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Research Paper

Many Positive Solutions for Iterative System on Nabla derivatives of Boundary Value Problems on Time Scales

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Abstract–*This paper establishes the existence of positive solutions for a nabla derivative second order iterative system of singular two-point boundary value problems on time scales. By applying H¨older's inequality and Guo–Krasnoselskii's fixed point theorem for operator on a cone, we derive sufficient conditions for the existence of positive solutions. Finally, we provide an example to check the validity of our obtained results.*

Keywords: *Nabla derivative, Iterative system, Time scale, Green's function, Singularity, Cone, Positive solutions.*

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I. INTRODUCTION

Time scale calculus, introduced to unify continuous and discrete analysis, provides an effective mathematical framework for investigating dynamic processes on different time domains. A time scale is a nonempty closed subset of the real numbers, allowing continuous intervals, isolated points, or combinations of continuous and discrete structures. Consequently, dynamic equations on time scales encompass both differential and difference equations as special cases. This unified perspective has encouraged extensive research into the qualitative analysis of dynamic equations and their associated boundary value problems(BVPs).

In 1994, Erbe and Wang [9] considered the second-order BVP

$$u'' + a(t)f(u) = 0,\ 0 < t < 1,$$

$$\alpha u(0) - \beta u'(0) = 0, \gamma u + \delta u'(1) = 0,$$

established existence of positive solutions. Recently in 2022, Prasad et al. [16] considered the following two-point iterative BVP on time scales

$$u_l^{\Delta\Delta}(t) + \lambda(t)\mathfrak{f}_l(u_{l+1}(t)) = 0,\ 1 \leqslant l \leqslant m,\quad t \in (0, \mathfrak{T})_{\mathbb{T}},$$

$$u_{m+1}(t) = u_1(t),\quad t \in (0, \mathfrak{T})_{\mathbb{T}},$$

$$u_l(0) = u_l^{\Delta}(0), u_l(\mathfrak{T}) = -u_l^{\Delta}(\mathfrak{T}),\ 1 \leqslant l \leqslant m,$$

where $m \in \mathbb{N}$, $\lambda(t) = \prod_{i=1}^{m} \lambda_i(t)$ and each $\lambda_i(t) \in \mathrm{L}_{\Delta}^{\mathrm{p}_i}([0, \mathfrak{T}]_{\mathbb{T}})$, $(\mathrm{p}_i \geqslant 1)$ has m-singularities in the interval $(0, \mathfrak{T})_{\mathbb{T}}$ and established infinitely many positive solutions. Some of the works along these lines [9, 12, 13, 14, 16, 18] for ordinary differential equations and [6, 19, 17] for dynamic equations on time scales.

Inspired by the aforementioned works, in this article, we establish existence of many positive solutions of nabla derivative second order iterative two-point BVP on time scales

$$\left.\begin{aligned} u_{\mathrm{j}}^{\nabla\nabla}(t) + \lambda(t)\mathfrak{f}_{\mathrm{j}}(u_{\mathrm{j}+1}(t)) = 0,\ 1 \leqslant \mathrm{j} \leqslant n,\quad t \in (0,1)_{\mathbb{T}}, \\ u_{n+1}(t) = u_1(t),\quad t \in (0,1)_{\mathbb{T}}, \end{aligned}\right\} \tag{1}$$

$$u_j(0) - \beta u_j^{\nabla}(0) = 0, \quad u_j(1) + \delta u_j^{\nabla}(1) = 0, \tag{2}$$

where $n \in \mathbb{N}$, $\lambda(t) = \prod_{k=1}^{s} \lambda_k(t)$, β, δ are constants and each $\lambda_k(t) \in L_{\Delta}^{p_k}([0,1]_{\mathbb{T}})$, $(p_k \geqslant 1)$ has s–singularities in the interval $(0,1)_{\mathbb{T}}$.

We assume the following conditions are true in the entire paper:

(Z1) $f_j : [0,\infty) \to [0,\infty)$ is continuous, $1 \leqslant j \leqslant n$.

(Z2) $\lim_{t \to t_k} \lambda_k(t) = \infty$, where $0 < t_n < t_{n-1} < \cdots < t_1 < 1$.

The rest of the paper is organized as follows. In Section 2, we construct Green's function for the homogeneous BVP corresponding to (1)-(2) and obtain bounds for the Green's function. In Section 3, we develop criteria for the presence of many positive solutions of (1)-(2) by using Guo–Krasnosel'skii fixed point theorem. In the last section, we give an example to illustrate our results.

2. GREEN'S FUNCTION AND BOUNDS

In this section, the Green's function for the homogeneous BVP corresponding to (1)-(2) is determined and certain lemmas are established, which are required for later discussion.

Lemma 1 *Let $h_1(t) \in C_{ld}([0,1])_{\mathbb{T}}$. Then the BVP*

$$u_1^{\nabla\nabla}(t) + h(t) = 0, \quad t \in (0,1)_{\mathbb{T}}, \tag{3}$$

$$u_1(0) - \beta u_1^{\nabla}(0) = 0, u_1(1) + \delta u_1^{\nabla}(1) = 0, \tag{4}$$

has one and only one solution

$$u_1(t) = \int_0^1 N(t,\tau) h(\tau) \nabla\tau, \tag{5}$$

where

$$N(t,\tau) = \frac{1}{A} \begin{cases} (\rho(\tau) + \beta)(1 + \delta - t), & if \quad \rho(\tau) \leqslant t, \\ (t + \beta)(1 - \rho(\tau) + \delta), & if \quad t \leqslant \rho(\tau), \end{cases} \tag{6}$$

and

$$A = 1 + \beta + \delta$$

Proof: Integrating (3) from 0 to t, we get

$$u_1(t) = -\int_0^t h(\tau)\nabla\tau + c_1.$$

Again integrating from 0 to t, we have

$$u_1(t) = -\int_0^t (t - \rho(\tau)) h_1(\tau)\nabla\tau + c_1 t + c_2.$$

By using boundary conditions (4), we get

$$c_1 = \frac{1}{A}\int_0^1 [\delta + 1 - \rho(\tau)] h(\tau)\nabla\tau \quad \text{and} \quad c_2 = \beta c_1.$$

Then,

$$u_1(t) = -\int_0^t [t-\rho(\tau)]\mathfrak{h}(\tau)\nabla\tau + \frac{(t+\beta)}{A}\int_0^1 [(1-\rho(\tau))+\delta]\mathfrak{h}(\tau)\nabla\tau$$
$$= \int_0^1 \mathsf{N}(t,\tau)\mathfrak{h}(\tau)\nabla\tau.$$

This completes the proof. □

Lemma 2 *Let* $\mathcal{G}(\eta) = \min\left\{\frac{\eta+\beta}{1+\beta}, \frac{\eta+\delta}{1+\delta}\right\} < 1$, *where* $\eta \in (0,\frac{1}{2})_T$. *Then* $\mathsf{N}(t,\tau)$ *has the following properties:*

(i) $0 \leqslant \mathsf{N}(t,\tau) \leqslant \mathsf{N}(\tau,\tau)$, *for all* $t,\tau \in [0,1]_{\mathbb{T}}$,

(ii) $\mathcal{G}(\eta)\mathsf{N}(\tau,\tau) \leqslant \mathsf{N}(t,\tau)$, *for all* $t \in [\eta,1-\eta]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$.

Proof: We can easily establish the inequality (i). For the inequality (ii), let $t \in [\eta,1-\eta]_{\mathbb{T}}$ and $t \leq \tau$, we obtain

$$\frac{\mathsf{N}(t,\tau)}{\mathsf{N}(\tau,\tau)} = \frac{t+\beta}{\tau+\beta} \geqslant \frac{\eta+\beta}{1+\beta} \geqslant \mathcal{G}(\eta).$$

Let $t \in [\eta,1-\eta]_{\mathbb{T}}$, and $t \geq \tau$, we obtain

$$\frac{\mathsf{N}(t,\tau)}{\mathsf{N}(\tau,\tau)} = \frac{1-t+\delta}{1-\tau+\delta} \geqslant \frac{\eta+\delta}{1+\delta} \geqslant \mathcal{G}(\eta).$$

This completes the proof.
□ Note that a n-tuple $\big(u_1(t), u_2(t), ..., u_n(t)\big)$ is a solution of (1)-(2) if and only if

$$u_{\mathrm{j}}(t) = \int_0^1 \mathsf{N}(t,\tau)\lambda(\tau)\mathfrak{f}_{\mathrm{j}}(u_{\mathrm{j}+1}(\tau))\nabla\tau,\ t \in (0,1)_{\mathbb{T}},\ 1 \leqslant \mathrm{j} \leqslant n,$$
$$u_{n+1}(t) = u_1(t),\ t \in (0,1)_{\mathbb{T}},\ t \in (0,1)_{\mathbb{T}}.$$

Equivalently,

$$u_1(t) = \int_0^1 \mathsf{N}(t,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\bigg(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\lambda(\tau_2)_2\bigg(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)\mathfrak{f}_3 ...$$
$$\mathfrak{f}_{n-1}\bigg(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n)\nabla\tau_n\bigg)...\nabla\tau_3\bigg)\nabla\tau_2\bigg)\nabla\tau_1.$$

3. MANY POSITIVE SOLUTIONS

In this section, we establish the existence of many positive solutions for (1)-(2) by using Guo–Krasnosel'skii theorem and Hölder's inequality.

Let $\mathcal{B}$ be a Banach space $\mathrm{C}_{ld}((0,1)_{\mathbb{T}},\mathbb{R})$ with norm $\|u\| = \max_{t\in[0,1]_{\mathbb{T}}} |u(t)|$. For $\eta \in (0,\frac{1}{2})_{\mathbb{T}}$, we define the cone $\mathbb{K}_\eta \subset \mathcal{B}$ as

$$\mathbb{K}_\eta = \big\{u \in \mathcal{B} : u(t) \text{ is nonnegative, } \min_{t\in[\eta,1-\eta]_{\mathbb{T}}} \{u(t)\} \geqslant \mathcal{G}(\eta)\|u\|_{\mathcal{B}}\big\}.$$

For any $u_1 \in \mathbb{K}_\eta$, define an operator $\Omega : \mathbb{K}_\eta \to \mathcal{B}$ by

$$(\Omega u_1)(t) = \int_0^1 \mathsf{N}(t,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\Big(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\lambda(\tau_2)\mathfrak{f}_2\Big(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)\mathfrak{f}_3 ...$$

$$\mathfrak{f}_{n-1}\Big(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n)\nabla\tau_n\Big)...\nabla\tau_3\Big)\nabla\tau_2\Big)\nabla\tau_1.$$

Lemma 3 *Assume that* (Z1) – (Z3) *hold. Then for each* $\eta \in (0,\frac{1}{2})_{\mathbb{T}}$, $\Omega(\mathbb{K}_\eta) \subset \mathbb{K}_\eta$ *and* $\Omega : \mathbb{K}_\eta \to \mathbb{K}_\eta$ *is completely continuous.*

Proof: From Lemma 2, we have $\mathsf{N}(t,\tau) \geqslant 0$, for all $t,\tau \in (0,1)_{\mathbb{T}}$ and so, $\Omega(u_1) \geqslant 0$. For any $u_1 \in \mathbb{K}_\eta$, we have

$$\begin{aligned}\|\Omega u_1\| &= \max_{t\in[0,1]_{\mathbb{T}}} \int_0^1 \mathsf{N}(t,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\Big(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\lambda(\tau_2)\mathfrak{f}_2\Big(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)\mathfrak{f}_3 ...\\ &\quad \mathfrak{f}_{n-1}\Big(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n)\nabla\tau_n\Big)...\nabla\tau_3\Big)\nabla\tau_2\Big)\nabla\tau_1\\ &\leqslant \int_0^1 \mathsf{N}(\tau_1,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\Big(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\lambda(\tau_2)\mathfrak{f}_2\Big(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)\mathfrak{f}_3 ...\\ &\quad \mathfrak{f}_{n-1}\Big(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n)\nabla\tau_n\Big)...\nabla\tau_3\Big)\nabla\tau_2\Big)\nabla\tau_1\end{aligned}$$

Again, from Lemma 2, we get

$$\begin{aligned}\min_{t\in[\eta,1-\eta]_{\mathbb{T}}}(\Omega u)(t) &\geqslant \mathcal{G}(\eta)\int_0^1 \mathsf{N}(\tau_1,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\Big(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\lambda(\tau_2)\mathfrak{f}_2\Big(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)\mathfrak{f}_3 ...\\ &\quad \mathfrak{f}_{n-1}\Big(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n)\nabla\tau_n\Big)...\nabla\tau_3\Big)\nabla\tau_2\Big)\nabla\tau_1\end{aligned}$$

It follows from the inequalities that

$$\min_{t\in[\eta,1-\eta]_{\mathbb{T}}}(\Omega u)(t) \geq \mathcal{G}(\eta)\|\Omega u_1\|.$$

Therefore, $\Omega u_1 \in \mathbb{K}_\eta$ and $\Omega(\mathbb{K}_\eta) \subset \mathbb{K}_\eta$ and using Arzela-Ascoli theorem, it can be easily prove that Ω is completely continuous. The proof is completed. □

Theorem 1 *(Ḧolder's inequality [3, 4, 15]) Let* $f \in \mathrm{L}^{\mathfrak{p}_\mathtt{k}}_\nabla[0,1]$ *with* $\mathfrak{p}_\mathtt{k} > 1$, *for* $\mathtt{k} = 1,2,...,n$ *and* $\sum_{\mathtt{k}=1}^{n}\frac{1}{\mathfrak{p}_\mathtt{k}} = 1$. *Then* $\prod_{\mathtt{k}=1}^{n} f_\mathtt{k} \in \mathrm{L}^1_\nabla[0,1]$ *and* $\Big\|\prod_{\mathtt{k}=1}^{n} f_\mathtt{k}\Big\|_1 \leqslant \prod_{\mathtt{k}=1}^{n}\|f_\mathtt{k}\|_{\mathfrak{p}_\mathtt{k}}$. *Further, if* $f \in \mathrm{L}^1_\nabla[0,1]$ *and* $g \in \mathrm{L}^\infty_\nabla[0,1]$, *then* $fg \in \mathrm{L}^1_\nabla[0,1]$ *and* $\|fg\|_1 \leqslant \|f\|_1\|g\|_\infty$.

Theorem 2 *[11] Let* $\mathbb{K}_\eta$ *be a cone in a Banach space* $\mathbb{B}$ *and* $\mathbb{R}_1, \mathbb{R}_2$ *are open sets with* $0 \in \mathbb{R}_1, \overline{\mathbb{R}}_1 \subset \mathbb{R}_2$. *Let* $\Omega : \mathbb{K}_\eta \cap (\overline{\mathbb{R}}_2 \setminus \mathbb{R}_1) \to \mathbb{K}_\eta$ *be completely continuous operator such that*

(a_1) $\|\Omega r\| \leqslant \|r\|, r \in \mathbb{K}_\eta \cap \partial\mathbb{R}_1$, *and* $\|\Omega r\| \geqslant \|r\|, r \in \mathbb{K}_\eta \cap \partial\mathbb{R}_2$, *or*

(a_2) $\|\Omega r\| \geqslant \|r\|, r \in \mathbb{K}_\eta \cap \partial\mathbb{R}_1$, *and* $\|\Omega r\| \leqslant \|r\|, r \in \mathbb{K}_\eta \cap \partial\mathbb{R}_2$.

Then Ω *has a fixed point in* $\mathbb{K}_\eta \cap (\overline{\mathbb{R}}_2 \setminus \mathbb{R}_1)$.

We will use the below conditions in the main results:

(Z3) there exists $\mu_i > 0$ such that $\lambda_i(t) > \mu_i, (i = 1,2,...,s)$ for $t \in [0,1]_{\mathbb{T}}$.

We consider the cases for $\lambda_\mathtt{k} \in \mathrm{L}^{\mathfrak{p}_\mathtt{k}}_\nabla(0,1)_{\mathbb{T}}$:

(i) $\sum_{\mathtt{k}=1}^{s}\frac{1}{\mathfrak{p}_\mathtt{k}} < 1$, (ii) $\sum_{\mathtt{k}=1}^{s}\frac{1}{\mathfrak{p}_\mathtt{k}} = 1$, (iii) $\sum_{\mathtt{k}=1}^{s}\frac{1}{\mathfrak{p}_\mathtt{k}} > 1$.

Firstly, we seek many positive solutions for the case $\sum_{\mathtt{k}=1}^{s}\frac{1}{\mathfrak{p}_\mathtt{k}} < 1$.

Theorem 3 *Suppose that* (Z1) – (Z3) *hold. Let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $\eta_r \in (t_{r+1}, t_r)$, $0 < \eta_1 < \frac{1}{2}$. *Let* $\{\mathfrak{D}_r\}_{r=1}^{\infty}$ *and* $\{\mathfrak{N}_r\}_{r=1}^{\infty}$ *be such that*

$$\mathfrak{D}_{r+1} < \mathcal{G}(\eta_r)\mathfrak{N}_r < \mathfrak{N}_r < \phi\ \mathfrak{N}_r < \mathfrak{D}_r,\ \ r \in \mathbb{N},$$

where

$$\phi = \max\left\{\left[\mathcal{G}(\eta_1)\prod_{\mathtt{k}=1}^{s}\mu_{\mathtt{k}}\int_{\eta_1}^{1-\eta_1}\mathsf{N}(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\},$$

Assume that $\mathfrak{f}$ *satisfies*

(H1) $\mathfrak{f}_{\mathtt{j}}(u) \leqslant \mathfrak{h}_1\mathfrak{D}_r$, *for all* $t \in (0,1)_{\mathbb{T}}$, $0 \leqslant u \leqslant \mathfrak{D}_r$, *where*

$$\mathfrak{h}_1 < \left[\|\mathsf{N}\|_{\mathrm{L}^{\mathfrak{q}}_{\nabla}}\prod_{\mathtt{k}=1}^{s}\|\lambda_{\mathtt{k}}\|_{\mathrm{L}^{\mathtt{p}_{\mathtt{k}}}_{\nabla}}\right]^{-1},$$

(H2) $\mathfrak{f}_{\mathtt{j}}(u) \geqslant \phi\mathfrak{N}_r$, *for all* $t \in [\eta_r, 1-\eta_r]_{\mathbb{T}}$, $\mathcal{G}(\eta_r)\mathfrak{N}_r \leqslant u \leqslant \mathfrak{N}_r$,

Then (1)-(2) *has many positive solutions* $\{(u_1^{[r]}, u_2^{[r]}, \ldots, u_n^{[r]})\}_{r=1}^{\infty}$ *such that* $u_{\mathtt{j}}^{[r]}(t) \geqslant 0$ *on* $(0,1)_{\mathbb{T}}$, $\mathtt{j} = 1,2,\ldots,n$, $r \in \mathbb{N}$.

Proof: Let

$$\mathbb{R}_{1,r} = \{u \in \mathcal{B} : \|u\| \leqslant \mathfrak{D}_r\}, \quad \mathbb{R}_{2,r} = \{u \in \mathcal{B} : \|u\| \leqslant \mathfrak{N}_r\},$$

be open subsets of $\mathcal{B}$. From the hypothesis, we have

$$t^* < t_{r+1} < \eta_r < t_r < \frac{1}{2}, \text{ for all } r \in \mathbb{N}.$$

Denote

$$\mathbb{K}_{\eta_r} = \{u \in \mathcal{B} : u(t) \text{ is nonnegative and } \min_{t\in[\eta_r,1-\eta_r]_{\mathbb{T}}}\{u(t)\} \geqslant \mathcal{G}(\eta_r)\|u\|_{\mathcal{B}}\}.$$

Let $u_1 \in \mathbb{K}_{\eta_r} \cap \partial\mathbb{R}_{1,r}$, Then $u_1(\tau) \leqslant \mathfrak{D}_r = \|u_1\|$ for all $\tau \in (0,1)_{\mathbb{T}}$.
By (H_1) and for $\tau_{n-1} \in (0,1)_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n &\leqslant \int_0^1 \mathsf{N}(\tau_n,\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n\\ &\leqslant \mathfrak{h}_1\mathfrak{D}_r\int_0^1 \mathsf{N}(\tau_n,\tau_n)\lambda(\tau_n)\nabla\tau_n\\ &\leqslant \mathfrak{h}_1\mathfrak{D}_r\int_0^1 \mathsf{N}(\tau_n,\tau_n)\prod_{\mathtt{k}=1}^{s}\lambda_{\mathtt{k}}(\tau_n)\nabla\tau_n.\end{aligned}$$

Since $\sum_{\mathtt{k}=1}^{s}\frac{1}{\mathtt{p}_{\mathtt{k}}} < 1$, there exists $\mathfrak{q} > 1$ such that $\frac{1}{\mathfrak{q}} + \sum_{\mathtt{k}=1}^{s}\frac{1}{\mathtt{p}_{\mathtt{k}}} = 1$. So,

$$\begin{aligned}\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n &\leqslant \mathfrak{h}_1\mathfrak{D}_r\|\mathsf{N}\|_{\mathrm{L}^{\mathfrak{q}}_{\nabla}}\left\|\prod_{\mathtt{k}=1}^{s}\lambda_{\mathtt{k}}\right\|_{\mathrm{L}^{\mathtt{p}_{\mathtt{k}}}_{\nabla}}\\ &\leqslant \mathfrak{D}_r.\end{aligned}$$

Similarly for $\tau_{n-2} \in (0,1)_{\mathbb{T}}$,

$$\begin{aligned}&\int_0^1 \mathsf{N}(\tau_{n-2},\tau_{n-1})\mu(\tau_{n-1})\mathfrak{f}_{n-1}\left(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n\right)\nabla\tau_{n-1}\\ &\qquad\leqslant \int_0^1 \mathsf{N}(\tau_{n-1},\tau_{n-1})\mu(\tau_{n-1})\mathfrak{f}_{n-1}(\mathfrak{D}_r)\nabla\tau_{n-1}\\ &\qquad\leqslant \mathfrak{D}_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega_1 u_1)(t) = \int_0^1 \mathsf{N}(t,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\Bigg(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\mu(\tau_2)\mathfrak{f}_2\Bigg(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)\ldots\\ \mathfrak{f}_{n-1}\Bigg(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n\Bigg)\ldots\nabla\tau_3\Bigg)\nabla\tau_2\Bigg)\nabla\tau_1\\ \leqslant \mathfrak{D}_r.\end{aligned}$$

Since $\mathfrak{D}_r = \|u_1\|$ for $u_1 \in \mathbb{K}_{\eta_r} \cap \partial\mathbb{R}_{1,r}$, we get

$$\|\Omega_1 u_1\| \leqslant \|u_1\|. \tag{7}$$

Let $t \in [\eta_r, 1-\eta_r]_{\mathbb{T}}$, then

$$\mathfrak{N}_r = \|u_1\| \geqslant u_1(t) \geqslant \min_{t\in[\eta_r,1-\eta_r]_{\mathbb{T}}} u_1(t) \geqslant \mathcal{G}(\eta_r)\|u_1(t)\| \geqslant \mathcal{G}(\eta_r)\mathfrak{N}_r.$$

By (H2) and for $\tau_{n-1} \in [\eta_r, 1-\eta_r]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_\eta)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n &\geqslant \mathcal{G}(\eta_r)\int_{\eta_r}^{1-\eta_r} \mathsf{N}(\tau_n,\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n\\ &\geqslant \mathcal{G}(\eta_r)\phi\ \mathfrak{N}_r\int_{\eta_r}^{1-\eta_r} \mathsf{N}(\tau_n,\tau_n)\lambda(\tau_n)\nabla\tau_n\\ &\geqslant \mathcal{G}(\eta_r)\phi\ \mathfrak{N}_r\prod_{\mathtt{k}=1}^{s}\mu_{\mathtt{k}}\int_{\eta_1}^{1-\eta_1} \mathsf{N}(\tau_n,\tau_n)\nabla\tau_n\\ &\geqslant \mathfrak{N}_r.\end{aligned}$$

Continue in this way,

$$(\Omega_1 u_1)(t) = \int_0^1 \mathsf{N}(t,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\Bigg(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\lambda(\tau_2)\mathfrak{f}_2\Bigg(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)...$$
$$\mathfrak{f}_{n-1}\Bigg(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n\Bigg)...\nabla\tau_3\Bigg)\nabla\tau_2\Bigg)\nabla\tau_1$$
$$\geqslant \mathfrak{N}_r.$$

Thus if $u_1 \in \mathbb{P}_{\eta_r} \cap \partial\mathbb{R}_{2,r}$, then

$$\|\Omega_1 u_1\| \geqslant \|u_1\|. \tag{8}$$

It is evident that $0 \in \partial\mathbb{R}_{2,r} \subset \partial\overline{\mathbb{R}}_{2,r} \subset \partial\mathbb{R}_{1,r}$. Using (7) and (8), it follows from Theorem 2 that Ω has a fixed point $u_1^{[r]} \in \mathbb{K}_{\eta_r} \cap (\overline{\mathbb{R}}_{1,r} \setminus \mathbb{R}_{2,r})$ such that $u_1^{[r]}(t) \geqslant 0$ on $[0,1]_{\mathbb{T}}$ and $r \in \mathbb{N}$.

Next setting $u_{n+1} = u_1$, we obtain many positive solutions $\{(u_1^{[r]}, u_2^{[r]}, \ldots, u_n^{[r]})\}_{r=1}^{\infty}$ of (1)-(2) given by

$$u_{\mathsf{j}}(t) = \int_0^1 \mathsf{N}(t,\tau)\lambda(\tau)\mathfrak{f}_{\mathsf{j}}(u_{\mathsf{j}+1}(\tau))\nabla\tau,\ t \in (0,1)_{\mathbb{T}},\ 1 \leqslant \mathsf{j} \leqslant n.$$

The proof is complete. □

For $\sum_{\mathsf{k}=1}^{s} \frac{1}{\mathsf{p}_{\mathsf{k}}} = 1$, we have the following theorem.

Theorem 4 *Suppose that* (Z1) – (Z3) *hold. Let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $\eta_r \in (t_{r+1}, t_r)$ *and* $0 < \eta_1 < \frac{1}{2}$. *Let* $\{\mathfrak{D}_r\}_{r=1}^{\infty}$ *and* $\{\mathfrak{N}_r\}_{r=1}^{\infty}$ *be such that*

$$\mathfrak{D}_{r+1} < \mathcal{G}(\eta_r)\mathfrak{N}_r < \mathfrak{N}_r < \phi\ \mathfrak{N}_r < \mathfrak{D}_r,\ r \in \mathbb{N},$$

Assume that $\mathfrak{f}$ *satisfies* (H2), (H4) *and*

(H5) $\mathfrak{f}_{\mathsf{j}}(u) \leqslant \mathfrak{h}_2\mathfrak{D}_r$, *for all* $t \in (0,1)_{\mathbb{T}}$, $0 \leqslant u \leqslant \mathfrak{D}_r$, *where*

$$\mathfrak{h}_2 < \left[\|\mathsf{N}\|_{\mathrm{L}_{\nabla}^{\infty}} \prod_{\mathsf{k}=1}^{n} \|\lambda_{\mathsf{k}}\|_{\mathrm{L}_{\nabla}^{\mathsf{p}_{\mathsf{k}}}}\right]^{-1},$$

Then (1)-(2) *has many positive solutions* $\{(u_1^{[r]}, u_2^{[r]}, \ldots, u_n^{[r]})\}_{r=1}^{\infty}$ *such that* $u_{\mathsf{j}}^{[r]}(t) \geqslant 0$ *on* $(0,1)_{\mathbb{T}}$, $\mathsf{j} = 1,2,\ldots,n, r \in \mathbb{N}$.

Proof: For a fixed r, let $\mathbb{R}_{1,r}, \mathbb{R}_{2,r}$ be as in the proof of Theorem 3. Let $u_1 \in \mathbb{K}_{\eta_r} \cap \partial\mathbb{R}_{1,r}$, Then $u_1(\tau) \leqslant \mathfrak{D}_r = \|u_1\|$ for all $\tau \in (0,1)_{\mathbb{T}}$. By (H_1) and for $\tau_{n-1} \in (0,1)_{\mathbb{T}}$, we have

$$\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n \leqslant \int_0^1 \mathsf{N}(\tau_n,\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n$$
$$\leqslant \mathfrak{h}_2\mathfrak{D}_r \int_0^1 \mathsf{N}(\tau_n,\tau_n)\lambda(\tau_n)\nabla\tau_n$$
$$\leqslant \mathfrak{h}_2\mathfrak{D}_r \int_0^1 \mathsf{N}(\tau_n,\tau_n)\prod_{\mathsf{k}=1}^{s}\lambda_{\mathsf{k}}(\tau_n)\nabla\tau_n.$$

Since $\sum_{\mathsf{k}=1}^{s} \frac{1}{\mathsf{p}_{\mathsf{k}}} = 1$, there exists $\mathfrak{q} > 1$ such that $\frac{1}{\mathfrak{q}} + \sum_{\mathsf{k}=1}^{s} \frac{1}{\mathsf{p}_{\mathsf{k}}} = 1$. So,

$$\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n \leqslant \mathfrak{h}_2\mathfrak{D}_r\|\mathsf{N}\|_{\mathrm{L}_{\nabla}^{\infty}} \left\|\prod_{\mathsf{k}=1}^{s}\lambda_{\mathsf{k}}\right\|_{\mathrm{L}_{\nabla}^{\mathsf{p}_{\mathsf{k}}}}$$
$$\leqslant \mathfrak{D}_r.$$

Similarly for $\tau_{n-2} \in (0,1)_{\mathbb{T}}$,

$$\int_0^1 \mathsf{N}(\tau_{n-2},\tau_{n-1})\mu(\tau_{n-1})\mathfrak{f}_{n-1}\Bigg(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n\Bigg)\nabla\tau_{n-1}$$
$$\leqslant \int_0^1 \mathsf{N}(\tau_{n-1},\tau_{n-1})\mu(\tau_{n-1})\mathfrak{f}_{n-1}(\mathfrak{D}_r)\nabla\tau_{n-1}$$
$$\leqslant \mathfrak{D}_r.$$

Continuing with this bootstrapping argument, we get

$$(\Omega_1 u_1)(t) = \int_0^1 \mathsf{N}(t,\tau_1)\lambda(\tau_1)\mathfrak{f}_1\Big(\int_0^1 \mathsf{N}(\tau_1,\tau_2)\mu(\tau_2)\mathfrak{f}_2\Big(\int_0^1 \mathsf{N}(\tau_2,\tau_3)\lambda(\tau_3)...$$
$$\mathfrak{f}_{n-1}\Big(\int_0^1 \mathsf{N}(\tau_{n-1},\tau_n)\lambda(\tau_n)\mathfrak{f}_n(u_1(\tau_n))\nabla\tau_n\Big)...\nabla\tau_3\Big)\nabla\tau_2\Big)\nabla\tau_1$$
$$\leqslant \mathfrak{D}_r.$$

Since $\mathfrak{D}_r = \|u_1\|$ for $u_1 \in \mathbb{K}_{\eta_r} \cap \partial\mathbb{R}_{1,r}$, we get

$$\|\Omega_1 u_1\| \leqslant \|u_1\|. \tag{9}$$

Let $t \in [\eta_r, 1-\eta_r]_{\mathbb{T}}$. Then

$$\mathfrak{N}_r = \|u_1\| \geqslant u_1(t) \geqslant \min_{t\in[\eta_r,1-\eta_r]_{\mathbb{T}}} u_1(t) \geqslant \mathcal{G}(\eta_r)\|u_1(t)\| \geqslant \mathcal{G}(\eta_r)\mathfrak{N}_r.$$

Then, the argument leading to (8) can be done to the present case and hence, the theorem. □

Lastly, the case $\sum_{k=1}^{n} \frac{1}{p_k} > 1$.

Theorem 5 *Suppose that* (Z1) – (Z3) *hold. Let* $\{\eta_r\}_{r=1}^{\infty}$ *be a sequence with* $\eta_r \in (t_{r+1}, t_r)$ *and* $0 < \eta_1 < \frac{1}{2}$. *Let* $\{\mathfrak{D}_r\}_{r=1}^{\infty}$ *and* $\{\mathfrak{N}_r\}_{r=1}^{\infty}$ *be such that*

$$\mathfrak{D}_{r+1} < \mathcal{G}(\eta_r)\mathfrak{N}_r < \mathfrak{N}_r < \phi\ \mathfrak{N}_r < \mathfrak{D}_r,\ r \in \mathbb{N},$$

Assume that $\mathfrak{f}$ *satisfies* (H2), (H4) *and*

(H5) $\mathfrak{f}_j(u) \leqslant \mathfrak{h}_3\mathfrak{D}_r$, *for all* $t \in (0,1)_{\mathbb{T}}$, $0 \leqslant u \leqslant \mathfrak{D}_r$, *where*

$$\mathfrak{h}_3 < \left[\|\mathsf{N}\|_{L^{\infty}_{\nabla}} \prod_{k=1}^{s} \|\lambda_k\|_{L^1_{\nabla}}\right]^{-1},$$

Then (1)-(2) *has many positive solutions* $\{(u_1^{[r]}, u_2^{[r]}, \ldots, u_n^{[r]})\}_{r=1}^{\infty}$ *such that* $u_j^{[r]}(t) \geqslant 0$ *on* $(0,1)_{\mathbb{T}}$, $j = 1,2,\ldots,n, r \in \mathbb{N}$.

Proof: The proof is similar to the proof of Theorem 3. Therefore, we omit the details here. □

4. EXAMPLE

Let us consider an example to illustrate our results.

Example 1 *Consider the following BVP on* $\mathbb{T} = [0,1]$.

$$\left.\begin{aligned} &u_j^{\nabla\nabla}(t) + \lambda(t)\mathfrak{f}_j(u_{j+1}(t)) = 0,\ t \in [0,1],\ j = 1,2,\\ &u_3(t) = u_1(t),\\ &u_j(0) - u_j^{\nabla}(0) = 0,\\ &u_j(1) + u_j^{\nabla}(1) = 0,,\quad j = 1,2, \end{aligned}\right\} \tag{10}$$

where $\lambda(t)=\lambda_1(t)\lambda_2(t)$, *in which* $\lambda_1(t)=\dfrac{1}{|t-\frac{1}{3}|^{\frac{1}{2}}}, \lambda_2(t)=\dfrac{1}{|t-\frac{2}{3}|^{\frac{1}{2}}}$,

$$\mathfrak{f}_1(u)=\mathfrak{f}_2(u)=\begin{cases}0.05\times10^{-4}, & u\in(10^{-4},\infty),\\ \dfrac{26\times10^{-(4r+3)}-0.05\times10^{-(4r)}}{10^{-(4r+3)}-10^{-(4r)}}(u-10^{-(4r)})+0.05\times10^{-(8r)}, & \\ \qquad u\in[10^{-(4r+3)},10^{-(4r)}],\\ 26\times10^{-(4r+3)}, & u\in(\dfrac{1}{5}\times10^{-(4r+3)},10^{-(4r+3)}),\\ \dfrac{26\times10^{-(4r+3)}-0.05\times10^{-(8r)}}{0.05\times10^{-(4r+3)}-10^{-(4r+4)}}(u-10^{-(4r+4)})+0.05\times10^{-(8r)}, & \\ \qquad u\in(10^{-(4r+4)},\dfrac{1}{5}\times10^{-(4r+3)}],\\ 0, & u=0,\end{cases}$$

Let $t_r=\frac{31}{64}-\sum\limits_{\mathrm{k}=1}^{r}\dfrac{1}{4(\mathrm{k}+1)^4}, \quad \eta_r=\dfrac{1}{2}(t_r+t_{r+1}),\ r=1,2,3,\ldots,$

then $\eta_1=\frac{15}{32}-\frac{1}{648}<\frac{15}{32}$ *and* $t_{r+1}<\eta_r<t_r, \eta_r>\frac{1}{5}, r=1,2,3,\ldots.$

It is clear that $t_1=\frac{15}{32}<\frac{1}{2}, \quad t_r-t_{r+1}=\dfrac{1}{4(r+2)^4},\ r=1,2,3,\ldots$

Since $\sum\limits_{\mathrm{k}=1}^{\infty}\dfrac{1}{\mathrm{k}^4}=\dfrac{\pi^4}{90}$ *and* $\sum\limits_{\mathrm{k}=1}^{\infty}\dfrac{1}{\mathrm{k}^2}=\dfrac{\pi^2}{6}$, *it follows that*

$$t^*=\lim_{r\to\infty}t_r=\frac{31}{64}-\sum_{\mathrm{k}=1}^{\infty}\frac{1}{4(\mathrm{k}+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}>\frac{1}{5}.$$

Since $\lambda_1,\lambda_2\in \mathrm{L}^{\mathrm{p}}[0,1]$, *for all* $0<p<2$.

$\mathcal{G}(\eta_1)=0.7336033951, \quad \mu_1=\mu_2=(\frac{4}{3})^{\frac{1}{4}},$

$$\int_{\eta_1}^{1-\eta_1}\mathsf{N}(\tau,\tau)\nabla\tau=\int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}}\frac{(2-\tau)(1+\tau)}{3})\nabla\tau=0.04918197801.$$

$$\phi=\max\left\{\left[\mathcal{G}(\eta_1)\prod_{\mathrm{k}=1}^{s}\mu_{\mathrm{k}}\int_{\eta_1}^{1-\eta_1}\mathsf{N}(\tau,\tau)d\tau\right]^{-1},\ 1\right\}=\ 25.792770387985.$$

Next, let $0<a<1$ *be fixed. Then* $\lambda_1,\lambda_2\in \mathrm{L}^{1+\mathrm{a}}[0,1]$ *and* $\frac{2}{1+a}>1$ *for* $0<a<1$.
It follows that

$$\prod_{\mathrm{k}=1}^{s}\|\lambda_{\mathrm{k}}\|_{\mathrm{L}^{\mathrm{p_k}}}\approx 6.67.$$

and $\|\mathsf{N}\|_\infty=\frac{2}{3}$. *So for* $0<a<1$, *we have*

$$\mathrm{h}_2<\left[\|\mathsf{N}\|_\infty\prod_{\mathrm{k}=1}^{s}\|\mu_{\mathrm{k}}\|_{\mathrm{L}_\nabla^{\mathrm{p_k}}}\right]^{-1}\approx 0.2247.$$

Taking $\mathrm{h}_2=\frac{1}{5}$. *In addition if we take* $\mathfrak{D}_r=10^{-4r}, \quad \mathfrak{N}_r=10^{-(4r+3)}$, *then*

$$\mathfrak{D}_{r+1}=10^{-(4r+4)}<\frac{1}{5}\times10^{-(4r+3)}<\mathcal{G}(\eta_r)\mathfrak{N}_r<\mathfrak{N}_r=10^{-(4r+3)}<\mathfrak{D}_r=10^{-4r},$$

$$\phi\ \mathfrak{N}_r=25.792770387985\times10^{-(4r+3)}<\frac{1}{5}\times10^{-4r}=\mathrm{h}_2\mathfrak{D}_r,\ r=1,2,3,\ldots,$$

and $\mathfrak{f}_1,\mathfrak{f}_2$ *satisfies the following growth conditions:*

$$\mathfrak{f}_1(u)=\mathfrak{f}_2(u)\leqslant \mathrm{h}_2\ \mathfrak{D}_r=\tfrac{1}{4}\times10^{-4r},\ u\in[0,10^{-4r}],$$

$$\mathfrak{f}_1(u)=\mathfrak{f}_2(u)\geqslant \phi\ \mathfrak{N}_r=22.879165774714\times10^{-(4r+3)},\ u\in\left[\tfrac{1}{5}\times10^{-(4r+3)},10^{-(4r+3)}\right].$$

Hence, all conditions in Theorem 3 are satisfied. Therefore, by Theorem 3, the system of iterative BVP (10) has many positive solutions $\left\{(u_1^{[r]},u_2^{[r]})\right\}_{r=1}^{\infty}$ *such that* $u_{\mathrm{j}}^{[r]}(t)\geqslant 0$ *on* $[0,1]$, $\mathrm{j}=1,2$ *and* $r\in\mathbb{N}$.

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