsilon)\sqrt{\log N}\right) + LN^{-1}\int_{N^{\frac{1}{2}}}^{N} t\exp\left(-(1+2\epsilon)\sqrt{\log t}\right)dt.$$

Thus,

$$\sum_{\substack{p_1\in P_{V_1}\backslash P_{V_0}\\ p_1=r_1'' \bmod (1+2\epsilon)}} \sum_m e^{2\pi i\theta.Q_m(u_m,p_1,\tilde{p})}\log p_1 = \sum_m \frac{1}{\varphi_m(1+2\epsilon)}\sum_{V_0<v_1\le V_1} e^{2\pi i\theta\cdot Q_m(u_m,v_1,\tilde{p})} + O\left(NL\exp\left(-(1+2\epsilon)\sqrt{\log N}\right)\right).$$

In view of (4.1), similar arguments applied to the sums over $p_2,\dots,p_{k''}$ lead to

$$\sum_{\substack{u_m\in\mathbb{N}^{k'}\\ u_m=r' \bmod (1+2\epsilon)}} \sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''}\\ (1+\epsilon)=r''\bmod (1+2\epsilon)}} \sum_m e^{2\pi i\xi^m.Q_m(u_m,1+\epsilon)}\mathbb{1}_{B_N}(u_m,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)$$

$$= \sum_m \frac{1}{\varphi_m(1+2\epsilon)^{k''}} \sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} e^{2\pi i\theta.Q_m((1+2\epsilon)u_m+r',v)} \mathbb{1}_{B_N}((1+2\epsilon)u_m+r',v) + O\left(N^k L \exp\left(-(1+2\epsilon)\sqrt{\log N}\right)\right).$$

By [12, Proposition 3.1], the number of lattice points in B_N at the distance $< (1+2\epsilon)$ from the boundary of B_N is $O((1+2\epsilon)N^{k-1})$. Moreover, for each $(x,y)\in[0,1]^k$, and $((1+2\epsilon)u_m+(1+2\epsilon)x, v+y)\in B_N$, we have

$$\sum_m |\theta\cdot Q_m((1+2\epsilon)u_m+(1+2\epsilon)x,v+y)-\theta\cdot Q_m((1+2\epsilon)u_m,v)| \le (1+\epsilon)(1+2\epsilon)\sum_{\gamma\in\Gamma}|\theta_\gamma|N^{|\gamma|-1} \lesssim (1+2\epsilon)N^{-1}L.$$

Hence, by (4.6) and (4.1),

$$\sum_{\substack{u_m\in\mathbb{N}^{k'}\\ u_m=r' \bmod (1+2\epsilon)}} \sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''}\\ (1+\epsilon)=r'' \bmod (1+2\epsilon)}} \sum_m e^{2\pi i\theta.Q_m(u_m,1+\epsilon)} \mathbb{1}_{B_N}(u_m,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)$$
$$= \sum_m \frac{1}{\varphi_m(1+2\epsilon)^{k''}} \sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} e^{2\pi i\theta.Q_m((1+2\epsilon)u_m+r',v)} \mathbb{1}_{B_N}((1+2\epsilon)u_m+r',v)$$
$$+ O\left(N^k L\exp\left(-(1+2\epsilon)\sqrt{\log N}\right)\right)$$
$$= \sum_m \frac{1}{\varphi_m(1+2\epsilon)^{k''}} \sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} e^{2\pi i\theta.Q_m((1+2\epsilon)u_m,v)} \mathbb{1}_{B_N}((1+2\epsilon)u_m,v)$$
$$+ O\left(N^k L\exp\left(-(1+2\epsilon)\sqrt{\log N}\right)\right).$$

Finally, another application of the mean value theorem allows us to replace the sums by the corresponding integrals. Indeed, we have

$$\Bigg|\sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \sum_m e^{2\pi i\theta.Q_m((1+2\epsilon)u_m,v)} \mathbb{1}_{B_N}((1+2\epsilon)u_m,v)$$
$$- \iint_{\mathbb{R}_+^k} \sum_m e^{2\pi i\theta.Q_m((1+2\epsilon)x,y)} \mathbb{1}_{B_N}((1+2\epsilon)x,y)\,dx\,dy\Bigg|$$
$$= \Bigg|\sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \int_{u_m+(0,1]^{k''}} \int_{v+(0,1]^{k''}} \sum_m \left(e^{2\pi i\theta.Q_m((1+2\epsilon)u_m,v)} \mathbb{1}_{B_N}((1+2\epsilon)u_m,v)\right.$$
$$\ldots$$
$$= \Bigg|\sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \int_{u_m+(0,1]^{k''}} \int_{v+(0,1]^{k''}} \sum_m \left(e^{2\pi i\theta.Q_m((1+2\epsilon)u_m,v)} \mathbb{1}_{B_N}((1+2\epsilon)u_m,v)\right.$$
$$\left. - e^{2\pi i\theta.Q_m((1+2\epsilon)x,y)} \mathbb{1}_{B_N}((1+2\epsilon)x,y)\right)dx\,dy\Bigg|$$
$$\lesssim \sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \iint_{(0,1]^k} \sum_m \left|e^{2\pi i\theta.Q_m((1+2\epsilon)u_m,v)} - e^{2\pi i\theta.Q_m((1+2\epsilon)u_m+(1+2\epsilon)x,v+y)}\right| \mathbb{1}_{B_N}((1+2\epsilon)u_m,v)\,dx\,dy$$
$$+ \sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \int_{(0,1]^k} \left|\mathbb{1}_{B_N}((1+2\epsilon)u_m,v) - \mathbb{1}_{B_N}((1+2\epsilon)(u_m+x,v+y)\right|dx\,dy,$$

which is again bounded by $(1+2\epsilon)N^{k-1}L$. Therefore,

$$\left|\sum_m \vartheta_B(N)\pi_N(\xi^m) - \sum_m G((1+\epsilon)/(1+2\epsilon))\right| B|N^k\Phi_N(\xi^m-(1+\epsilon)/(1+2\epsilon))|$$
$$\le (1+\epsilon)N^k L\exp(-(1+2\epsilon)\sqrt{\log N}).$$

In particular, taking $\xi^m=0, \epsilon=-1$ and $L=1$, we obtain

$$\vartheta_B(N) = |B|N^k\left(1+O\left(\exp\left(-(1+2\epsilon)\sqrt{\log N}\right)\right)\right). \tag{4.10}$$

This completes the proof.

Lemma 3 (see [27]). For each $\alpha>0$ there is $\epsilon\ge 0$ such that for all $N\in\mathbb{N}$, and $\xi^m\in\mathbb{T}^d$ satisfying

$$\left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right| \le N^{-|\gamma|}L, \quad \text{for all } \gamma\in\Gamma, \tag{4.11}$$

where $1\le 1+2\epsilon\le L$, $(1+\epsilon)\in\mathbf{A}_{1+2\epsilon}$, and $1\le L\le\exp((1+2\epsilon)\sqrt{\log N})(\log N)^{-\alpha}$, we have

$$\left|\sum_m (\mathrm{m}_0)_N(\xi^m) - G((1+\epsilon)/(1+2\epsilon))\Phi_N \sum_m (\xi^m - (1+\epsilon)/(1+2\epsilon))\right| \le (1+\epsilon)(\log N)^{-\alpha}.$$

Proof. Given $\alpha > 0$, let $\beta' \ge d\beta_\alpha$, where β_α is the value determined in Theorem 2 .

Suppose that (4.11) holds for some $(\log N)^{\beta'} < 1+2\epsilon \le L$ and $(1+\epsilon) \in \mathbf{A}_{1+2\epsilon}$. For each $\gamma \in \Gamma$, by Dirichlet's principle there are coprime integers $(1+\epsilon)'_\gamma$ and $(1+2\epsilon)'_\gamma$ such that $1 \le (1+\epsilon)'_\gamma \le (1+2\epsilon)'_\gamma \le N^{|\gamma|}L^{-1}(\log N)^{-\beta'/d}$, and satisfying

$$\left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)'_\gamma}{(1+2\epsilon)'_\gamma}\right| \le \frac{1}{(1+2\epsilon)'_\gamma} N^{-|\gamma|} L (\log N)^{\beta'/d}.$$

Assume that for some $\gamma \in \Gamma$, $(\log N)^{\beta'/d} \le (1+2\epsilon)'_\gamma \le N^{|\gamma|}L^{-1}(\log N)^{-\beta'/d}$. Then, by Theorem 2 , we have

$$|(\mathrm{m}_0)_N(\xi^m)| \le (1+\epsilon)(\log N)^{-\alpha}.$$

If for all $\gamma \in \Gamma, 1 \le (1+2\epsilon)'_\gamma \le (\log N)^{\mathrm{B}'/d}$, then we set $(1+2\epsilon)'' = \mathrm{lcm}\big((1+2\epsilon)'_\gamma : \gamma \in \Gamma\big)$ and $(1+\epsilon)''_\gamma = (1+\epsilon)'_\gamma (1+2\epsilon)''/(1+2\epsilon)'_\gamma$ getting $1 \le (1+2\epsilon)'' \le (\log N)^{\beta'}$ and $(1+\epsilon)'' \in \mathbf{A}_{(1+2\epsilon)''}$ with

$$\left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)'_\gamma}{(1+2\epsilon)'_\gamma}\right| = \left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)''_\gamma}{(1+2\epsilon)''}\right| \le N^{-|\gamma|} L (\log N)^{\frac{\beta'}{d}}.$$

Since $(1+\epsilon)'/(1+2\epsilon)' \ne (1+\epsilon)/(1+2\epsilon)$,

$$\begin{aligned}(\log N)^{-\beta'} L^{-1} \le \frac{1}{(1+2\epsilon)''(1+2\epsilon)} \le \left|\frac{(1+\epsilon)''_\gamma}{(1+2\epsilon)''} - \frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right| &\le \left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)''_\gamma}{(1+2\epsilon)''}\right| + \left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right| \\ &\le N^{-|\gamma|} L \left(1 + (\log N)^{\frac{\beta'}{d}}\right),\end{aligned}$$

which is possible only for finite number of $N\,'s$.

Finally, in the case when $1 \le 1+2\epsilon \le (\log N)^{\beta'}$, by Proposition 4.1, we obtain

$$(\mathrm{m}_0)_N(\xi^m) = G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\Phi_N\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right) + O((\log N)^{-\alpha}),$$

which concludes the proof.

Lemma 4 (see [27]). For all $0 \le \epsilon < \infty, N_1, N_2 \in \mathbb{N}, N_1 < N_2$, and any $f_m \in \ell^{1+\epsilon}(\mathbb{Z}^d)$,

$$\left\|\sum_{n=N_1}^{N_2-1} \sum_m |M_{n+1}f_m - M_n f_m|\right\|_{\ell^{1+\epsilon}} \le C_{1+\epsilon} N_1^{-k}\big(\vartheta_B(N_2) - \vartheta_B(N_1)\big) \sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

Proof. Let us denote by m_n the convolution kernel corresponding to $(M_m)_n$. Consider $(x,y) \in \mathbb{N}^{k'} \times \mathbb{P}^{k''}$. If $(x,y) \in B_{N_1}$ then

$$\sum_{n=N_1}^{N_2-1} |m_{n+1}(x,y) - m_n(x,y)| = \left(\frac{1}{\vartheta_B(N_1)} - \frac{1}{\vartheta_B(N_2)}\right)\prod_{j=1}^{k''} \log y_j.$$

If $(x,y) \in B_{N_2} \setminus B_{N_1}$ then by setting

$$n_0 = min\{n \in \mathbb{N} : x \in B_n\},$$

we have

$$\begin{aligned}\sum_{n=N_1}^{N_2-1} |m_{n+1}(x,y) - m_n(x,y)| &= \left(\frac{1}{\vartheta_B(n_0)} + \sum_{n=n_0}^{N_2-1} \frac{1}{\vartheta_B(n)} - \frac{1}{\vartheta_B(n+1)}\right)\prod_{j=1}^{k''} \log y_j \\ &= \left(\frac{2}{\vartheta_B(n_0)} - \frac{1}{\vartheta_B(N_2)}\right)\prod_{j=1}^{k''} \log y_j.\end{aligned}$$

Therefore,

$$\left\|\sum_{n=N_1}^{N_2-1} |m_{n+1} - m_n|\right\|_{\ell^1} \le \left(\frac{1}{\vartheta_B(N_1)} - \frac{1}{\vartheta_B(N_2)}\right)\vartheta_B(N_1) + \frac{2}{\vartheta_B(N_1)}\big(\vartheta_B(N_2) - \vartheta_B(N_1)\big),$$

and hence, by Young's inequality,

$$\left\|\sum_{n=N_1}^{N_2-1}\sum_m |M_{n+1}f_m - M_n f_m|\right\|_{\ell^{1+\epsilon}} \le \left\|\sum_{n=N_1}^{N_2-1}|m_{n+1}-m_n|\right\|_{\ell^1}\sum_m \| f_m \|_{\ell^{1+\epsilon}}$$
$$\lesssim \frac{\vartheta_B(N_2)-\vartheta_B(N_1)}{\vartheta_N(N_1)}\sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

which finishes the proof since $\vartheta_B(N_1) \simeq N_1^k$.

4.2. Truncated discrete singular operators. We investigate the asymptotic of Fourier multipliers corresponding to the truncated discrete singular operators H_N with a kernel K_m satisfying (1.2) and (1.3) (see [27]).

For b_N be the Fourier multiplier corresponding to H_N, that is for a finitely supported function $f_m: \mathbb{Z}^d \to \mathbb{C}$,
$$H_N f_m = \mathcal{F}^{-1}(b_N \hat{f_m}),$$
where
$$\mathfrak{b}_N(\xi^m) = \sum_{n\in Z^{k'}}\sum_{(1+\epsilon)\in(\pm\mathbb{P})^{k''}}\sum_m e^{2\pi i\xi^m\cdot Q_m(n,1+\epsilon)}K_m(n,1+\epsilon)\mathbb{1}_{B_N}(n,1+\epsilon)\left(\prod_{j=1}^{k''}\log|p_j|\right).$$
We also define
$$\Psi_N(\xi^m) = \text{ p.v. } \iint_{B_N}\sum_m e^{2\pi i\xi^m\cdot Q_m(x,y)}K_m(x,y)\mathrm{d}x\,\mathrm{d}y.$$
In view of a multi-dimensional version of van der Corput's lemma (see [22, Proposition 2.1]), for $< N' \le 2N$,
$$\left|\sum_m \Psi_N(\xi^m) - \sum_m \Psi_{N'}(\xi^m)\right| \lesssim \min\sum_m \left\{1, |N^A\xi^m|_\infty^{-\frac{1}{d}}\right\}.$$
Moreover, by (1.3),
$$\left|\sum_m \Psi_N(\xi^m) - \sum_m \Psi_{N'}(\xi^m)\right| \lesssim \min\sum_m \{1, |N^A\xi^m|_\infty\}.$$
Hence,
$$\left|\sum_m \Psi_N(\xi^m) - \sum_m \Psi_{N'}(\xi^m)\right| \lesssim \min\sum_m \left\{|N^A\xi^m|_\infty, |N^A\xi^m|_\infty^{-\frac{1}{d}}\right\}. \tag{4.12}$$
We start with a proposition analogous to Proposition 4.1.

Proposition 4.2 (see [27]). For each $\beta' > 0$ there $\epsilon \ge 0$ such that for all $N < N' \le 2N$, and $\xi^m \in \mathbb{T}^d$ satisfying
$\left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right| \le N^{-|\gamma|}L,$ for all $\gamma \in \Gamma$,
where $1 \le 1+2\epsilon \le (\log N)^{\beta'}, (1+\epsilon) \in \mathbf{A}_{1+2\epsilon}$, and $1 \le L \le \exp((1+2\epsilon)\sqrt{\log N})$, we have
$$\sum_m |\mathrm{b}_{N'}(\xi^m) - \mathrm{b}_N(\xi^m) - G((1+\epsilon)/(1+2\epsilon))(\Psi_{N'}(\xi^m - (1+\epsilon)/(1+2\epsilon)) - \Psi_N(\xi^m - (1+\epsilon)/(1+2\epsilon))|$$
$$\le (1+\epsilon)L\exp(-(1+2\epsilon)\sqrt{\log N}).$$
Proof. For a prime number $1+\epsilon, (1+\epsilon) \mid (1+2\epsilon)$ if and only if $((1+\epsilon)\mathrm{mod}(1+2\epsilon), 1+2\epsilon) > 1$. Therefore, by (1.1), (1.2), and the prime number theorem, for any $s \in \{1, \ldots, k''\}$,
$$\left|\sum_{u_m\in\mathbb{N}^{k'}}\sum_{\substack{r''\in\mathbb{N}_{1+2\epsilon}^{k''}\\(r_s'',1+2\epsilon)>1}}\sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''}\\(1+\epsilon)=r''\,mod\,(1+2\epsilon)}}\sum_m e^{2\pi i\xi.Q_m(n,1+\epsilon)}K_m(u_m,1+\epsilon)\,\mathbb{1}_{B_{N'}\setminus B_N}(u_m,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)\right|$$
$$\lesssim N^{-1}\sum_{(1+\epsilon)|(1+2\epsilon)}\log(1+\epsilon) \lesssim N^{-1}\log(1+2\epsilon).$$
To simplify the notations, for $(x,y) \in \mathbb{R}^k \setminus \{0\}$, we set
$$F_m(x,y) = e^{2\pi i\theta\cdot Q_m(x,y)}K_m(x,y),$$
where $\theta = \xi^m - (1+\epsilon)/(1+2\epsilon)$. For any $(u_m, 1+\epsilon) \in \mathbb{N}^{k'} \times \mathbb{P}^{k''}$ such that $u_m \equiv r'\mathrm{mod}(1+2\epsilon)$, and $(1+\epsilon) \equiv r''\mathrm{mod}(1+2\epsilon)$, we have

$$\xi_\gamma^m u_m^{\gamma'} p^{\gamma''} \equiv \frac{(1+\epsilon)_\gamma}{1+2\epsilon}(r')^{\gamma'}(r'')^{\gamma''} + \theta_\gamma u_m^{\gamma'} p^{\gamma''} \ (\text{mod}1),$$

thus

$$\sum_{u_m \in \mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathbb{P}^{k''}} \sum_m e^{2\pi i \xi . Q_m(u_m,1+\epsilon)} K_m(u_m,1+\epsilon)\, \mathbb{1}_{B_{N'}/B_N}(u_m,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)$$

$$= \sum_{u_m\in\mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathbb{P}^{k''}} \sum_m e^{2\pi i((1+\epsilon)/1+2\epsilon)\cdot Q_m(r',r'')} \sum_{\substack{u_m\in\mathbb{N}^{k'} \\ u_m = r' \bmod (1+2\epsilon)}} \sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''} \\ (1+\epsilon)=r'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,1+\epsilon)\mathbb{1}_{B_{N'}\setminus B_N}(u_m,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)$$

$$+O(N^{-1}\log\log N).$$

Fix $u_m \in \mathbb{N}^{k'}, \tilde{p} \in \mathbb{P}^{k''-1}$ and $r_1'' \in A_{1+2\epsilon}$. Then

$$\{v \in \mathbb{N}: (u_m, v, \tilde{p}) \in B_{N'} \setminus B_N\} = (V_0+1,\dots,V_1),$$

for some $1 \le V_0 \le V_1 \le N' \le 2N$. Let $\tilde{V}_0 = \max\{N^{1/2}, V_0\}$ and $\tilde{V}_1 = \max\{N^{1/2}, V_1\}$. We have

$$\sum_{\substack{P_1\in P_{V_1}\setminus P_{V_0} \\ p_1 = r_1'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,p_1,\tilde{p})\log p_1 = \sum_{\substack{P_1\in P_{\tilde{V}_1}\setminus P_{\tilde{V}_0} \\ p_1 = r_1'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,p_1,\tilde{p})\log p_1 + O(N^{-k+1/2}).$$

By the partial summation

$$\sum_{\substack{P_1\in P_{\tilde{V}_1}\setminus P_{\tilde{V}_0} \\ p_1 = r_1'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,p_1,\tilde{p})\log p_1 = \sum_{\substack{\tilde{V}_0<v_1\le\tilde{V}_1 \\ v_1 = r_1'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,v_1,\tilde{p})\mathbb{1}_P(v_1)\log v_1$$

$$= \sum_m \vartheta(\tilde{V}_1;1+2\epsilon,r_1'')F_m(u_m,\tilde{V}_1,\tilde{p}) - \vartheta(\tilde{V}_0;1+2\epsilon,r_1'')F_m(u_m,\tilde{V}_0,\tilde{p}) - \int_{\tilde{V}_0}^{\tilde{V}_1}\sum_m \vartheta(t;1+2\epsilon,r_1'')\frac{d}{dt}F_m(u_m,t,\tilde{p})dt.$$

Analogously, we have

$$\sum_{V_0<v_1\le V_1}\sum_m F_m(u_m,v_1,\tilde{p}) = \sum_m \tilde{V}_1 F_m(u_m,\tilde{V}_1,\tilde{p}) - \sum_m \tilde{V}_0 F_m(n,\tilde{V}_0,\tilde{p}) - \int_{\tilde{V}_0}^{\tilde{V}_1}\sum_m t\frac{\mathrm{d}}{\mathrm{d}t}F_m(u_m,t,\tilde{p})\mathrm{d}t + O(N^{-k+1/2}).$$

Hence, by (4.9) and (1.2), we obtain

$$\left|\sum_{\substack{P_1\in P_{V_1}\setminus P_{V_0} \\ p_1 = r_1'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,p_1,\tilde{p})\log p_1 - \sum_m \frac{1}{\varphi_m(1+2\epsilon)}\sum_{V_0<v_1\le V_1} F_m(u_m,v_1,\tilde{p})\right|$$

$$\lesssim N^{-k+1/2} + \left|\vartheta(\tilde{V}_1;1+2\epsilon,r_1'') - \sum_m \frac{\tilde{V}_1}{\varphi_m(1+2\epsilon)}\right|N^{-k} + \left|\vartheta(\tilde{V}_0;1+2\epsilon,r_1'') - \sum_m \frac{\tilde{V}_0}{\varphi_m(1+2\epsilon)}\right|N^{-k}$$

$$+\left(\sum_{\gamma\in\Gamma}|\theta_\gamma|N^{-\mathrm{k}-1+|\gamma|} + N^{-k-1}\right)\int_{\tilde{V}_0}^{\tilde{V}_1}\sum_m \left|\vartheta(t;1+2\epsilon,r_1'') - \frac{t}{\varphi_m(1+2\epsilon)}\right|\mathrm{d}t$$

$$\lesssim N^{-k+1}\exp\left(-(1+2\epsilon)\sqrt{\log N}\right) + LN^{-k-1}\int_{N^{\frac{1}{2}}}^{N} t\exp\left(-(1+2\epsilon)\sqrt{\log t}\right)dt.$$

Therefore,

$$\sum_{\substack{P_1\in P_{V_1}\setminus P_{V_0} \\ p_1 = r_1'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,p_1,\tilde{p})\log p_1$$

$$= \sum_m \frac{1}{\varphi_m(1+2\epsilon)}\sum_{V_0<v_1\le V_1} F_m(u_m,v_1,\tilde{p}) + O\left(N^{-k+1}L\exp\left(-(1+2\epsilon)\sqrt{\log N}\right)\right).$$

By similar reasonings applied to the sums over $p_2,\dots,p_{k''}$, one can show that

$$\sum_{\substack{u_m\in\mathbb{N}^{k'} \\ n=r' \bmod (1+2\epsilon)}} \sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''} \\ (1+\epsilon)=r'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m,1+\epsilon)\mathbb{1}_{B_{N'}\setminus B_N}(u_m,1+\epsilon)\left(\prod_{j=1}^{k''}\log p_j\right)$$

$$= \sum_m \frac{1}{\varphi_m(1+2\epsilon)^{k''}} \sum_{u_m \in \mathbb{N}^{k'}} \sum_{(1+\epsilon)\in\mathbb{P}^{k''}} F_m((1+2\epsilon)u_m + r', v)\, \mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)u_m + r', v)$$
$$+ O\big(L \exp(-(1+2\epsilon)\sqrt{\log N})\big).$$

Since for each $(x,y) \in [0,1]^k$ and $((1+2\epsilon)u_m + (1+2\epsilon)x, v+y) \in B_{N'} \setminus B_N$, we have

$$\left|\sum_m \theta \cdot Q_m((1+2\epsilon)u_m + (1+2\epsilon)x, v+y) - \sum_m \theta \cdot Q_m((1+2\epsilon)u_m, v)\right|$$
$$\lesssim (1+2\epsilon) \sum_{\gamma\in\Gamma} |\theta_\gamma| N^{|\gamma|-1} \leqslant (1+2\epsilon) N^{-1} L,$$

and

$$|K_m((1+2\epsilon)u_m + (1+2\epsilon)x, v+y) - K_m((1+2\epsilon)u_m, v)| \lesssim (1+2\epsilon)N^{-k-1},$$

thus by the mean value theorem, we obtain

$$|F_m((1+2\epsilon)u_m + (1+2\epsilon)x, v+y) - F_m((1+2\epsilon)u_m, v)| \lesssim (1+2\epsilon)N^{-k-1}L.$$

Moreover, in view of [12, Proposition 3.1], the number of lattice points in B_N of distance $< (1+2\epsilon)$ from the boundary of B_N is $O((1+2\epsilon)N^{k-1})$. Therefore,

$$\sum_{\substack{u_m \in \mathbb{N}^{k'} \\ n = r' \bmod (1+2\epsilon)}} \sum_{\substack{(1+\epsilon)\in\mathbb{P}^{k''} \\ (1+\epsilon) = r'' \bmod (1+2\epsilon)}} \sum_m F_m(u_m, 1+\epsilon)\mathbb{1}_{B_{N'}\setminus B_N}(u_m, 1+\epsilon)\left(\prod_{j=1}^{k''} \log p_j\right)$$
$$= \sum_m \frac{1}{\varphi_m(1+2\epsilon)^{k''}} \sum_{u_m \in \mathbb{N}_0^{k'}} \sum_{v\in\mathbb{P}^{k''}} F_m((1+2\epsilon)u_m, v)\, \mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)u_m + r', v)$$
$$+ O\big(L \exp(-(1+2\epsilon)\sqrt{\log N})\big)$$
$$= \sum_m \frac{1}{\varphi_m(1+2\epsilon)^{k''}} \sum_{u_m \in \mathbb{N}_0^{k'}} \sum_{v\in\mathbb{P}^{k''}} F_m((1+2\epsilon)u_m, v)\, \mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)u_m, v) + O\big(L \exp(-(1+2\epsilon)\sqrt{\log N})\big)$$

Lastly, we can replace the sums by the corresponding integrals because

$$\sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \sum_m F_m((1+2\epsilon)u_m, v)\mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)u_m, v) - \iint_{\mathbb{R}_+^k} \sum_m F_m((1+2\epsilon)x, y)\mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)x, y)\,dx\,dy \,|$$
$$\leq \sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \int_{(0,1]^k} \sum_m |F_m((1+2\epsilon)u_m, v) - \sum_m F_m((1+2\epsilon)u_m + (1+2\epsilon)x, v+y)|\mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)u_m, v)\,dx\,dy$$
$$+ \sum_{u_m\in\mathbb{N}_0^{k'}} \sum_{v\in\mathbb{N}^{k''}} \int_{(0,1]^k} \sum_m \left|F_m((1+2\epsilon)u_m + (1+2\epsilon)x, v+y)\left(\mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)u_m, v) - \mathbb{1}_{B_{N'}\setminus B_N}((1+2\epsilon)u_m + (1+2\epsilon)x, v+y)\right)\right| dx\,dy,$$

which is bounded by $(1+2\epsilon)N^{-1}L$.

Analogously to Lemma 3, we can prove the following statement.

Lemma 5 [27]. For each $\alpha > 0$ there is $\epsilon \geq 0$ such that for all $N \leq N' \leq 2N$, and $\xi^m \in \mathbb{T}^d$ satisfying

$$\left|\xi_\gamma^m - \frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right| \leq N^{-|\gamma|}L, \quad \text{for all } \gamma \in \Gamma,$$

where $1 \leq 1+2\epsilon \leq L$, $(1+\epsilon) \in \mathbf{A}_{1+2\epsilon}$, and $1 \leq L \leq \exp((1+2\epsilon)\sqrt{\log N})(\log N)^{-\alpha}$,

$$\sum_m |b_{N'}(\xi^m) - b_N(\xi^m) - G((1+\epsilon)/(1+2\epsilon))(\Psi_{N'}(\xi^m - (1+\epsilon)/(1+2\epsilon)) - \Psi_N(\xi^m - (1+\epsilon)/(1+2\epsilon)))| \leq (1+\epsilon)(\log N)^{-\alpha}.$$

Lemma 6 (see [27]). For all $0 \leq \epsilon < \infty, N_1, N_2 \in \mathbb{N}, N_1 < N_2$, and any $f_m \in \ell^{1+\epsilon}(\mathbb{Z}^d)$,

$$\left\|\sum_{n=N_1}^{N_2-1} \sum_m |H_{n+1}f_m - H_n f_m|\right\|_{\ell^{1+\epsilon}} \leq C_{1+\epsilon} N_1^{-k}\big(\vartheta_B(N_2) - \vartheta_B(N_1)\big) \sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

Proof. Let h_n denote the convolution kernel corresponding to H_n. Observe that for $(x,y) \in Z^{k'} \times (\pm\mathbb{P})^{k''}$, if $(x,y) \in B_{N_2} \setminus B_{N_1}$ then

$$\sum_{n=N_1}^{N_2-1} |h_{n+1}(x,y) - h_n(x,y)| = |K_m(x,y)| \prod_{j=1}^{k''} \log|y_j|,$$

otherwise the sum equals zero. Thus, by (1.2), we obtain

$$\left\|\sum_{n=N_1}^{N_2-1} |h_{n+1} - h_n|\right\|_{\ell^1} \lesssim N_1^{-k}\big(\vartheta_B(N_2) - \vartheta_B(N_1)\big),$$

hence, by Young's inequality,

$$\left\|\sum_{n=N_1}^{N_2-1}\sum_m |H_{n+1}f_m - H_n f_m|\right\|_{\ell^{1+\epsilon}} \le \left\|\sum_{n=N_1}^{N_2-1} |h_{n+1}-h_n|\right\|_{\ell^1}\sum_m \| f_m \|_{\ell^{1+\epsilon}}$$

$$\lesssim N_1^{-k}\big(\vartheta_B(N_2)-\vartheta_B(N_1)\big)\sum_m \| f_m \|_{\ell^{1+\epsilon}},$$

which completes the proof.

5. Variational Estimates

Following [27] we present the estimates for $\ell^{1+\epsilon}(\mathbb{Z}^d)$ norm of the $(2+\epsilon)$-variational seminorm for the averaging operators $(M_N : N \in \mathbb{N})$ and the truncated discrete singular operators $(H_N : N \in \mathbb{N})$. In order to give a unified approach, we set $(Y_N : N \in \mathbb{N})$ to be any of them. By $(\mathfrak{y}_N : N \in \mathbb{N})$ we denote the corresponding discrete Fourier multipliers and by $(\Upsilon_N : N \in \mathbb{N})$ its continuous counterparts. We start by listing properties that are sufficient to obtain $(2+\epsilon)$-variational estimates. Let $\rho \in (0,1)$ and set $N_n = \lfloor 2^{n^\rho} \rfloor$.

Property 1. In view of [11] (see also [9]) for each $0<\epsilon<\infty$ there is $C_{1+\epsilon}>0$ such that for all $0<\epsilon<\infty$ and any function $f_m \in L^{1+\epsilon}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$,

$$\left\|\sum_m V_{2+\epsilon}\big(\mathcal{F}^{-1}(\Upsilon_N \mathcal{F} f_m) : N\in\mathbb{N}\big)\right\|_{L^{1+\epsilon}} \le C_{1+\epsilon}\frac{2+\epsilon}{\epsilon}\sum_m \| f_m \|_{L^{1+\epsilon}},$$

and

$$\left\|\left(\sum_{n\ge 0}\sum_m V_2\big(\mathcal{F}^{-1}(\Upsilon_N \mathcal{F} f_m) : N\in[2^n,2^{n+1})\big)^2\right)^{1/2}\right\|_{L^{1+\epsilon}} \le C_{1+\epsilon}\sum_m \| f_m \|_{L^{1+\epsilon}}.$$

Property 2. By (4.3) and (4.12), for each $n\in\mathbb{N}$,

$$\sum_m \left|\Upsilon_{N_n}(\xi^m)-\Upsilon_{N_{n+1}}(\xi^m)\right| \lesssim \min\sum_m \left\{|N_n^A\xi^m|_\infty, |N_n^A\xi^m|_\infty^{-\frac{1}{d}}\right\}, \tag{5.1}$$

where A is the matrix defined in (2.8).

Property 3. By Lemma 4 and Lemma 6 we deduce that for each $0<\epsilon<\infty$ and any $f_m \in \ell^{1+\epsilon}(Z^d)$,

$$\left\|\sum_{N=N_n}^{N_{n+1}-1}\sum_m |Y_{N+1}f_m - Y_N f_m|\right\|_{\ell^{1+\epsilon}} \le C_{1+\epsilon,\rho}n^{\rho-1}\sum_m \| f_m \|_{\ell^{1+\epsilon}}, \tag{5.2}$$

because by (4.10),

$$N_n^{-k}\big(\vartheta_B(N_{n+1})-\vartheta_B(N_n)\big) \lesssim 2^{k(n+1)^\rho - kn^\rho} - 1 + e^{-(1+2\epsilon)n^{\rho/2}} \lesssim n^{\rho-1},$$

In particular,

$$\|Y_{N_{n+1}} - Y_{N_n}\|_{\ell^{1+\epsilon}\to\ell^{1+\epsilon}} \le (1+\epsilon). \tag{5.3}$$

Property 4. By Theorem 2 and partial summation for each $\alpha>0$, there is $\beta_\alpha>0$ so that for any $\beta>\beta_\alpha$, and $n\in\mathbb{N}$, if there is $\gamma_0\in\Gamma$, such that

$$\left|\xi_{\gamma_0}^{\mathrm{m}} - \frac{1+\epsilon}{1+2\epsilon}\right| \le \frac{1}{(1+2\epsilon)^2},$$

for some coprime numbers $(1+\epsilon)$ and $(1+2\epsilon)$ such that $1\le 1+\epsilon\le 1+2\epsilon$, and $(\log N_n)^\beta \le 1+2\epsilon \le N_n^{|\gamma_0|}(\log N_n)^{-\beta}$, then

$$\sum_m \left|\mathfrak{y}_{N_{n+1}}(\xi^m) - \mathfrak{y}_{N_n}(\xi^m)\right| \le (1+\epsilon)(\log N_n)^{-\alpha}.$$

Property 5. By Proposition 4.1 and Proposition 4.2, for each $\beta'>0$ there is $\epsilon\ge 0$ such that for all $n\in\mathbb{N}$, and $\xi^m\in\mathbb{T}^d$, satisfying

$$\left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right| \le N^{-|\gamma|}L,\quad \text{for all } \gamma\in\Gamma,$$

where $1\le 1+2\epsilon\le(\log N_n)^{\beta'}$, $(1+\epsilon)\in\mathbf{A}_{1+2\epsilon}$, and $1\le L\le\exp\big((1+2\epsilon)\sqrt{\log N_n}\big)$, we have

$$\mathfrak{y}_{N_{n+1}}(\xi^m) - \mathfrak{y}_{N_n}(\xi^m) = G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\left(\Upsilon_{N_{n+1}}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right) - \Upsilon_{N_n}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right)\right) + O\big(L\exp\big(-(1+2\epsilon)\sqrt{\log N_n}\big)\big). \tag{5.4}$$

Property 6. By Lemma 3 and Lemma 5 , for each $\alpha>0$, all $n\in\mathbb{N}$, and $\xi^m\in\mathbb{T}^d$, satisfying

$$\left|\xi_\gamma^{m}-\frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right|\le N^{-|\gamma|}L,\ \ \text{for all } \gamma\in\Gamma,$$

where $1\le 1+2\epsilon\le L$, $(1+\epsilon)\in \mathbf{A}_{1+2\epsilon}$, and $1\le L\le \exp\left((1+2\epsilon)\sqrt{\log N_n}\right)(\log N_n)^{-\alpha}$, we have

$$\eta_{N_{n+1}}(\xi^m)-\eta_{N_n}(\xi^m)=G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\left(\Upsilon_{N_{n+1}}\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right)-\Upsilon_{N_n}\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right)\right)+O((\log N_n)^{-\alpha}).\quad (5.5)$$

Before we embark on proving variational estimates, we show the following auxiliary result.

Proposition 5.1 [27]. For each $0<\epsilon<\infty$ there is $\epsilon\ge 0$, such that

For every $j_1,j_2\in\mathbb{N}$, $j_1<j_2$ such that

$$n_{j_1-1}<2^n\le n_{j_1}<n_{j_2}<2^{n+1}\le n_{j_2+1},$$

we estimate

$$\sum_{j=j_1+1}^{j_2}\sum_m\left|\mathcal{F}^{-1}\left(\left(\Upsilon_{n_j}-\Upsilon_{n_{j-1}}\right)\mathcal{F}f_m\right)\right|^2\le\sum_m V_2\left(\mathcal{F}^{-1}(\Upsilon_N\mathcal{F}f_m):N\in[2^n,2^{n+1})\right)^2.$$

Hence, for some increasing sequence of integers $\left((m_0)_j:j\in\mathbb{N}\right)$, we have

$$\sum_{j=1}^{\infty}\sum_m\left|\mathcal{F}^{-1}\left(\left(\Upsilon_{n_{j+1}}-\Upsilon_{n_j}\right)\mathcal{F}f_m\right)\right|^2\lesssim\sum_{j=1}^{\infty}\sum_m V_2\left(\mathcal{F}^{-1}(\Upsilon_N\mathcal{F}f_m):N\in\left[2^j,2^{j-1}\right)\right)^2$$
$$+\sum_{j=1}^{\infty}\sum_m\left|\mathcal{F}^{-1}\left(\left(\Upsilon_{2^{(m_0)_j}}-\Upsilon_{2^{(m_0)_{j+1}}}\right)\mathcal{F}f_m\right)\right|^2.$$

The conclusion now follows by [5] and Property 1.

We prove the following theorem.

Theorem 4. For each $0<\epsilon<\infty$ there is $\epsilon\ge 0$ such that for any finitely supported function $f_m:\mathbb{Z}^d\to\mathbb{C}$,

$$\left\|\sum_m V_{2+\epsilon}(Y_Nf_m:N\in\mathbb{N})\right\|_{\ell^{1+\epsilon}}\le(1+\epsilon)\frac{2+\epsilon}{\epsilon}\sum_m\|f_m\|_{\ell^{1+\epsilon}}.$$

We split a variational seminorm into two parts long variations $V_{2+\epsilon}^L$, and short variations $V_{2+\epsilon}^S$, where

$$V_{2+\epsilon}^L(Y_Nf_m:N\in\mathbb{N})=V_{2+\epsilon}\left(Y_{N_n}f_m:n\in\mathbb{N}_0\right),$$

and

$$\sum_m V_{2+\epsilon}^S(Y_Nf_m:N\in\mathbb{N})=\left(\sum_{n\ge 0}\sum_m V_{2+\epsilon}\left(Y_Nf_m:N\in[N_n,N_{n+1})\right)^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}},$$

respectively. Then

$$\sum_m V_{2+\epsilon}(Y_Nf_m:N\in\mathbb{N})\lesssim\sum_m V_{2+\epsilon}^L(Y_Nf_m:N\in\mathbb{N})+V_{2+\epsilon}^S(Y_Nf_m:N\in\mathbb{N}).\quad (5.6)$$

We first estimate $\ell^{1+\epsilon}$-norm of long variations (see [27]).

5.1. Long variations. Let $\beta\in\mathbb{N}$ which value will be determined later. Take $\rho\in(0,1)$ and $0<\chi<\frac{1}{10}\min\{1,1+2\epsilon\}$ where $(1+2\epsilon)$ is the constant from Lemma 3. For each $n\in\mathbb{N}$, we define the multiplier

$$\Xi_n^\beta(\xi^m)=\sum_{\frac{1+\epsilon}{1+2\epsilon}\in\mathcal{U}_{[n^\rho]}^\beta}\sum_m\eta_n\left(\xi^m-(1+\epsilon)/(1+2\epsilon)\right),$$

where the sets $\mathcal{U}_{[n^\rho]}^\beta$ are given by (2.9) and

$$\eta_n(\xi^m)=\eta\left(2^{-\chi^{\sqrt{\log N_n}}}N_n^A\xi^m\right).$$

We write

$$\left\|\sum_m V_{2+\epsilon}(Y_{N_n}f_m:n\in\mathbb{N})\right\|_{\ell^{1+\epsilon}}=\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=1}^n\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\hat{f}_m\right):n\in\mathbb{N}\right)\right\|_{\ell^{1+\epsilon}}$$
$$\le\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=1}^n\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_j^\beta\hat{f}_m\right):n\in\mathbb{N}\right)\right\|_{\ell^{1+\epsilon}}\quad (5.7)$$
$$+\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=1}^n\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\left(1-\Xi_j^\beta\right)\hat{f}_m\right):n\in\mathbb{N}\right)\right\|_{\ell^{1+\epsilon}}.$$

We now separately estimate each term on the right-hand side of (5.7). We notice that in view of (5.3) and (2.10), we have

$$\left\| \sum_m \mathcal{F}^{-1}\left(\left(\mathfrak{y}_{N_{n+1}} - \mathfrak{y}_{N_n} \right)\left(1 - \Xi_{n+1}^{\beta}\right) \hat{f}_m \right) \right\|_{\ell^{1+\epsilon}} \lesssim \sum_m \| f_m \|_{\ell^{1+\epsilon}} + \sum_m \left\| \mathcal{F}^{-1}\left(\Xi_{n+1}^{\beta} \hat{f}_m \right) \right\|_{\ell^{1+\epsilon}} \leqslant \log(n+1) \sum_m \| f_m \|_{\ell^{1+\epsilon}}. \tag{5.8}$$

In fact, for $\epsilon = 1$, we can gain some decay in n. Given $\alpha > 0$, we select β_α to be determined by Property 4 .

Let $\beta > d\beta_\alpha$. Take any $\xi^m \in \mathbb{T}^d$. By Dirichlet's principle, for each $\gamma \in \Gamma$, there are coprime integers $(1+\epsilon)_\gamma$ and $(1+2\epsilon)_\gamma$, such that $1 \le (1+\epsilon)_\gamma \le (1+2\epsilon)_\gamma \le N_n^{|\gamma|}(\log N_n)^{-\beta/d}$, and

$$\left| \xi_\gamma^m - \frac{(1+\epsilon)_\gamma}{(1+2\epsilon)_\gamma} \right| \le \frac{1}{(1+2\epsilon)_\gamma} N_n^{-|\gamma|} (\log N_n)^{\beta/d}.$$

Suppose that $1 \le (1+2\epsilon)_\gamma \le (\log N_n)^{\beta/d}$, for all $\gamma \in \Gamma$. We set $(1+2\epsilon)' = \operatorname{lcm}\big((1+2\epsilon)_\gamma : \gamma \in \Gamma\big)$ and $(1+\epsilon)'_\gamma = (1+\epsilon)_\gamma (1+2\epsilon)'/(1+2\epsilon)_\gamma$. Observe that for all $\gamma \in \Gamma$, we have

$$\left| \xi_\gamma^m - \frac{(1+\epsilon)_\gamma}{(1+2\epsilon)_\gamma} \right| = \left| \xi_\gamma^m - \frac{(1+\epsilon)'_\gamma}{(1+2\epsilon)'} \right| \le N_n^{-|\gamma|} (\log N_n)^{\beta/d} \le \frac{1}{32d} N_n^{-|\gamma|} \cdot 2^{\chi\sqrt{\log N_n}},$$

provided that

$$32d(\log N_n)^{\beta/d} \le 2^{\chi\sqrt{\log N_n}},$$

which excludes only a finite number of $n's$ depending on β and ρ. In particular, $\eta_n(\xi^m - (1+\epsilon)'/(1+2\epsilon)') = 1$. Since $1 \le (1+2\epsilon)' \le (\log N_n)^\beta, (1+\epsilon)' \in \mathbf{A}_{(1+2\epsilon)'}$, we conclude that $\Xi_{n+1}^\beta(\xi^m) = 1$. Hence, the condition $\Xi_{n+1}^\beta(\xi^m) < 1$ implies that $(\log N_n)^{\beta/d} \le (1+2\epsilon)_\gamma \le N_n^{|\gamma|}(\log N_n)^{-\beta/d}$ for some $\gamma \in \Gamma$. Now, by Property 4 , we obtain

$$\left| \mathfrak{y}_{N_{n+1}}(\xi^m) - \mathfrak{y}_{N_n}(\xi^m) \right| \lesssim (\log N_n)^{-\alpha},$$

which entails that

$$\left\| \sum_m \mathcal{F}^{-1}\left(\left(\eta_{N_{n+1}} - \eta_{N_n} \right)\left(1 - \Xi_{n+1}^{\beta}\right) \hat{f}_m \right) \right\|_{\ell^2} \lesssim \sum_m (\log N_n)^{-\alpha} \| f_m \|_{\ell^2}. \tag{5.9}$$

Interpolation between (5.8) and (5.9), shows that for each $0 < \epsilon < \infty$ and $\alpha > 0$ there is $\beta_{1+\epsilon,\alpha} > 0$ such that for all $\beta > \beta_{1+\epsilon,\alpha}$ and $n \in \mathbb{N}$, we have

$$\left\| \sum_m \mathcal{F}^{-1}\left(\left(\eta_{N_{n+1}} - \eta_{N_n} \right)\left(1 - \Xi_{n+1}^{\beta}\right) \hat{f}_m \right) \right\|_{\ell^{1+\epsilon}} \le C_\alpha (\log N_n)^{-\alpha} \sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

Taking $\beta > \beta_{1+\epsilon, 2\rho^{-1}}$, we get

$$\left\| V_{2+\epsilon} \sum_m \left(\mathcal{F}^{-1}\left(\left(\eta_{N_j} - \eta_{N_{j-1}} \right)\left(1 - \Xi_j^{\beta}\right) \hat{f}_m \right) : n \in \mathbb{N} \right) \right\|_{\ell^{1+\epsilon}} \lesssim \sum_{n \ge 1} \sum_m \left\| \mathcal{F}^{-1}\left(\left(\eta_{N_{n+1}} - \eta_{N_n} \right)\left(1 - \Xi_{n+1}^{\beta}\right) \hat{f}_m \right) \right\|_{\ell^{1+\epsilon}} \lesssim \left(\sum_{n \ge 1} n^{-2} \right) \sum_m \| f_m \|_{\ell^{1+\epsilon}}. \tag{5.10}$$

We now turn to bounding the first term on the right-hand side of (5.7). For each $n \in \mathbb{N}$ and $s \in \{0, \dots, n-1\}$ let us define the multiplier

$$\Xi_{n,s}^\beta(\xi^m) = \sum_{\frac{1+\epsilon}{1+2\epsilon} \in R_s^\beta} \eta_n\left(\xi^m - (1+\epsilon)/(1+2\epsilon)\right),$$

where $\mathcal{R}_s^\beta = \mathcal{U}_{\lfloor (s+1)^\beta \rfloor}^\beta \setminus \mathcal{U}_{\lfloor s^\rho \rfloor}^\beta$. By the triangle inequality we can write

$$\left\| \sum_m V_{2+\epsilon}\left(\sum_{j=1}^n \sum_{s=0}^{j-1} \mathcal{F}^{-1}\left(\left(\mathfrak{y}_{N_j} - \mathfrak{y}_{N_{j-1}} \right) \Xi_{j,s}^\beta \hat{f}_m \right) : n \in \mathbb{N} \right) \right\|_{\ell^{1+\epsilon}} \le \sum_{s=0}^{\infty} \left\| \sum_m V_{2+\epsilon}\left(\sum_{j=s+1}^n \mathcal{F}^{-1}\left(\left(\mathfrak{y}_{N_j} - \mathfrak{y}_{N_{j-1}} \right) \Xi_{j,s}^\beta \hat{f}_m \right) : s < n \right) \right\|_{\ell^{1+\epsilon}}. \tag{5.11}$$

Thus, the aim is to show that for each $\beta \in \mathbb{N}, 0 < \epsilon < \infty, s \in \mathbb{N}_0$,

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=s+1}^{n} \mathcal{F}^{-1}\left(\left(\mathfrak{y}_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right): s<n\right)\right\|_{\ell^{1+\epsilon}} \lesssim (s+1)^{-2}\sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

We split the variational seminorm into two parts: $s < n \le 2^{\kappa_s}$ and $2^{\kappa_s} < n$, where

$$\kappa_s = 20d\left(\lfloor \rho^{-1}(s+1)^{\rho/10}\rfloor + 1\right).$$

We begin with $\epsilon = 1$ and $s < n \le 2^{\kappa_s}$.

Theorem 5 (see [27]). For each $\beta \in \mathbb{N}$ there is $\epsilon \ge 0$ such that for all $s \in \mathbb{N}_0$, $0 < \epsilon < \infty$ and any finitely supported function $f_m: \mathbb{Z}^d \to \mathbb{C}$, we have

$$\left\|V_{2+\epsilon}\left(\sum_m \sum_{j=s+1}^{n} \mathcal{F}^{-1}\left(\left(\mathfrak{y}_{N_j}-\mathfrak{y}_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right): s<n\le 2^{(K_m)_s}\right)\right\|_{\ell^2} \le (1+\epsilon)(s+1)^{-\delta\beta\rho+2}\sum_m \| f_m \|_{\ell^2},$$

where δ is determined in Theorem 3.

Proof. First, let us see that for each $m_0 > s$, supports of functions $\eta_{m_0}(\cdot -(1+\epsilon)/(1+2\epsilon))$ are disjoint while $(1+\epsilon)/(1+2\epsilon)$ varies over $\mathcal{R}_s^{\beta}$. Indeed, otherwise there would be $(1+\epsilon)/(1+2\epsilon), (1+\epsilon)'/(1+2\epsilon)' \in \mathcal{A}_s^{\beta}, (1+\epsilon)'/(1+2\epsilon)' \neq (1+\epsilon)/(1+2\epsilon)$ and $\xi^m \in \mathrm{T}^d$, such that $\eta_{m_0}(\xi^m - (1+\epsilon)/(1+2\epsilon)) > 0$ and $\eta_{m_0}(\xi^m - (1+\epsilon)'/(1+\epsilon)') > 0$. Hence,

$$e^{-2(s+1)^{\rho/10}} \le \frac{1}{(1+2\epsilon)(1+2\epsilon)'} \le \left|\frac{1+\epsilon}{1+2\epsilon} - \frac{(1+\epsilon)'}{(1+2\epsilon)'}\right| \le \left|\xi_\gamma^{\mathrm{m}} - \frac{1+\epsilon}{1+2\epsilon}\right| + \left|\xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)'_\gamma}{(1+2\epsilon)'}\right| \le 2^{-m_0^{\rho}|\gamma|}2^{\chi m_0^{\frac{\rho}{2}}},$$

which is impossible.

Next, we consider the following multiplier

$$\Lambda_{n,s}^{\beta}(\xi^m) = \sum_{\frac{1+\epsilon}{1+2\epsilon}\in R_s^{\beta}} \sum_m G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\left(\Upsilon_{N_n}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right) - \Upsilon_{N_{n-1}}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right)\right)\eta_n\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right).$$

Let us see that $\Lambda_{n,s}^{\beta}$ is sufficiently close to $\left(\mathfrak{y}_{N_n} - \mathfrak{y}_{N_{n-1}}\right)\Xi_{j,s}^{\beta}$. For each $(1+\epsilon)/(1+2\epsilon) \in \mathcal{R}_s^{\beta}$, we have $1+2\epsilon \le \exp\left(\frac{1+2\epsilon}{2}\sqrt{\log N_n}\right)$, thus by (5.5), on the support of $\eta_n(-(1+\epsilon)/(1+2\epsilon))$ we can write

$$\begin{aligned}\left(\mathfrak{y}_{N_n}(\xi^m) - \mathfrak{y}_{N_{n-1}}(\xi^m)\right) = \ & G((1+\epsilon)/(1+2\epsilon))\left(\Upsilon_{N_n}(\xi^m - (1+\epsilon)/(1+2\epsilon)) - \Upsilon_{N_{n-1}}(\xi^m - (1+\epsilon)/(1+2\epsilon)\right) \\ & + O\left((\log N_n)^{-\rho^{-1}-\beta\delta}\right).\end{aligned}$$

Therefore,

$$\left\|\sum_m \mathcal{F}^{-1}\left(\left(\left(\mathfrak{y}_{N_n} - \mathfrak{y}_{N_{n-1}}\right)\Xi_{n,s}^{\beta} - \Lambda_{n,s}^{\beta}\right)\hat{f_m}\right)\right\|_{\ell^2} \le (1+\epsilon)n^{-1-\beta\delta\rho}\sum_m \| f_m \|_{\ell^2},$$

and hence,

$$\|V_{2+\epsilon}\| \le \left\|\sum_m V_{2+\epsilon}\left(\sum_{j=s+1}^{n} \mathcal{F}^{-1}\left(\Lambda_{j,s}^{\beta}\hat{f_m}\right): s<n\le 2^{\kappa_s}\right)\right\|_{\ell^2} + \left(\sum_{n=s+1}^{\infty} n^{-1-\beta\delta\rho}\right)\sum_m \| f_m \|_{\ell^2}. \quad (5{,}12)$$

Therefore, our task is reduced to showing boundedness of the first term on the right-hand side of (5.12). Observe that for $n > s, \eta_n = \eta_n\eta_s$, thus we can write

$$\Lambda_{n,s}^{\beta}\hat{f_m} = \Theta_{n,s}^{\beta}\hat{F_m}.$$

where

$$\Theta_{n,s}^{\beta}(\xi^m) = \sum_{\frac{1+\epsilon}{1+2\epsilon}\in R_s^{\beta}} \sum_m \left(\Upsilon_{N_n}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right) - \Upsilon_{N_{n-1}}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right)\right)\eta_n\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right),$$

and

$$\hat{F_m}(\xi^m) = \sum_{\frac{1+\epsilon}{1+2\epsilon}\in R_s^{\beta}} G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\eta_s\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right)\hat{f_m}(\xi^m).$$

Now, in view of Lemma 1 ,

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=s+1}^{n} \mathcal{F}^{-1}\left(\Theta_{j,s}^{\beta}\hat{f_m}\right): s<n\le 2^{(K_m)_s}\right)\right\|_{\ell^2}$$

$$\leq \sqrt{2}\sum_{i=0}^{(K_m)_s}\left\|\left(\sum_{j=0}^{2^{k_s-i}-1}\left|\sum_{m_0\in I_j^i+s+1}\sum_m \mathcal{F}^{-1}\left(\Theta_{m_0,s}^{\beta}\hat{F_m}\right)\right|^2\right)^{\frac{1}{2}}\right\|_{\ell^2}, \tag{5.13}$$

where $I_j^i=\{j2^i,j2^i+1,\dots,(j+1)2^i-1\}$. Let us consider a fixed $i\in\{0,\dots,\kappa_s\}$. To bound the norm of the square function on the right-hand side of (5.13), we first study its continuous counterpart, that is

$$\left(\sum_{j=0}^{2^{\varkappa_s-i}-1}\left|\sum_{m_0\in I_j^i+s+1}\sum_m \mathcal{F}^{-1}\left(\left(\Upsilon_{N_{m_0}}-\Upsilon_{N_{m_0-1}}\right)\eta_{m_0}\hat{f_m}\right)\right|^2\right)^{1/2}.$$

If $\eta_{m_0}(\xi^m)<1$ then

$|\xi_\gamma^m|\geq \frac{1}{32d}N_{m_0}^{-|\gamma|}2^{\chi\sqrt{\log N_{m_0}}}$, for some $\gamma\in\Gamma$,

thus by Property 2,

$$\sum_m\left|\Upsilon_{N_{m_0}}(\xi^m)-\Upsilon_{N_{m_0-1}}(\xi^m)\right|\lesssim\sum_m\left|N_{m_0}^A\xi^m\right|^{-\frac{1}{d}}\lesssim 2^{-\frac{\chi\sqrt{\log N_{m_0}}}{d}}.$$

Therefore,

$$\left\|\left(\sum_{j=0}^{2^{k_s-i}-1}\left|\sum_{m_0\in I_j^i+s+1}\sum_m \mathcal{F}^{-1}\left(\left(\Upsilon_{N_{m_0}}-\Upsilon_{N_{m_0-1}}\right)(\eta_{m_0}-1)\mathcal{F}f_m\right)\right|^2\right)^{\frac{1}{2}}\right\|_{L^2}$$

$$\leq\sum_{m_0=s}^{2^{k_s}}\sum_m\left\|\mathcal{F}^{-1}\left(\left(\Upsilon_{N_{m_0}}-\Upsilon_{N_{m_0-1}}\right)(\eta_{m_0}-1)\mathcal{F}f_m\right)\right\|_{L^2}$$

$$\lesssim\left(\sum_{m_0\geq 1}2^{-\frac{\chi\sqrt{\log N_{m_0}}}{d}}\right)\sum_m\|f_m\|_{L^2}.$$

Now, by Proposition 5.1, we have

$$\left\|\left(\sum_{j=0}^{2^{k_s-i}-1}\left|\sum_{m_0\in I_j^i+s+1}\sum_m \mathcal{F}^{-1}\left(\left(\Upsilon_{N_{m_0}}-\Upsilon_{N_{m_0-1}}\right)\mathcal{F}f_m\right)\right|^2\right)^{1/2}\right\|_{L^2}\lesssim\sum_m\|f_m\|_{L^2},$$

thus, in view of (2.10), we conclude that

$$\left\|\left(\sum_{j=0}^{2^{k_s-i}-1}\left|\sum_{m_0\in I_j^i+s+1}\sum_m \mathcal{F}^{-1}\left(\Theta_{m_0,s}^{\beta}\hat{F_m}\right)\right|^2\right)^{1/2}\right\|_{\ell^2}\lesssim\log(s+2)\sum_m\|F_m\|_{\ell^2}.$$

Therefore, by (5.13), we arrive at the

$$\left\|V_{2+\epsilon}\left(\textstyle\sum_{j=s+1}^n\sum_m \mathcal{F}^{-1}\left(\Theta_{j,s}^{\beta}\hat{f_m}\right): s<n\leq 2^{(K_m)_s}\right)\right\|_{\ell^2} s\kappa_s\log(s+2)\sum_m\|F_m\|_{\ell^2}. \tag{5.14}$$

Finally, by Plancherel's theorem

$$\|F_m\|_{\ell^2}^2=\sum_{\frac{1+\epsilon}{1+2\epsilon}\in R_s^\beta}\sum_m|G((1+\epsilon)/(1+2\epsilon))|^2\int_{\mathbb{T}^d}\eta_s(\xi^m-(1+\epsilon)/(1+2\epsilon))^2|\hat{f_m}(\xi^m)|^2\,d\xi^m,$$

and hence, by Theorem 3 ,

$$\|F_m\|_{\ell^2}\lesssim(s+1)^{-\beta\delta\rho}\sum_m\|f_m\|_{\ell^2},$$

which together with (5.14) and (5.12) concludes the proof.

Theorem 6 (see [27]). For each $\beta \in \mathbb{N}$ and $0 < \epsilon < \infty$ there is $\epsilon \geq 0$, such that for all $s \in \mathbb{N}_0, 0 < \epsilon < \infty$, and any finitely supported function $f_m: \mathbb{Z}^d \to \mathbb{C}$, we have

$$\left\| V_{2+\epsilon} \sum_m \left(\sum_{j=s+1}^{n} \mathcal{F}^{-1}\left(\left(\eta_{N_j} - \eta_{N_{j-1}} \right) \Xi_{j,s}^{\beta} \hat{f_m} \right) : s < n \leq 2^{\kappa_s} \right) \right\|_{\ell^{1+\epsilon}} \leq (1+\epsilon)(s+1)\log(s+2) \sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

Proof. For the proof, let us consider the following multiplier

$$\Pi_{n,s}^{\beta}(\xi^m) = \sum_{\frac{1+\epsilon}{1+2\epsilon} \in R_s^{\beta}} \sum_m \left(\Upsilon_{N_n}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right) - \Upsilon_{N_{n-1}}\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right) \right) \eta_s(\xi^m - (1+\epsilon)/(1+2\epsilon)).$$

Fix $s < n_1 < n_2 \leq \min\{2^{\kappa_x}, 2n_1\}$. Let $J_{n_1} = N_{n_1} 2^{-3\chi\sqrt{\log N_{n_1}}}$. We claim the following holds true.

Claim 2. For each $\beta \in \mathbb{N}$ and $0 < \epsilon < \infty$ there is $\epsilon \geq 0$, such that for all $n_1 \leq n \leq n_2 \leq 2n_1$,

$$\left\| \sum_m \mathcal{F}^{-1}\left(\left((\eta_{N_n} - \eta_{N_{n-1}}) \Xi_{n,s}^{\beta} - (\mathrm{m}_0)_{J_{n_1}} \Pi_{n,s}^{\beta} \right) \hat{f_m} \right) \right\|_{\ell^{1+\epsilon}} \leq (1+\epsilon) n^{-2} \sum_m \| f_m \|_{\ell^{1+\epsilon}}. \qquad (5.15)$$

The constant $(1+\epsilon)$ is independent of n_1 and n_2.

Let us first observe that, by (5.3), we can write

$$\begin{aligned} &\| \sum_m \mathcal{F}^{-1}\left(\left((\eta_{N_n} - \eta_{N_{n-1}}) \Xi_{n,s}^{\beta} - (\mathrm{m}_0)_{J_{n_1}} \Pi_{n,s}^{\beta} \right) \hat{f_m} \right) \|_{\ell^{1+\epsilon}} \\ &\quad \leq \sum_m \left\| \mathcal{F}^{-1}\left((\eta_{N_n} - \eta_{N_{n-1}}) \Xi_{n,s}^{\beta} \hat{f_m} \right) \right\|_{\ell^{1+\epsilon}} + \sum_m \left\| \mathcal{F}^{-1}\left((\mathrm{m}_0)_{J_{n_1}} \Pi_{n,s}^{\beta} \hat{f_m} \right) \right\|_{\ell^{1+\epsilon}} \\ &\quad \leqq \sum_m \left\| \mathcal{F}^{-1}\left(\Xi_{n,s}^{\beta} \hat{f_m} \right) \right\|_{\ell^{1+\epsilon}} + \sum_m \left\| \mathcal{F}^{-1}\left(\Pi_{n,s}^{\beta} \hat{f_m} \right) \right\|_{\ell^{1+\epsilon}}, \end{aligned}$$

thus, by (2.10),

$$\left\| \sum_m \mathcal{F}^{-1}\left(\left((\eta_{N_n} - \eta_{N_{n-1}}) \Xi_{n,s}^{\beta} - (\mathrm{m}_0)_{J_{n_1}} \Pi_{n,s}^{\beta} \right) \hat{f_m} \right) \right\|_{\ell^{1+\epsilon}} \lessdot \log(n+1) \sum_m \| f_m \|_{\ell^{1+\epsilon}}. \qquad (5.16)$$

We can improve the estimate for $\epsilon = 1$. Namely, we are going to show that for each $\alpha > 0$, and $n_1 \leq n \leq n_2 \leq 2n_1$,

$$\left\| \sum_m \mathcal{F}^{-1}\left(\left((\eta_{N_n} - \eta_{N_{n-1}}) \Xi_{n,s}^{\beta} - (\mathrm{m}_0)_{J_{n_1}} \Pi_{n,s}^{\beta} \right) \hat{f_m} \right) \right\|_{\ell^2} \lessdot n^{-\alpha\rho} \sum_m \| f_m \|_{\ell^2}. \qquad (5.17)$$

Given $\alpha > 0$, let $(1+2\epsilon)$ be the minimal value among those determined in Lemma 3 and Lemma 5 . Then for each $\frac{1+\epsilon}{1+2\epsilon} \in R_s^{\beta}$,

$$\begin{aligned} &\left(\eta_{N_n}(\xi) - \eta_{N_{n-1}}(\xi^m) \right) \eta_n(\xi^m - 1 + \epsilon/1 + 2\epsilon) \\ &\quad - m_{J_{n_1}}(\xi^m) \left(\Upsilon_{N_n}(\xi^m - 1 + \epsilon/1 + 2\epsilon) - \Upsilon_{N_{n-1}}(\xi^m - 1 + \epsilon/1 + 2\epsilon) \right) \eta_s(\xi^m - 1 + \epsilon/1 + 2\epsilon) \\ &= G(1 + \epsilon/1 + 2\epsilon) \left(\Upsilon_{N_n}(\xi^m - 1 + \epsilon/1 + 2\epsilon) - \Upsilon_{N_{n-1}}(\xi^m - 1 + \epsilon/1 + 2\epsilon) \right) (1 \\ &\quad - \Phi_{J_{n_1}}(\xi^m - 1 + \epsilon/1 + 2\epsilon)) \eta_n(\xi^m - 1 + \epsilon/1 + 2\epsilon) \\ &+ (m_0)_{J_{n_1}}(\xi^m) \left(\Upsilon_{N_n}(\xi^m - 1 + \epsilon/1 + 2\epsilon) - \Upsilon_{N_{n-1}}(\xi^m - 1 + \epsilon/1 + 2\epsilon) \right) (\eta_s(\xi^m - 1 + \epsilon/1 + 2\epsilon) \\ &\quad - \eta_n(\xi^m - 1 + \epsilon/1 + 2\epsilon)) \\ &\quad + O((\log N_n)^{-\alpha}) \eta_s(\xi^m - 1 + \epsilon/1 + 2\epsilon). \end{aligned}$$

If $\eta_s(\xi^m - 1 + \epsilon/1 + 2\epsilon) - \eta_n(\xi^m - 1 + \epsilon/1 + 2\epsilon) \neq 0$, then $\left| \xi_\gamma^m - \frac{(1+\epsilon)_\gamma}{1+2\epsilon} \right| \geq \frac{1}{32d} N_n^{-|\gamma|} 2^{\chi\sqrt{\log N_n}}$, for some $\gamma \in \Gamma$,

thus, by Property 2,

$$\begin{aligned} \sum_m \left| \Upsilon_{N_n}(\xi^m - 1 + \epsilon/1 + 2\epsilon) - \Upsilon_{N_{n-1}}(\xi^m - 1 + \epsilon/1 + 2\epsilon) \right| &\lesssim \sum_m \left| N_n^A \left(\xi^m - \frac{1+\epsilon}{1+2\epsilon} \right) \right|_\infty^{-\frac{1}{d}} \\ &\lesssim 2^{-\frac{\chi\sqrt{\log N_n}}{d}}. \end{aligned}$$

Moreover, if $\eta_n(\xi^m - 1 + \epsilon/1 + 2\epsilon) > 0$ then

$$\sum_m \left| \xi_\gamma^m - \frac{(1+\epsilon)_\gamma}{1+2\epsilon} \right| \leq \frac{1}{16d} N_n^{-|\gamma|} 2^{\chi\sqrt{\log N_n}} \leq J_{n_1}^{-|\gamma|} 2^{-\chi\sqrt{\log N_{n_1}}}, \text{ for all } \gamma \in \Gamma,$$

hence, by (4.2), we obtain

$$\sum_m \left|1-\Phi_{J_{n_1}}(\xi^m-1+\epsilon/1+2\epsilon)\right| \lesssim \sum_m \left|J_{n_1}^A\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right)\right|_\infty \lesssim 2^{-\chi\sqrt{\log N_{n_1}}}.$$

Therefore,

$$\begin{aligned}&\left(\eta_{N_n}(\xi)-\eta_{N_{n-1}}(\xi^m)\right)\eta_n(\xi^m-1+\epsilon/1+2\epsilon)\\&\quad-(m_0)_{J_{n_1}}(\xi^m)\left(\Upsilon_{N_n}(\xi^m-1+\epsilon/1+2\epsilon)-\Upsilon_{N_{n-1}}(\xi^m-1+\epsilon/1+2\epsilon)\right)\eta_s(\xi^m-1+\epsilon/1+2\epsilon)\\&\quad+O((\log N_n)^{-\alpha})\eta_s(\xi^m-1+\epsilon/1+2\epsilon).\end{aligned}\tag{5.18}$$

Since the functions $\eta_s(-1+\epsilon/1+2\epsilon)$ have disjoint supports provided that $(1+\epsilon)\in \mathbf{A}_{1+2\epsilon}$ and $1\le 1+2\epsilon\le e^{(s+1)^{\rho/10}}$, by (5.18) and Plancherel's theorem we conclude (5.17). Now, by interpolation between (5.17) and (5.16) we arrive at (5.15).

With a help of Claim 2, we obtain

$$\begin{aligned}&\left\|V_{2+\epsilon}\left(\sum_{j=n_1}^{n}\sum_m \mathcal{F}^{-1}\left(\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}-(m_0)_{J_{n_1}}\Pi_{j,s}^{\beta}\right)\hat{f_m}\right):n_1\le n\le n_2\right)\right\|_{\ell^{1+\epsilon}}\\&\lesssim \sum_{n=n_1}^{n_2}\sum_m \left\|\mathcal{F}^{-1}\left(\left((\eta_{N_n}-\eta_{N_{n-1}})\Xi_{n,s}^{\beta}-(m_0)_{J_{n_1}}\Pi_{j,s}^{\beta}\right)\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}}\\&\lesssim \left(\sum_{n=1}^{\infty} n^{-2}\right)\sum_m \|f_m\|_{\ell^{1+\epsilon}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=n_1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):n_1\le n\le n_2\right)\right\|_{\ell^{1+\epsilon}}\\&\lesssim \sum_m \|f_m\|_{l^{1+\epsilon}}+\sum_m \left\|V_{2+\epsilon}\left(\sum_{j=n_1}^{n}\mathcal{F}^{-1}\left((m_0)_{J_{n_1}}\Pi_{j,s}^{\beta}\hat{f_m}\right):n_1\le n\le 2n_1\right)\right\|_{\ell^{1+\epsilon}},\end{aligned}\tag{5.19}$$

with an implied constant independent of n_1. We next claim that the following holds true.

Claim 3. For each $\beta\in\mathbb{N}$ and $0<\epsilon<\infty$ there is $\epsilon\ge 0$, such that for all $s\in\mathbb{N}_0$. we have

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=0}^{n}\mathcal{F}^{-1}\left(\Pi_{j,s}^{\beta}\hat{f_m}\right):0\le n\le 2^{(K_m)_s}\right)\right\|_{\ell^{1+\epsilon}}\le (1+\epsilon)\kappa_s\log(s+2)\sum_m \|f_m\|_{\ell^{1+\epsilon}}.\tag{5.20}$$

Let us see that (5.20) suffices to finish the proof of the theorem. Indeed, (5.19) together with (5.20) imply that

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=n_1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):n_1\le n\le n_2\right)\right\|_{\ell^{1+\epsilon}}\lesssim \kappa_s\log(s+2)\sum_m \|f_m\|_{\ell^{1+\epsilon}}.$$

Therefore, by (2.2) and Minkowski's inequality

$$\begin{aligned}&\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=n_1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):n_1\le n\le n_2\right)\right\|_{\ell^{1+\epsilon}}\\&\lesssim \kappa_s^{\frac{1+\epsilon}{2+\epsilon}}\left(\sum_{\log_2 s\le m_0\le k_s}\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=n_1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):2^{m_0}\le n\le 2^{m_0+1}\right)\right\|_{\ell^{1+\epsilon}}^{2+\epsilon}\right)^{1/2+\epsilon}\\&\lesssim \kappa_s\max_{\substack{s<n_1<n_2\le 2n_1\\ n_2\le 2^{k_s}}}\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=n_1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):n_1\le n\le n_2\right)\right\|_{\ell^{1+\epsilon}}\\&\lesssim \kappa_s^2\log(s+2)\sum_m \|f_m\|_{\ell^{1+\epsilon}}.\end{aligned}$$

It remains to prove Claim 3. By Lemma 1, we can write

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=0}^{n}\mathcal{F}^{-1}\left(\Pi_{j,s}^{\beta}\hat{f_m}\right): 0\le n\le 2^{(K_m)_s}\right)\right\|_{\ell^{1+\epsilon}}$$

$$\le \sqrt{2}\sum_{i=0}^{(K_m)_s}\left\|\sum_m\left(\sum_{j=0}^{2^{\kappa_s-i}-1}\left|\sum_{m_0\in I_i^i}\mathcal{F}^{-1}\left(\left(\Pi_{m_0+1,s}^{\beta}-\Pi_{m_0,s}^{\beta}\right)\hat{f_m}\right)\right|^2\right)^{1/2}\right\|_{\ell^{1+\epsilon}}, \quad (5.21)$$

where $I_j^i=\{j2^i, j2^i+1,\dots,(j+1)2^i-1\}$. Let us fix $i\in\{0,1,\dots\kappa_s\}$. In view of Proposition 5.1,

$$\left\|\left(\sum_{j=0}^{2^{\kappa_s-i}-1}\sum_m\left|\sum_{m_0\in I_j^i}\mathcal{F}^{-1}\left(\left(\Upsilon_{N_{m_0+1}}-\Upsilon_{N_{m_0}}\right)\hat{f_m}\right)\right|^2\right)^{1/2}\right\|_{L^{1+\epsilon}}\lesssim\sum_m \| f_m \|_{L^{1+\epsilon}},$$

where the implied constant is independent of i. Hence, by (2.10), we obtain

$$\left\|\left(\sum_{j=0}^{2^{\kappa s-i}-1}\sum_m\left|\sum_{m_0\in I_j^i}\mathcal{F}^{-1}\left(\left(\Pi_{m_0+1,s}^{\beta}-\Pi_{m_0,s}^{\beta}\right)\hat{f_m}\right)\right|^2\right)^{1/2}\right\|_{\ell^{1+\epsilon}}\lesssim\log(s+2)\sum_m \| f_m \|_{l^{1+\epsilon}},$$

which together with (5.21) implies (5.20).

We now turn to studying the part of the variational seminorm where $2^{(K_m)_s}<n$. For $s\in\mathbb{N}_0$ we set

$$(Q_m)_s=\left(\left\lfloor e^{(s+1)^{\rho/10}}\right\rfloor\right)!$$

Theorem 7 (see [27]). For each $\beta\in\mathbb{N}$ there is $\epsilon\ge 0$, such that for all $0<\epsilon<\infty, s\in\mathbb{N}_0$, and any finitely supported function $f_m:\mathbb{Z}^d\to\mathbb{C}$, we have

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=2^{ks}+1}^{n}\mathcal{F}^{-1}\left(\left(\mathfrak{v}_{N_j}-\mathfrak{v}_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right): 2^{\kappa_s}<n\right)\right\|_{\ell^2}\le(1+\epsilon)\frac{2+\epsilon}{\epsilon}(s+1)^{-\delta\beta\rho}\sum_m \| f_m \|_{\ell^2},$$

where δ is determined in Theorem 3 .

Proof. Let us define

$$\Omega_{n,s}^{\beta}=\sum_{\frac{1+\epsilon}{1+2\epsilon}\in R_s^{\beta}}\sum_m G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\left(\Upsilon_{N_n}\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right)-\Upsilon_{N_{n-1}}\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right)\right)\varrho_s\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right),$$

where

$$\varrho_s(\xi^m)=\eta\left((Q_m)_{s+1}^{3dA}\xi^m\right).$$

Our first goal is to show that the multipliers $\Omega_{n,s}^{\beta}$ approximate $\left(\mathfrak{v}_{N_n}-\mathfrak{v}_{N_{n-1}}\right)\Xi_{n,s}^{\beta}$ well.

Claim 4. For each $\beta\in\mathbb{N}$ there is $\epsilon\ge 0$, such that for all $s\in\mathbb{N}_0$, and $n>2^{(K_m)_s}$,

$$\left\|\sum_m\mathcal{F}^{-1}\left(\left(\left(\eta_{N_n}-\eta_{N_{n-1}}\right)\Xi_{n,s}^{\beta}-\Omega_{n,s}^{\beta}\right)\hat{f_m}\right)\right\|_{\ell^2}\le(1+\epsilon)\cdot 2^{-\frac{\chi\sqrt{\log N_n}}{d}}\sum_m \| f_m \|_{\ell^2}. \quad (5.22)$$

Since $n>2^{\kappa_s}$, for each $(1+\epsilon)/(1+2\epsilon)\in\mathcal{R}_s^{\beta}$ we have $1+2\epsilon\le\log N_n$. Therefore, by (5.4), we obtain

$$\left(\mathrm{v}_{N_n}(\xi^m)-\mathrm{v}_{N_{n-1}}(\xi^m)\right)\eta_n(\xi^m-1+\epsilon/1+2\epsilon)-G(1+\epsilon/1+2\epsilon)\left(\Upsilon_{N_n}(\xi^m-1+\epsilon/1+2\epsilon)-\Upsilon_{N_{n-1}}(\xi^m-1+\epsilon/1+2\epsilon)\right)\varrho_s(\xi^m-1+\epsilon/1+2\epsilon)$$
$$=G(1+\epsilon/1+2\epsilon)\left(\Upsilon_{N_n}(\xi^m-1+\epsilon/1+2\epsilon)-\Upsilon_{N_{n-1}}(\xi^m-1+\epsilon/1+2\epsilon)\right)\left(\eta_n(\xi^m-1+\epsilon/1+2\epsilon)-\varrho_s(\xi^m-1+\epsilon/1+2\epsilon)\right)$$
$$+O\left(\exp\left((\chi\log 2-1+2\epsilon)\sqrt{\log N_n}\right)\right).$$

Next, if $\varrho_s(\xi^m-1+\epsilon/1+2\epsilon)-\eta_n(\xi^m-1+\epsilon/1+2\epsilon)\ne 0$, then

$\left|\xi_{\gamma}^{m}-\frac{(1+\epsilon)_\gamma}{1+2\epsilon}\right|\ge\frac{1}{32d}N_n^{-|\gamma|}2^{\chi\sqrt{\log N_n}}$, for some $\gamma\in\Gamma$,

and thus, by (5.1), we have

$$\sum_m\left|\Upsilon_{N_n}(\xi^m-1+\epsilon/1+2\epsilon)-\Upsilon_{N_{n-1}}(\xi^m-1+\epsilon/1+2\epsilon)\right|\lesssim\sum_m\left|N_n^A\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right)\right|_\infty^{-\frac{1}{d}}\lesssim 2^{-\frac{\chi\sqrt{\log N_n}}{d}}.$$

Hence,

$$\left(\eta_{N_n}(\xi^m)-\eta_{N_{n-1}}(\xi^m)\right)\eta_n(\xi^m-1+\epsilon/1+2\epsilon)=G(1+\epsilon/1+2\epsilon)\left(\Upsilon_{N_n}(\xi^m-1+\epsilon/1+2\epsilon)-\Upsilon_{N_{n-1}}(\xi^m-1+\epsilon/1+2\epsilon)\right)\varrho_s(\xi^m-1+\epsilon/1+2\epsilon)$$
$$+O\left(2^{-\chi\sqrt{\log N_n}/d}\right).$$

Since the functions $\eta_s(\cdot-(1+\epsilon)/(1+2\epsilon))$ have disjoint supports while $1+\epsilon/1+2\epsilon$ varies over $\mathcal{R}_s^{\beta}$, by Plancherel's theorem we obtain (5.22).

Now, by applying Claim 4 ,

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=2^{ks}+1}^{n} \mathcal{F}^{-1}\left(\left(\mathfrak{y}_{N_j}-\mathfrak{y}_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f}_m\right): 2^{K_s}<n\right)\right\|_{\ell^2}$$

$$\lesssim (s+1)^{-\delta\beta\rho}\sum_m\left(\sum_{n=2^{k_s}+1}^{\infty} 2^{-\frac{\chi\sqrt{\log N_n}}{2d}}\right)\| f_m \|_{\ell^2},$$

thus

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=2^{ks}+1}^{n} \mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f}_m\right): 2^{(K_m)_s}<n\right)\right\|_{\ell^2}$$

$$\le (s+1)^{-\delta\beta\rho}\sum_m \| f_m \|_{\ell^{1+\epsilon}}+\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=2^{ks}+1}^{n} \mathcal{F}^{-1}\left(\Omega_{j,s}^{\beta}\hat{f}_m\right): 2^{(K_m)_s}<n\right)\right\|_{\ell^2}.$$

Our next task is to show that there is $\epsilon \ge 0$ such that

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=2^{Ks}+1}^{n} \mathcal{F}^{-1}\left(\Omega_{j,s}^{\beta}\hat{f}_m\right): 2^{(K_m)_s}<n\right)\right\|_{\ell^2} \le (1+\epsilon)(s+1)^{-\delta\beta\rho}\sum_m \| f_m \|_{\ell^2}. \quad (5.23)$$

For the proof, let us define

$$I(x,y)=V_{2+\epsilon}\left(\sum_{1+\epsilon/1+2\epsilon\in R_s^{\beta}}\sum_m G(1+\epsilon/1+2\epsilon)e^{-2\pi i(1+\epsilon/1+2\epsilon)\cdot x}\sum_{j=2^{ks}+1}^{n}\mathcal{F}^{-1}\left(\left(\Upsilon_{N_j}-\Upsilon_{N_{j-1}}\right)\varrho_s\hat{f}_m(\cdot+1+\epsilon/1+2\epsilon)\right)(y): 2^{(K_m)_s}<n\right),$$

and

$$J(x,y)=\sum_{\frac{1+\epsilon}{1+2\epsilon}\in R_s^{\beta}}\sum_m G\left(\frac{1+\epsilon}{1+2\epsilon}\right)e^{-2\pi i\left(\frac{1+\epsilon}{1+2\epsilon}\right)\cdot x}\mathcal{F}^{-1}\left(\varrho_s\hat{f}_m\left(\cdot+\frac{1+\epsilon}{1+2\epsilon}\right)\right)\Bigg)(y).$$

By Plancherel's theorem, for any $u_m \in \mathbb{N}^d_{(Q_m)_s}$ and $1+\epsilon/1+2\epsilon \in \mathcal{R}_s^{\beta}$, we have

$$\Bigg\|\sum_m \mathcal{F}^{-1}\left(\left(\Upsilon_{n_j}-\Upsilon_{n_{j-1}}\right)\varrho_s\hat{f}_m(\cdot+1+\epsilon/1+2\epsilon)\right)(x+u_m)$$

$$-\sum_m \mathcal{F}^{-1}\left(\left(\Upsilon_{n_j}-\Upsilon_{n_{j-1}}\right)\varrho_s\hat{f}_m(\cdot+1+\epsilon/1+2\epsilon)\right)(x)\Bigg\|_{\ell^2(x)}$$

$$=\sum_m \left\|\left(1-e^{-2\pi i\xi^m\cdot u_m}\right)\left(\Upsilon_{n_j}(\xi)-\Upsilon_{n_{j-1}}(\xi^m)\right)\varrho_s(\xi^m)\hat{f}_m(\xi^m+1+\epsilon/1+2\epsilon)\right\|_{L^2(d\xi^m)}$$

$$\lesssim \sum_m N_j^{-1/d}|u_m|\cdot\left\|\varrho_s(\cdot+1+\epsilon/1+2\epsilon)\hat{f}_m\right\|_{L^2},$$

because in view of (5.1), for each $\xi^m \in \mathbb{T}^d$,

$$\sum_m |\xi^m|\cdot\left|\Upsilon_{N_j}(\xi^m)-\Upsilon_{N_{j-1}}(\xi^m)\right| \lesssim \sum_m N_j^{-1/d}|\xi^m|_\infty^{\mathrm{d}-1/d} \lesssim N_j^{-1/d}.$$

Therefore,

$$\left|\| I(x,x+u_m) \|_{\ell^2(x)} - \| I(x,x) \|_{\ell^2(x)}\right| \lesssim \sum_m |u_m|\left(\sum_{j=2^{(K_m)_s+1}}^{\infty} N_j^{-\frac{1}{d}}\right)\sum_{\frac{1+\epsilon}{1+2\epsilon}\in R_s^{\beta}}\left\|\varrho_s\left(\cdot-\frac{1+\epsilon}{1+2\epsilon}\right)\hat{f}_m\right\|_{L^2}.$$

Since the set $\mathcal{R}_s^{\beta}$ has at most $e^{(d+1)(s+1)^{\rho/10}}$ elements, and

$$(d+1)(s+1)^{\frac{\rho}{10}}+(s+1)^{\frac{\rho}{10}}e^{(s+1)^{\frac{\rho}{10}}}-\frac{\log 2}{2d}2^{(K_m)_s} \le -(s+1)^{\rho},$$

we obtain

$$\| I(x,x) \|_{\ell^2(x)} \lesssim \sum_m \| I(x,x+u_m) \|_{\ell^2(x)} + 2^{-(s+1)^{\rho}}\sum_m \| f_m \|_{\ell^2}.$$

Hence,

$$\left\| V_{2+\epsilon}\left(\sum_{j=2^{\kappa_s}+1}^{n}\sum_m \mathcal{F}^{-1}\left(\Omega_{j,s}^{\beta}\hat{f_m}\right): 2^{\kappa_s} < n\right)\right\|_{\ell^2}^2 \lesssim \sum_m \frac{1}{(Q_m)_s^d}\sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}} \| I(x,x+u_m)\|^2_{\ell^2(x)} + 2^{-2(s+1)^\rho}\sum_m \| f_m\|^2_{\ell^2}. \quad (5.24)$$

Let us observe that the functions $x \mapsto I(x,y)$ and $x \mapsto J(x,y)$ are $(Q_m)_s Z^d$-periodic. Therefore, by repeated change of variables, we get

$$\sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}}\sum_m \| I(x,x+u_m)\|^2_{\ell^2(x)} = \sum_{x\in Z^d}\sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}}\sum_m I(x-u_m,x)^2 = \sum_{x\in Z^d}\sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}}\sum_m I(u_m,x)^2 = \sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}}\sum_m \| I(u_m,x)\|^2_{\ell^2(x)}.$$

By [12, Proposition 4.1] (see also [15, Proposition 3.2]), Property 1 entails that for each $u_m \in \mathbb{N}^d_{(Q_m)_s}$, we have

$$\begin{aligned}\| I(u_m,x)\|_{\ell^2(x)} &= \left\| V_{2+\epsilon}\left(\sum_{j=2^{K_s}+1}^{n}\sum_m \mathcal{F}^{-1}\left(\left(\Upsilon_{N_j}-\Upsilon_{N_{j-1}}\right)J\left(u_m,\tilde{\cdot}\right)\right): 2^{(K_m)_s} < n\right)\right\|_{\ell^2(x)} \\ &\le (1+\epsilon)\frac{2+\epsilon}{\epsilon}\sum_m \| J(u_m,x)\|_{\ell^2(x)}.\end{aligned}$$

Observe that

$$\sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}}\sum_m \| J(u_m,x)\|^2_{\ell^2(x)} = \sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}}\sum_m \| J(x,x+u_m)\|^2_{\ell^2(x)}.$$

Since by Theorem 3 and disjointness of supports of $\varrho_s(\cdot - 1+\epsilon/1+2\epsilon)$ while $1+\epsilon/1+2\epsilon$ varies over R_s^β, we get

$$\begin{aligned}\|J(x,x+u_m)\|^2_{\ell^2(x)} &= \int_{\mathbb{T}^d}\left|\sum_{\frac{1+\epsilon}{1+2\epsilon}\in\mathcal{A}_s^B}\sum_m G\left(\frac{1+\epsilon}{1+2\epsilon}\right)e^{2\pi i\left(\frac{1+\epsilon}{1+2\epsilon}\right)\cdot u_m}(Q_m)_s\left(\xi^m-\frac{1+\epsilon}{1+2\epsilon}\right)\right|^2\left|\hat{f}_m(\xi^m)\right|^2 d\xi^m \\ &\lesssim (s+1)^{-2\delta\beta\rho}\sum_m \|f_m\|^2_{\ell^2(x)},\end{aligned}$$

we obtain

$$\sum_{u_m\in\mathbb{N}^d_{(Q_m)_s}}\sum_m \| I(x,x+u_m)\|^2_{\ell^2(x)} \lesssim \left(\frac{2+\epsilon}{\epsilon}\right)^2(s+1)^{-2\delta\beta\rho}\sum_m (Q_m)_s^d \| f_m\|^2_{\ell^2},$$

which together with (5.24) implies (5.23) and the proof of theorem is completed.

Theorem 8 (see [27]). For each $\beta\in\mathbb{N}$ and $0<\epsilon<\infty$ there is $\epsilon \ge 0$, such that for all $s\in\mathbb{N}_0$, $0<\epsilon<\infty$, and any finitely supported function $f_m:\mathbb{Z}^d\to\mathbb{C}$,

$$\left\| V_{2+\epsilon}\left(\sum_{j=2^{k_s+1}}^{n}\sum_m \mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right): 2^{(K_m)_s} < n\right)\right\|_{\ell^{1+\epsilon}} \le (1+\epsilon)\frac{2+\epsilon}{\epsilon}\log(s+2)\sum_m \| f_m\|_{\ell^{1+\epsilon}}.$$

Proof. First, we are going to refine Claim 4.

Claim 5. For each $\beta\in\mathbb{N}$ and $0<\epsilon<\infty$ there is $c_{1+\epsilon}>0$ such that for all $s\in\mathbb{N}_0$, and $n>2^{\kappa_s}$,

$$\left\|\sum_m \mathcal{F}^{-1}\left(\left(\left(\eta_{N_n}-\eta_{N_{n-1}}\right)\Xi_{n,s}^{\beta}-\Omega_{n,s}^{\beta}\right)\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}} \le (1+\epsilon)\cdot 2^{-\chi c_{1+\epsilon}\sqrt{\log N_n}}\sum_m \| f_m\|_{\ell^{1+\epsilon}}. \quad (5.25)$$

We notice the following trivial bound

$$\left\|\sum_m \mathcal{F}^{-1}\left(\Omega_{n,s}^{\beta}\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}} \lesssim e^{(d+1)(s+1)^{\frac{\rho}{10}}}\sum_m \| f_m\|_{\ell^{1+\epsilon}} \le (\log N_n)\sum_m \| f_m\|_{\ell^{1+\epsilon}},$$

thus, by (5.3) and (2.10), we also have

$$\begin{aligned}\left\|\sum_m \mathcal{F}^{-1}\left(\left(\left(\eta_{N_n}-\eta_{N_{n-1}}\right)\Xi_{n,s}^{\beta}-\Omega_{n,s}^{\beta}\right)\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}} &\le \left\|\sum_m \mathcal{F}^{-1}\left(\left(\eta_{N_n}-\eta_{N_{n-1}}\right)\Xi_{n,s}^{\beta}\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}} + \left\|\sum_m \mathcal{F}^{-1}\left(\Omega_{n,s}^{\beta}\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}} \\ &\lesssim \sum_m (\log N_n)\sum_m \|f_m\|_{\ell^{1+\epsilon}}. \quad (5.26)\end{aligned}$$

Now, interpolation between (5.26) and (5.22) leads to (5.25).

Next, using Claim 5, we obtain

$$\left\|\sum_{m} V_{2+\epsilon}\left(\sum_{j=2^{(K_m)_s}+1}^{n} \mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}-\Omega_{j,s}^{\beta}\hat{f_m}\right): 2^{\kappa_s}<n\right)\right\|_{\ell^{1+\epsilon}}$$
$$\lesssim \sum_{n=2^{k_s}+1}^{\infty}\sum_{m}\left\|\mathcal{F}^{-1}\left(\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{n,s}^{\beta}-\Omega_{n,s}^{\beta}\right)\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}}$$
$$\lesssim \left(\sum_{n=2^{K_s}+1}^{\infty}\sum_{m} 2^{-\chi c_{1+\epsilon}}\sqrt{\log N_n}\right)\| f_m \|_{\ell^{1+\epsilon}}.$$

Hence,

$$\left\|V_{2+\epsilon}\left(\sum_{j=2^{ks}}^{n}\sum_{m}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right): 2^{(K_m)_s}<n\right)\right\|_{\ell^{1+\epsilon}}$$
$$\leqslant \sum_{m}\| f_m \|_{\ell^{1+\epsilon}}+\left\|V_{2+\epsilon}\left(\sum_{m}\sum_{j=2^{ks}+1}^{n}\mathcal{F}^{-1}\left(\Omega_{j,s}^{\beta}\hat{f_m}\right): 2^{(K_m)_s}<n\right)\right\|_{\ell^{1+\epsilon}},$$

and the proof is reduced to showing the following claim.

Claim 6. For each $\beta\in\mathbb{N}$ and $0<\epsilon<\infty$ there is $\epsilon\geq 0$ such that for all $0<\epsilon<\infty$, and $s\in\mathbb{N}_0$,

$$\left\|V_{2+\epsilon}\left(\sum_{m}\sum_{j=2^{(K_m)_s}+1}^{n}\mathcal{F}^{-1}\left(\Omega_{j,s}^{\beta}\hat{f_m}\right): 2^{\kappa_s}<n\right)\right\|_{\ell^{1+\epsilon}} \leq (1+\epsilon)\frac{2+\epsilon}{\epsilon}\log(s+2)\sum_{m}\| f_m \|_{\ell^{1+\epsilon}}.$$

For any $1+\epsilon/1+2\epsilon\in\mathcal{H}_s^{\beta}, x\in\mathbb{Z}^k$ and $m_0\in\mathbb{N}_{(Q_m)_s}^k$, we have

$$\mathcal{F}^{-1}\left(\left(\Upsilon_{N_n}(\cdot-1+\epsilon/1+2\epsilon)-\Upsilon_{N_{n-1}}(\cdot-1+\epsilon/1+2\epsilon)\right)\varrho_s(\cdot-1+\epsilon/1+2\epsilon)\hat{f_m}\right)((Q_m)_s x+m_0)$$
$$=\mathcal{F}^{-1}\left(\left(\Upsilon_{N_n}-\Upsilon_{N_{n-1}}\right)\varrho_s\hat{f_m}(\cdot+1+\epsilon/1+2\epsilon)\right)((Q_m)_s x+m_0)e^{-2\pi i(1+\epsilon/1+2\epsilon)\cdot m_0}.$$

Therefore,

$$\left\|V_{2+\epsilon}\left(\sum_{j=2^{k_s}+1}^{n}\sum_{m}\mathcal{F}^{-1}\left(\Omega_{j,s}^{\beta}\hat{f_m}\right): 2^{\kappa_s}<n\right)\right\|_{\ell^{1+\epsilon}}^{1+\epsilon}$$
$$=\sum_{m_0\in\mathbb{N}_{(Q_m)_s}^k}\sum_{m}\left\|V_{2+\epsilon}\left(\sum_{j=2^{k_s}+1}^{n}\mathcal{F}^{-1}\left(\left(\Upsilon_{N_j}-\Upsilon_{N_{j-1}}\right)\varrho_s F_m(\cdot;m_0)\right)((Q_m)_s x+m_0): 2^{\kappa_s}<n\right)\right\|_{\mathcal{R}^{1+\epsilon}(x)}^{1+\epsilon},$$

where

$$F_m(\xi^m;m_0)=\sum_{1+\epsilon/1+2\epsilon\in R_s^{\beta}}\sum_{m} G(1+\epsilon/1+2\epsilon)\hat{f_m}(\xi^m+1+\epsilon/1+2\epsilon)e^{-2\pi i(1+\epsilon/1+2\epsilon)\cdot m_0}.$$

By [12, Proposition 4.2] (see also [15, Proposition 3.2]), we can write

$$\sum_{m_0\in\mathbb{N}_{(Q_m)_s}^k}\sum_{m}\| V_{2+\epsilon}\left(\sum_{j=2^{k_s}+1}^{n}\mathcal{F}^{-1}\left(\left(\left(\Upsilon_{N_j}-\Upsilon_{N_{j-1}}\right)\varrho_s F_m(\cdot;m_0)\right)((Q_m)_s x+m_0): 2^{\kappa_s}<n\right)\|_{\ell^{1+\epsilon}(x)}^{1+\epsilon}$$
$$\leq\left(\frac{(1+\epsilon)(2+\epsilon)}{\epsilon}\right)^{1+\epsilon}\left\|\sum_{1+\epsilon/1+2\epsilon\in R_s^{\beta}}\sum_{m} G(1+\epsilon/1+2\epsilon)\mathcal{F}^{-1}\left(\varrho_s(\cdot-1+\epsilon/1+2\epsilon)\hat{f_m}\right)\right\|_{\mathcal{R}^{1+\epsilon}}^{1+\epsilon}.$$

Therefore, the problem is reduced to showing

$$\left\|\sum_{1+\epsilon/1+2\epsilon\in R_s^{\beta}}\sum_{m} G(1+\epsilon/1+2\epsilon)\mathcal{F}^{-1}\left(\varrho_s(\cdot-1+\epsilon/1+2\epsilon)\hat{f_m}\right)\right\|_{\ell^{1+\epsilon}} \leq (1+\epsilon)\log(s+2)\sum_{m}\| f_m \|_{\ell^{1+\epsilon}}. \quad (5.27)$$

For the proof, let $N=\left\lfloor e^{(s+1)^{\rho/10}}\right\rfloor+1$ and $J=2^N$. We write

$$\sum_{1+\epsilon/1+2\epsilon\in R_s^{\beta}}\sum_{m} G(1+\epsilon/1+2\epsilon)\mathcal{F}^{-1}\left(\varrho_s(\cdot-1+\epsilon/1+2\epsilon)f_m\right)\|_{\ell^{1+\epsilon}} \leq\sum_{m}\left\|\sum_{1+\epsilon/1+2\epsilon\in R_s^{\beta}}\mathcal{F}^{-1}\left(\left((m_0)_J-G(1+\epsilon/1+2\epsilon)\right)\varrho_s(\cdot-1+\epsilon/1+2\epsilon)f_m\right)\right\|_{\ell^{1+\epsilon}}$$
$$+\sum_{m}\left\|\sum_{1+\epsilon/1+2\epsilon\in R_s^{\beta}}\mathcal{F}^{-1}\left((m_0)_J\varrho_s(\cdot-1+\epsilon/1+2\epsilon)f_m\right)\right\|_{\ell^{1+\epsilon}}.$$

In view of (2.10), we have

$$\left\| \sum_{1+\epsilon/1+2\epsilon\in R_s^\beta} \sum_m \mathcal{F}^{-1}((\mathrm{m}_0)_J \varrho_s(\cdot -1+\epsilon/1+2\epsilon)\hat{f_m}) \right\|_{\ell^{1+\epsilon}} \le \sum_m \left\| \sum_{1+\epsilon/1+2\epsilon\in R_s^\beta} \mathcal{F}^{-1}(\varrho_s(\cdot -1+\epsilon/1+2\epsilon)\hat{f_m}) \right\|_{\ell^{1+\epsilon}} \lesssim \log(s+2) \sum_m \| f_m \|_{\ell^{1+\epsilon}}. \quad (5.28)$$

Next, we have the following trivial bound

$$\sum_{1+\epsilon/1+2\epsilon\in \mathcal{R}_s^\beta} \sum_m \mathcal{F}^{-1}\left(\left((\mathrm{m}_0)_J - G(1+\epsilon/1+2\epsilon)\right)\varrho_s(\cdot -1+\epsilon/1+2\epsilon)\hat{f_m}\right) \|_{\ell^{1+\epsilon}} \le e^{(d+1)(s+1)^{\rho/10}} \sum_m \| f_m \|_{\ell^{1+\epsilon}} \le (\log J)^{d+1} \sum_m \| f_m \|_{\ell^{1+\epsilon}}. \quad (5.29)$$

We want to improve the above estimate for $\epsilon = 1$. We have

$$\left\| \sum_{\frac{1+\epsilon}{1+2\epsilon}\in \mathcal{R}_s^\beta} \sum_m \mathcal{F}^{-1}\left(\left((\mathrm{m}_0)_J - G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\right) \varrho_s\left(\cdot - \frac{1+\epsilon}{1+2\epsilon}\right)\hat{f_m}\right) \right\|_{\ell^2}^2 = \sum_{\frac{1+\epsilon}{1+2\epsilon}\in \mathcal{R}_s^\beta} \int_{\mathbb{T}^d} \sum_m \mathcal{F}^{-1} \left| (\mathrm{m}_0)_J - G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\right|^2 \varrho_s\left(\xi - \frac{1+\epsilon}{1+2\epsilon}\right) \left|\hat{f_m}(\xi^m)\right|^2 d\xi^m. \quad (5.30)$$

Since each fraction $1+\epsilon/1+2\epsilon$ belonging to $\mathcal{R}_s^\beta$ has its denominator $(1+2\epsilon) \le e^{(s+1)^{\rho/10}} \le \log J$, by

$$(m_J(\xi^m) - G(1+\epsilon/1+2\epsilon))\varrho_s(\xi^m - 1+\epsilon/1+2\epsilon) = G(1+\epsilon/1+2\epsilon)(\Phi_J(\xi^m - 1+\epsilon/1+2\epsilon) - 1)\varrho_s(\xi^m - 1+\epsilon/1+2\epsilon) + O(\exp(-(1+2\epsilon)\sqrt{\log J})). \quad (5.31)$$

Proposition 4.1,

If $\varrho_s(\xi^m - 1+\epsilon/1+2\epsilon) > 0$ then

$$\left| \xi_\gamma^{\mathrm{m}} - \frac{(1+\epsilon)_\gamma}{1+2\epsilon} \right| \le (Q_m)_{s+1}^{-3d|\gamma|} \le J^{-2|\gamma|}, \quad \text{for all } \gamma \in \Gamma,$$

thus, by (4.2), we get

$$\left|\Phi_J(\xi^m - 1+\epsilon/1+2\epsilon) - 1\right| \lesssim \left| J^A \left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right)\right|_\infty \lesssim J^{-1}.$$

Hence, (5.31) takes the following form

$$\left((\mathrm{m}_0)_J(\xi^m) - G\left(\frac{1+\epsilon}{1+2\epsilon}\right)\right) \varrho_s\left(\xi^m - \frac{1+\epsilon}{1+2\epsilon}\right) = O\left(\exp(-(1+2\epsilon)\sqrt{\log J})\right).$$

Therefore, by (5.30), we get

$$\left\| \sum_{1+\epsilon/1+2\epsilon\in R_s^\beta} \sum_m \mathcal{F}^{-1}\left(\left((\mathrm{m}_0)_J - G(1+\epsilon/1+2\epsilon)\right)\varrho_s(\cdot -1+\epsilon/1+2\epsilon)\hat{f_m}\right) \right\|_{\ell^2} \lesssim e^{-(1+2\epsilon)\sqrt{\log J}} \sum_m \| f_m \|_{\ell^2}. \quad (5.32)$$

Now, interpolating (5.29) with (5.32), we obtain

$$\left\| \sum_{1+\epsilon/1+2\epsilon\in R_s^\beta} \sum_m \mathcal{F}^{-1}\left(\left((\mathrm{m}_0)_J - G(1+\epsilon/1+2\epsilon)\right)\varrho_s(\cdot -1+\epsilon/1+2\epsilon)\hat{f_m}\right) \right\|_{\ell^{1+\epsilon}} \lesssim \sum_m \| f_m \|_{\ell^{1+\epsilon}},$$

which together with (5.28) implies (5.27), and the proof of the theorem is completed.

Theorem 9 (see [27]). For each $0 < \epsilon < \infty$ and $\rho \in (0,1)$, there is $\epsilon \ge 0$ such that for all $0 < \epsilon < \infty$ and any finitely supported function $f_m: \mathbb{Z}^d \to \mathbb{C}$,

$$\left\| \sum_m V_{2+\epsilon}\left(Y_{N_n} f_m : n \in \mathbb{N}_0\right) \right\|_{\ell^{1+\epsilon}} \le (1+\epsilon)\frac{2+\epsilon}{\epsilon} \sum_m \| f_m \|_{\ell^{1+\epsilon}},$$

where $N_n = \lfloor 2^{n^\rho} \rfloor$.

Proof. In view of (5.7), (5.10) and (5.11), we have

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\hat{f_m}\right):n\in\mathbb{N}\right)\right\|_{\ell^{1+\epsilon}}$$
$$\leq C_{1+\epsilon,\beta}\sum_m \| f_m \|_{\ell^{1+\epsilon}}+\sum_{s=0}^{\infty}\sum_m \left\|V_{2+\epsilon}\left(\sum_{j=s+1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):s<n\right)\right\|_{\ell^{1+\epsilon}},$$

provided $\beta>\beta_{1+\epsilon,2\rho^{-1}}$. Next, we split the index set

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=s}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):s<n\right)\right\|_{\ell^{1+\epsilon}}$$
$$\lesssim\sum_m \left\|V_{2+\epsilon}\left(\sum_{j=s+1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):s<n\leq 2^{\kappa_s}\right)\right\|_{\ell^{1+\epsilon}}$$
$$+\sum_m \left\|V_{2+\epsilon}\left(\sum_{j=2^{k_s}+1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):2^{\kappa_s}<n\right)\right\|_{\ell^{1+\epsilon}}.$$

By interpolation between Theorem 5 and Theorem 6, and between Theorem 7 and Theorem 8, for β sufficiently larger we get

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=s+1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):s<n\leq 2^{(K_m)_s}\right)\right\|_{\ell^{1+\epsilon}}\leq C_{1+\epsilon}(s+1)^{-2}\sum_m \| f_m \|_{\ell^{1+\epsilon}},$$

and

$$\left\|\sum_m V_{2+\epsilon}\left(\sum_{j=2^{(K_m)_s}+1}^{n}\mathcal{F}^{-1}\left(\left(\eta_{N_j}-\eta_{N_{j-1}}\right)\Xi_{j,s}^{\beta}\hat{f_m}\right):2^{(K_m)_s}<n\right)\right\|_{\ell^{1+\epsilon}}\leq C_{1+\epsilon}\frac{2+\epsilon}{\epsilon}(s+1)^{-2}\sum_m \| f_m \|_{\ell^{1+\epsilon}},$$

and the theorem follows.

5.2. Short variations.

Theorem 10 (see [27]). For each $0<\epsilon<\infty$ there are $\rho\in(0,1)$ and $\epsilon\geq 0$ such that for all $0<\epsilon<\infty$ and any finitely supported function $f_m:\mathbb{Z}^d\to\mathbb{C}$, we have

$$\left\|\sum_m\left(\sum_{n\geq 0}V_{2+\epsilon}\left(\left(Y_N-Y_{N_n}\right)f_m:N\in[N_n,N_{n+1})\right)^{2+\epsilon}\right)^{1/2+\epsilon}\right\|_{\ell^{1+\epsilon}}\leq(1+\epsilon)\sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

Proof. Let $u_m=\min\{2,1+\epsilon\}$. By monotonicity and Minkowski's inequality, we get

$$\left\|\left(\sum_{n\geq 0}\sum_m V_{2+\epsilon}\left(Y_N f_m:N\in[N_n,N_{n+1})\right)^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right\|_{\ell^{1+\epsilon}}\leq\left\|\left(\sum_{n\geq 0}\left(\sum_{N=N_n}^{N_{n+1}-1}\sum_m |Y_{N+1}f_m-Y_N f_m|\right)^{u_m}\right)^{\frac{1}{u_m}}\right\|_{\ell^{1+\epsilon}}$$
$$\leq\left(\sum_{n\geq 0}\left\|\sum_{N=N_n}^{N_{n+1}-1}\sum_m |Y_{N+1}f_m-Y_N f_m|\right\|_{\ell^{1+\epsilon}}^{u_m}\right)^{\frac{1}{u_m}},$$

which together with (5.2) gives

$$\left\|\left(\sum_{n\geq 0}\sum_m V_{2+\epsilon}\left(Y_N f_m:N\in[N_n,N_{n+1})\right)^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right\|_{\ell^{1+\epsilon}}\lesssim\left(\sum_{n\geq 1}n^{-u_m(1-\rho)}\right)^{\frac{1}{u_m}}\sum_m \| f_m \|_{\ell^{1+\epsilon}}.$$

which is bounded whenever $0<\rho<\frac{u_m-1}{u_m}$.

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