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Research Paper

Equivalence Characterization of Generalized John Domains and the Minimization Property

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ABSTRACT: *In this paper, we investigate generalized John domains and the Ψ -min property in metric spaces, and establish an equivalence characterization between them. Under this characterization, domains satisfying the Ψ -min property also preserve the ψ' -min property under intersections and finite unions.*

KEYWORDS: *Generalized John domain, Ψ -min property, Intersection of domains, Finite union.*

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I. INTRODUCTION

To study the approximation theory and injectivity problem of bi-Lipschitz mappings in Euclidean spaces, John introduced the concept of a John domain in 1961 in reference [1]. This concept has provided a powerful tool for describing the geometric properties of mappings, especially in dealing with quasisymmetric, quasiconformal, and quasi-hyperbolic mappings, where it plays a crucial role. Shortly thereafter, Martio and Savas further explored the injectivity problem of mappings in 1979 in reference [2], introducing the concept of a "uniform domain", which provided a new perspective for studying the local properties of mappings. Due to the importance of John domains and uniform domains in various mathematical fields, the in-depth study of these concepts has not only broadened the theoretical foundation of related fields but also promoted their development in practical applications, especially in Euclidean and Banach spaces.

Currently, many important results concerning the properties of John domains have been confirmed in different spaces, with relevant research findings scattered throughout references [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. These studies provide important theoretical support for understanding the geometric and topological properties of mappings. Furthermore, inspired by the study of generalized disks, Guo and Koskela introduced the concept of a generalized John domain in references [3, 11]. This concept is a natural extension of the traditional John domain, bringing new ideas to the field and opening up broader avenues for research. The generalized John domain not only possesses stronger structural properties but also accommodates a wider variety of mappings and spaces, thereby advancing the fields of mapping theory, geometric analysis, and more.

With the introduction of the generalized John domain, researchers began to explore its properties in more general settings and its applicability in other spaces. This development has not only provided new challenges for mathematical theory but also offered potential solutions to various problems in applied mathematics and geometry. Future research may further uncover the role of these domains in complex space structures, particularly in the applications of high-dimensional spaces and non-Euclidean geometries.

In recent years, Qingshan Zhou and others studied the geometric properties of quasihyperbolic geodesics in John spaces in reference [13], solving an open problem posed by Heinonen and establishing the necessary and sufficient conditions for quasi-hyperbolic John spaces. In reference [14], the $\min$ - $\max$ property of domains was introduced, and based on this, the necessary and sufficient conditions for diameter uniform domains were established.

Definition 1.1. A domain (open and connected) D in X is said to be a C-uniform domain if for any pair of points $z_1, z_2 \in D$ that can be joined by a rectifiable arc $\gamma \in D$, there exists a constant $C \geq 1$ such that γ satisfies the following two conditions:

(1) (Double cone condition) $\min\left\{l(\gamma[z_1,z]), l(\gamma[z_2,z])\right\} \leq C\delta_D(z)$ for all $z \in \gamma$;

(2) (Quasiconvex condition) $l(\gamma) \leq C \mid z_1 - z_2 \mid$,

where $l(\gamma)$ denotes the arc-length of γ, $\gamma[z_i, z]$ is the subcurve of γ between z_i and z ,and $\delta_D(z)$ denotes the distance $\operatorname{dist}(z, \partial D)$. At this time, γ is said to be a double C-uniform curve. If the condition (1) is satisfied, not necessarily (2), then D is said to be a C-John domain and the arc γ is called a C- John curve.

The classes of John domains and of uniform domains in Euclidean space play an important role in many areas of modern mathematical analysis, see [19,24,25]. Inspired by the study on generalized quasidisks [23,20], Guo and Koskela [22] generalized the definition of John domain as follows.

Definition 1.2. Let $D \subseteq X$ be a bounded domain. Let a continuous and increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) \geq t$ for all $t > 0$, we say that D is φ- John domain if each pair points z_1, z_2 in D can be joined by a rectifiable curve γ in D satisfying

$$\min\{l(\gamma[z_1,u]), (\gamma[u,z_2])\} \leq \varphi(C\delta_D(u)),$$

for all $u \in \gamma$. At this time, γ is said to be a φ- John curve. he concept of dist φ-John domain and diam φ-John domain are defined analogously. corresponding curve is called a length (dist, diam) φ-John curve.

Our first purpose is to show that a domain G in metric space X is a φ-John domain f and only if G satisfies the ψ-min property. Our proof is based on a refinement of the method of Vasudevarao et. al. [26], we obtain a general result as follows.

Theorem 1.3. Suppose that $G \subset E$ is a domain, where E is a Banach space with dimension at least 2. Then G satisfies the ψ-min property if and only if G is a diameter φ_1-John domain, where ψ and φ_1 depend only on each other.

Corollary 1.4. Assume that G is a domain in a c-quasiconvex metric space. If G satisfies the ψ-min property, then G is a φ-John domain, where φ depends only on ψ and the constant c.

Theorem 1.5. Suppose that G_1 and G_2 are two domains in a metric space such that $G_1 \bigcap G_2 \neq \varnothing$. If G_1 and G_2 satisfy the ψ_1-min property and the ψ_2-min property, respectively, then $G_1 \bigcap G_2$ satisfies the ψ_{int}-min property, where ψ_{int} depends only on ψ_1 and ψ_2.

Theorem 1.6. Let E be a metric space, and let $G_1, G_2 \subseteq E$ be two c-quasiconvex domains with $c \geq 1$. Assume there exists a point $z_0 \in G_1 \bigcap G_2$ and a radius $r > 0$ such that

$$B(z_0, r) \subseteq G_1 \bigcup G_2 \quad \text{and} \quad \min\{\operatorname{diam}(G_1), \operatorname{diam}(G_2)\} \leq c_0 r$$

where $c_0 \geq 1$ and $\operatorname{diam}(G_i)$ denotes the diameter of G_i for $i = 1, 2$. If G_1 and G_2 satisfy the ψ_1-min property and the ψ_2-min property, respectively, then $G_1 \bigcup G_2$ satisfies the ψ_{uni}-min property, where ψ_{uni} depends only on ψ_1, ψ_2 and c.

Corollary 1.7. Let E be a metric space, and let $\{G_k\}_{k=1}^n$ be a sequence of c-quasiconvex domains in E, each satisfying the ψ_k-min property. Assume that for every $k \in \{1, 2, \ldots, n-1\}$, the pair (G_k, G_{k+1}) satisfies the conditions of Theorem 1.6 with the corresponding parameters, then the union $\bigcup_{k=1}^{n} G_k$ satisfies a Ψ_n -min property, where the function Ψ_n depends only on c, c_0, and the collection $\{\psi_k\}_{k=1}^{n}$.

II. PRELIMINARIES

Throughout the paper, we always assume that E and E' denote Banach spaces with dimension at least 2, $G \subseteq E$ and $G' \subseteq E'$ are domains (open and connected sets). The norm of a vector x in E is written as $|x|$, and for every pair of points x, y in E, the distance between them is denoted by $|x-y|$. Let $\delta_G(x)$ denotes the distance from a point x in the domain G to the boundary of G.

Throughout this paper, we denote the balls and spheres centered at $x \in D$ with radius $r > 0$ as

$$B(x,r) = \{y \in E : |x-y| < r\}, \qquad S(x,r) = \{y \in E : |x-y| = r\}$$

and

$$\overline{B}(x,r) = \{y \in E : |x-y| \le r\}.$$

For a set A in E, we use $\overline{A}$ to denote the metric completion of A, and we let $\partial A = \overline{A} \setminus A$ be its metric boundary.

Let $A \subset E$ be a set, and define the diameter of A as

$$\operatorname{diam}(A) = \max\{|x-y| : x, y \in A\}.$$

Let $\gamma : [a,b] \to G$ be a curve, and the length of γ is defined as

$$\ell(\gamma) = \sup\left\{\sum_{i=0}^{n} |\gamma(t_i) - \gamma(t_{i-1})|\right\},$$

where $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ is an arbitrary partition of the interval $[a,b]$. If $\ell(\gamma) < \infty$, then γ is said to be rectifiable. If for any pair of points $x, y \in G$, there exists a rectifiable curve in G connecting them, then G is said to be rectifiably connected. Assume $c \ge 1$, and for any two points $x, y \in G$, there exists a curve γ in G connecting them, satisfying

$$\ell(\gamma) \le c\,|x-y|$$

then G is said to be c-quasiconvex.

Definition 2.1. Let $G \subset E$ be a domain. We say that G has the *minimizing property* if there exists a family of curves Γ in G and a constant $C > 0$ such that for every pair of points $u, v \in G$, there exists a curve $\gamma \in \Gamma$ joining them that satisfies

$$\min_{j=1,2} |x_j - y| \le C\,|x-y|$$

for each triple of points $x_1, x, x_2 \in \gamma$ and any $y \in \partial G$.

If there exists an increasing function $\psi : [0,\infty) \to [0,\infty)$ that satisfies $\psi(0) = 0$ and $\psi(t) \ge t$ for all $t > 0$, such that

$$\min_{j=1,2} |x_j - y| \le \psi(C\,|x-y|),$$

then G is said to satisfy the ψ -min property.

Lemma 2.2.[16] Let E be a metric space and let $G_1, G_2 \subseteq E$ be two c-quasiconvex domains for some $c \ge 1$. Suppose that G_1 and G_2 are φ-John domains in E, and that there exist a point $z_0 \in G_1 \cap G_2$ and a radius $r > 0$ such that

$$B(z_0, r) \subseteq G_1 \cup G_2 \quad \text{and} \quad \min\{\operatorname{diam}(G_1), \operatorname{diam}(G_2)\} \le c_0 r,$$

where $c_0 \ge 1$ and $\operatorname{diam}(G_i)$ denotes the diameter of G_i for $i = 1, 2$. Then, $G_1 \cup G_2$ is a φ'-John domain, where

$$\varphi'(C't) = 4cc_0\varphi(Ct)$$

and $C' \geq 1$ is a constant. Moreover, φ' is a continuous, increasing function satisfying $\varphi'(0)=0$ and $\varphi'(t) \geq t$ for all $t > 0$.

III. PROOF OF MAIN RESULTS

3.1. Proof of Theorem 1.3.

We first prove the sufficiency.We know that $G \subset E$ is a domain, where E is a Banach space with dimension at least 2. And it is a diameter φ_1-John domain, for any $x_1, x_2 \in G$, so there exists a diameter φ_1-John arc γ connecting x_1 and x_2. For any $x \in \gamma$, we have

$$\min\{\mathrm{diam}(\gamma[x_1,x]), \mathrm{diam}(\gamma[x_2,x])\} \leq \varphi_1(C_1\delta_G(x)).$$

Here $C_1 \geq 1$ is a constant, and φ is a continuous, increasing function with $\varphi(0)=0$ and $\varphi(t) \geq t$ for all $t > 0$.

Therefore, we deduce that

$$\begin{aligned}\min\{|x_1-x|,|x_2-x|\} &\leq \min\{\mathrm{diam}(\gamma[x_1,x]), \mathrm{diam}(\gamma[x_2,x])\}\\ &\leq \varphi_1(C_1\delta_G(x)).\end{aligned}$$

Moreover, for any $y \in \partial G$, we have

$$\begin{aligned}\min\{|x_1-y|,|x_2-y|\} &\leq \min\{|x_1-x|,|x_2-x|\}+|x-y|\\ &\leq \varphi_1(C_1\delta_G(x))+|x-y|\\ &\leq \varphi_1(C_1|x-y|)+|x-y|.\end{aligned}$$

Thus, we obtain

$$\min\{|x_1-y|,|x_2-y|\} \leq \psi(C_1|x-y|),$$

which shows that G has the minimizing property, where

$$\psi(Ct) = \varphi_1(C_1 t) + t.$$

Here $C \geq 0$ is a constant, and ψ is a continuous, increasing function with $\psi(0)=0$ and $\psi(t) \geq t$ for all $t > 0$.Hence the sufficiency is true.

In the following, we prove the necessity. Let

$$\Gamma := \{\gamma : u \,\text{Ð}\, v : \gamma \text{ is a curve joining } u \text{ and } v\}.$$

Since G has the minimizing property, for any $\gamma \in \Gamma$ and $y \in \partial G$, as well as an ordered triple of points $x_1, x, x_2 \in \gamma$, we have

$$\min\{|x_1-y|,|x_2-y|\} \leq \psi\big(C|x-y|\big).$$

Here $C \geq 0$ is a constant, and ψ is a continuous, increasing function with $\psi(0)=0$ and $\psi(t) \geq t$ for all $t > 0$.We choose a point $z \in \partial G$ such that

$$|x-z| \leq 2\delta_G(x).$$

For $j=1$ or $j=2$, we claim that

$$\gamma[x_1,x] \subseteq B(z,\psi(2C\delta_G(x))) \quad \text{and} \quad \gamma[x_2,x] \subseteq B(z,\psi(2C\delta_G(x))).$$

Without loss of generality, we only need to prove that $\gamma[x_1,x] \subseteq B(z,\psi(2C\delta_G(x)))$, the proof of $\gamma[x_2,x] \subseteq B(z,\psi(2C\delta_G(x)))$ follows in a similar manner. Suppose that

$$\gamma[x_1,x] \,\text{Ú}\, B(z,\psi(2C\delta_G(x)))$$

holds. Then there exists a point $u_1 \in \gamma[x_1,x]$ such that

$$|u_1 - z| > \psi(2C\delta_G(x)).$$

For any $z \in \partial G$ with $|x-z| \leq 2\delta_G(x)$, it follows that

$$\psi(2C|x-z|) < |u_1 - z|.$$

which is a clear contradiction as minimizing property. Thus, we must have

$$\gamma[x_1,x]\subseteq B(z,\psi(2C\delta_G(x))).$$

For any $u_2\in\gamma[x_2,x]$, using a similar argument as in $\gamma[x_1,x]\subseteq B(z,\psi(2C\delta_G(x)))$, we can obtain

$$\gamma[x_2,x]\subseteq B(z,\psi(2C\delta_G(x))).$$

Therefore, it follows from the above fact that

$$\min\{\mathrm{diam}(\gamma[x_1,x]),\mathrm{diam}(\gamma[x_2,x])\}\le 2\psi(2C\delta_G(x)).$$

Then we see that G is a diameter φ_1-John domain, where

$$\varphi_1(C_1t)=2\psi(2Ct).$$

Here $C_1\ge 1$ is a constant, and φ_1 is a continuous, increasing function with $\varphi_1(0)=0$ and $\varphi_1(t)\ge t$ for all $t>0$.

Hence the statements in Theorem 1.3 are proved.

3.2. Proof of Corollary 1.4.

Since G satisfies the ψ-min property, for any $\gamma\in\Gamma$ and $y\in\partial G$, as well as for any ordered triple of points $x_1,x,x_2\in\gamma$. From the proof of Theorem 1.3, it follows that G is a diameter φ_1-John domain, where

$$\varphi_1(C_1t)=2\psi(2Ct),$$

thus

$$\min\{\mathrm{diam}(\gamma[x_1,x]),\mathrm{diam}(\gamma[x,x_2])\}\le 2\psi(2C\delta_G(x)) \tag{0.1}$$

holds. Moreover, since G is c-quasiconvex, we have

$$\ell(\gamma[x_j,x])\le c\,|x_j-x|. \tag{0.2}$$

By combining inequalities (1.1) and (1.2), we obtain

$$\begin{aligned}\min\{\ell(\gamma[x_1,x]),\ell(\gamma[x,x_2])\}&\le c\min\{|x_1-x|,|x-x_2|\}\\&\le c\min\{\mathrm{diam}(\gamma[x_1,x]),\mathrm{diam}(\gamma[x,x_2])\}\\&\le 2c\psi(2C\delta_G(x)).\end{aligned}$$

Therefore, we have proven that G is a φ-John domain, where $\varphi(C't)=2c\psi(2Ct)$. Here $C'\ge 1$ is a constant, and φ_1 is a continuous, increasing function satisfying $\varphi_1(0)=0$ and $\varphi_1(t)\ge t$ for all $t>0$.

Thus the assertions of Corollary 1.4 are established.

3.3. Proof of Theorem 1.5.

In this section, we may assume that $G_1\bigcap G_2\ne\varnothing$ .Suppose that

$$\partial(G_1\bigcap G_2)=\alpha_1\bigcup\beta_1,$$

where α_1 and β_1 denotes the subset of ∂G_1 and ∂G_2, respectively. Further, we set

$$\alpha_2=\partial G_1\setminus\alpha_1 \quad\text{and}\quad \beta_2=\partial G_2\setminus\beta_1.$$

Hence, it follows immediately that

$$\partial G_1=\alpha_1\bigcup\alpha_2,\partial G_2=\beta_1\bigcup\beta_2,\quad\text{and}\quad \partial(G_1\bigcup G_2)=\alpha_2\bigcup\beta_2.$$

We assume that $u,v\in G_1\bigcap G_2$. Let Γ_1 and Γ_2 denotes the families of curves joining u and v in G_1 and G_2, respectively, it follows that $\Gamma_1\bigcap\Gamma_2\ne\varnothing$.

Let

$$\Gamma=\Gamma_1\bigcap\Gamma_2.$$

Let $\gamma\in\Gamma$, and let $x_1,x,x_2\in\gamma$ be a triple of points. For any $y\in\partial(G_1\bigcap G_2)$, since $\partial(G_1\bigcap G_2)=\alpha_1\bigcup\beta_1$, we divide the discussions into two cases:

If $y\in\alpha_1$, since G_1 satisfy the ψ_1-min property, then we have

$$\min\{|x_1-y|,|x_2-y|\}\le\psi_1(C_{1'}\,|x-y|)\,. \tag{0.3}$$

If $y\in\beta_1$, since G_2 satisfy the ψ_2-min property, using a similar argument as in $y\in\alpha_1$, we can obtain

$$\min\{|x_1-y|,|x_2-y|\}\le\psi_2(C_{2'}\,|x-y|)\,. \tag{0.4}$$

Here $C_i'\ge 1$ are two constants, and $\psi_i:[0,\infty)\to[0,\infty)$ are two continuous, increasing functions with $\psi_i(0)=0$ and $\psi_i(t)\ge t$ for all $t>0$, where $i=1,2$.

Combining (1.3) and (1.4), it follows that

$$\min\{|x_1-y|,|x_2-y|\}\le\psi(C'\,|x-y|)\,,$$

where

$$\psi(C't)=\max\{\psi_1(C_{1'}t),\psi_2(C_{2'}t)\}\,.$$

Therefore, $G_1\bigcap G_2$ satisfies the ψ -min property. Hence, this completes the proof of Theorem 1.5.

3.4. Proof of Theorem 1.6.

Since G_1 and G_2 satisfy the ψ_1-min property and the ψ_2-min property, respectively, Theorem 1.3 implies that G_1 and G_2 are φ_1-John and φ_2-John domains, respectively, where $\varphi_i(C_it)=2c\psi(2C_{i'}t)$ for $i=1,2$.

Let

$$\varphi^*(Ct)=\max\{\varphi_1(C_1t),\varphi_2(C_2t)\}=2c\max\{\psi_1(2C_1't),\psi_2(2C_2't)\}$$

Then both G_1 and G_2 are φ^*-John domains. Moreover, by Lemma 2.2, the union $G_1\bigcup G_2$ is a φ_{u}-John domain, where

$$\varphi_{\mathrm{u}}(C_{\mathrm{u}}t)=4cc_0\varphi^*(Ct)=8c^2c_0\max\{\psi_1(2C_1't),\psi_2(2C_2't)\}\,.$$

Let $\gamma=\gamma[u,v]$ be a φ_{u}-John arc in $G_1\bigcup G_2$. Moreover, let $x_1,x_2\in\gamma$ and $x\in\gamma[x_1,x_2]$.

Since

$$\begin{aligned}\min\{\ell(\gamma[x_1,x]),\ell(\gamma[x,x_2])\}&\le\min\{\ell(\gamma[u,x]),\ell(\gamma[x,v])\}\\&\le\varphi_{\mathrm{u}}\left(C_{\mathrm{u}}\delta_{G_1\bigcup G_2}(x)\right),\end{aligned}$$

where $\delta_{G_1\bigcup G_2}(x)=\mathrm{dist}(x,\alpha_2\bigcup\beta_2)$. it follows that

$$\begin{aligned}\min\{|x_1-x|,|x-x_2|\}&\le\min\{\ell(\gamma[x_1,x]),\ell(\gamma[x,x_2])\}\\&\le\varphi_{\mathrm{u}}\left(C_{\mathrm{u}}\delta_{G_1\bigcup G_2}(x)\right).\end{aligned}$$

Therefore, for any $y\in\partial(G_1\bigcup G_2)$, we have

$$\begin{aligned}\min\{|x_1-y|,|x_2-y|\}&\le\min\{|x_1-x|,|x_2-x|\}+|x-y|\\&\le\varphi_{\mathrm{u}}(C_{\mathrm{u}}\,|x-y|)+|x-y|\,.\end{aligned}$$

Hence, $G_1\bigcup G_2$ satisfies the ψ_{u} -min property, where

$$\psi_u(C_u't)=\varphi_u(C_ut)+t=8c^2c_0\max\{\psi_1(2C_1't),\psi_2(2C_2't)\}+t.$$

where C_u', C_1',and C_2' are positive constants.

Therefore, the proof of Theorem 1.6 is complete.

3.5. Proof of Corollary 1.7.

We prove this corollary by induction on n.

Base case $n = 2$**.**

Theorem1.6 directly yields that $G_1 \bigcup G_2$ satisfies the ψ_{u} -min property.

Case $n = 3$**.**

Since each G_k ($k = 1, 2, 3$) satisfies the ψ_k -min property, Theorem1.3 implies that G_k is a φ_k -John domain, where $\varphi_k(C_k t) = 2c\psi_k(2C_k^{'} t)$. Applying Theorem 1.6 to G_1 and G_2, we obtain that $G_1 \bigcup G_2$ is a φ_{u_2} -John domain, with $\varphi_{u_2} = \varphi_{\mathrm{u}}$. Observing that $2c\psi_3(t) \le 8c^2 c_0 \psi_3(t)$, we define

$$\varphi_3^*(C_3 t) = 8c^2 c_0 \max\left\{\psi_1(2C_1^{'} t), \psi_2(2C_2^{'} t), \psi_3(2C_3^{'} t)\right\}.$$

Consequently, both $G_1 \bigcup G_2$ and G_3 are φ_3^* -John domains. By Lemma 2.3, the union $\bigcup_{k=1}^{3} G_k$ is therefore a φ_{u_3} -John domain, where

$$\varphi_{u_3}(C_{u_3} t) = 2^5 c^3 c_0^2 \max\left\{\psi_1(2C_1^{'} t), \psi_2(2C_2^{'} t), \psi_3(2C_3^{'} t)\right\}.$$

Hence, following the argument in the proof of Theorem1.6, we conclude that $\bigcup_{k=1}^{3} G_k$ satisfies the Ψ_3 -min property, with

$$\Psi_3(C_{u_3}^{'} t) = 2^5 c^3 c_0^2 \max\left\{\psi_1(2C_1^{'} t), \psi_2(2C_2^{'} t), \psi_3(2C_3^{'} t)\right\} + t.$$

This establishes the case $n = 3$.

Inductive step.

We now assume that the statement holds for some integer $n - 1 \ge 2$. By the induction hypothesis,

$$\varphi_{u_{n-1}}(C_{u_{n-1}} t) = 2^{2n-3} c^{n-1} c_0^{n-2} \max_{k=1,\ldots,n-1} \psi_k(2C_k^{'} t).$$

Following the same pattern as in the case $n = 3$, we define

$$\varphi_{u_n}^*(C_{u_n} t) = 2^{2n-3} c^{n-1} c_0^{n-2} \max_{k=1,\ldots,n} \psi_k(2C_k^{'} t).$$

Then, again by the reasoning in the proof of Theorem1.6, we conclude that $\bigcup_{k=1}^{n} G_k$ satisfies the Ψ_n -min property, where

$$\Psi_n(C_{u_n}^{'} t) = 2^{2n-1} c^n c_0^{n-1} \max_{k=1,\ldots,n} \psi_k(2C_k^{'} t) + t.$$

This completes the induction and the proof of Corollary1.7.

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REFERENCES

[1]. F. John, Rotation and strain. Comm. Pure Appl. Math., 14,1961, 391–413.

[2]. Martio, O. and Jukka, S., Injectivity theorems in plane and space. Annales Fennici Mathematici.,4(2), 1979, 383–401.

[3]. Guo, C., Uniform continuity of quasiconformal mappings onto generalized John domains. Annales Fennici Mathematici., 40(1), 2015, 183–202.

[4]. Huang, M. Saminathan, P. and Wang, X., Decomposition and removability properties of John domains. Proceedings Mathematical Sciences., 118, 2008, 357–370.

[5]. Kim, K. and Navah, L., Harmonic measure and hyperbolic distance in John disks. Mathematica Scandinavica., 1998, 283–299.

[6]. Väisälä, J., Unions of John domains. Proceedings of the American Mathematical Society., 2000, 1135–1140.

[7]. Zhou, Q. Li, L. and Antti, R., Generalized John Gromov hyperbolic domains and extensions of maps. Mathematica Scandinavica., 127.10.7146, 2021.

[8]. Gehring, F. W. and Bruce P. P., Quasiconformally homogeneous domains. Journal d'analyse Mathématique., 30(1), 1976, 172–199.

[9]. Gehring, F. W. and Brad G. O., *Uniform domains and the quasi-hyperbolic metric*. Journal d'Analyse Mathématique., 36(1), 1979, 50–74.

[10]. Guo, C., Generalized quasidisks and conformality II. Proceedings of the American Mathematical Society., 143(8), 2015, 3505–3517.

[11]. Guo, C. and Pekka, K., Generalized John disks. Open Mathematics., 12(2), 2014, 349–361.

[12]. Guo, C. Pekka, K. and Juhani, T., Generalized quasidisks and conformality. Publicacions Matematiques., 58, 2014, 193–212.

[13]. Zhou, Q. and Saminathan, P., Gromov hyperbolic John is quasihyperbolic John I. Annali Della Scuola Normale Superiore Di Pisa-classe Di Scienze., 2021, 19-19.

[14]. Zhou, Q. and Antti, R., Quasimobius invariance of uniform domains. Studia Mathematica., 261(1), 2021, 1-24.

[15]. Balogh, Z. M. and Stephen M. B., Geometric characterizations of Gromov hyperbolicity. Inventiones mathematicae., 153(2), 2003, 261–301.

[16]. Liu, H. Yan, F. Xia, L., The Properties of Generalized John Domains in Metric Spaces. AIMS MATHEMATICS., 9(6), 2024, 15875–15890.

[17]. Herron, D. A., Quasiconformal deformations and volume growth. Proceedings of the London Mathematical Society., 92(1), 2006, 161–199.

[18]. Koskela, P. Päivi, L. and Vesna, M., Gromov hyperbolicity and quasihyperbolic geodesics. Annales Scientifiques de l'Ecole Normale Superieure., 47(5), 2014, 975–990.

[19]. Heinonen J. Koskela P. Quasiconformal maps in metric spaces with controlled geometry. Acta Math., 181(1), 1998, 1–61.

[20]. Guo C. Generalized quasidisks and conformality II. Proceedings of the American Mathematical Society, 143(8), 2015, 3505–3517.

[21]. Guo C. Uniform continuity of quasiconformal mappings onto generalized John domains. Ann Acad Sci Fenn Ser A I Math., 40, 2015, 183–202.

[22]. Guo C. Koskela P. Generalized John disks. Cent. Eur. J. Math., 12, 2014, 349–361.

[23]. Guo C. Koskela P. Takkinen J. Generalized quasidisks and conformality. Publ. Mat., 58, 2014, 193–212.

[24]. Martio O. Sarvas J. Injectivity theorems in plane and space. Ann. Acad. Sci. Fenn. Ser. A I Math., 4, 1978-1979, 383–401.

[25]. Väisälä J. Quasisymmetric embeddings in Euclidean spaces. Trans. Amer. Math. Soc., 264(1) , 1981, 191–204.

[26]. Vasudevarao A. Alan P. J. On invariance of John domains under quasisymmetric mappings. arXiv:2309.15452v2 [math.CV].