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Research Paper

Mappings with p-contraction, (φ, Γ, β)-contraction on b-metric spaces and fixed point theorems

Diksha[1], Kamal Kumar[2,*]
[1, 2]Department of Physical Sciences-Mathematics, Baba Mastnath University, Rohtak-124021, Haryana, India
[1]Department of Mathematics, Government College, Matanhail (Jhajjar),124106, Haryana Email addresses-dikshaahlawat14@gmail.com, kamalkumar@bmu.ac.in
*(*Corresponding Author)*

Abstract: *The present study manifests some fixed point results focused on two type of contractive mappings in complete b-metric spaces. One is, the class of p-contraction with relate to a family of mappings and other is, (φ, Γ, β)-contraction mappings. Additionally, we prove uniqueness and a few fixed point theorems for these mentioned contractive mappings on b-metric spaces. These theorems make improvements and builds upon a number of well acknowledged concepts in a variety of current literature. To illustrate how the findings serve as suitable expansions of earlier results, an application is also given. A few examples from which the Banach fixed point theorem is not applicable are provided but these examples supports our findings. Therefore, the fixed point theorems from the existing literature are unified and generalized by our results.*

Keywords: *Fixed point, p-contraction, Euler Gamma function, (φ, Γ, β)-contraction, complete b-metric space, weakly Picard continuous mappings.*

I. Introduction:

A distance function that determines the separation between each member of a set is called metric space. The fixed point theory is one of the most significant tool in numerous fields of study, including economics, computer science, engineering, and the advancement of non-linear analysis. The Banach Contraction Principle, which had been confirmed by the Banach[1] in 1922, is the most famous and extensively applied theorem in fixed point theory. This theorem states that if (X,£) is a metric space, where X is non- empty set and let f is a continuous function with mapping $f: X \to X$ is known as the contraction if there exist $\mu > 0$ such that

$$£(fx, fy) \leq \mu\, £(x, y), \forall\, x, y \in X.$$

where μ is a Lipschitz constant which is less than 1 then f gives a unique fixed point. In addition to this, $x_0 \in X$ be arbitrary point and the Picard sequence $\{f^n x_0\}$ converges to the fixed point. Here, a question arises, the Banach Contraction Principle can be obtained by either weaken the continuity or non contraction. Many mathematicians worked on this. Then, Kannan[14] in 1968, extended the theorem Banach contraction theorem by weaken the condition of continuity i.e., $f: X \to X$ be a mapping such that

$$£(fx, fy) \leq \mu\, (£(x, fx) + £(y, fy),$$

where $0 \leq \mu \leq \frac{1}{2}$ and $x, y \in X$.

After that Chatterjea[7], Riech[17-18], Ciric[8] and many researchers extend this work by defining some other mappings. In continuation with Bernide[3] introduced a very interesting class of mappings which was class of almost contractions which includes Kannan's mapping and other classes of mapping. Khojasteh et al.[16] proved fixed point theorem with class of contractions having simulation function. Olaru and Branga[6] obtained fixed point results for generalized contractions in spaces. Recently, Pacurar and Popescu[17] invented a new class of generalized Chattergea-type mapping. Many authors have been contributed to fixed point theory.

Firstly, in 1989 the b-metric space idea was mentioned by the Bakhtin [2], that ends up into the generalization of metric space. First of all, the b-metric space appeared with in the work of Bakhtin[2] and Czerwik[9]. In b-metric space, we take a real no. which is greater than or equal to one, with in the triangular inequality condition. Normally, the generalization of usual metric space is the b-metric space.

Huang and Samet[13] introduced two new classes self mappings on a metric space. One is, the class of p-contraction with relate to a family of mappings and other is, (φ, Γ, β)-contraction on metric space and gave some fixed point results on complete metric space by using these contractions. Also, proved theorems for fixed points by weaken the continuity in complete metric space for these mapping.

Motived by their work, we have a tendency to extend these mappings on b-metric space and the existence & uniqueness of fixed point for p-contraction and (φ, Γ, β)-contraction mapping proved by taking Picard sequence on complete b-metric spaces. Several illustation also given to support our results and some of examples in which Banach theorem is invalid. Since then, many authors have proved several fixed point results in complete metric spaces and b-metric spaces see [4-5], [8-11], [14,20]. Here, we will also prove some new fixed point results p-contractions and (φ, Γ, β)-contraction in complete b-metric spaces.

II. Preliminaries:

In conjunction with introducing new notions and notations that are significant to fixed point theorems for p-contraction and (φ, Γ, β)-contraction mappings within b-metric spaces, we must go through some fundamental definitions below.

Definition 2.1. In 1989, Bakhtin [2] introduced the concept of b-metric space.
Let X is a non-empty set and $t \geq 1$ be a given real number. Let $£ : X \times Y \to [0, \infty)$ be a function satisfying the following conditions for each $u, v, w \in X$.
(B1) $£(u, v) = 0$ if and only if $u = v$;
(B2) $£(u, v) = £(v, u)$;
(B3) $£(u, w) \leq t(£(u, v) + £(v, w))$.
Then £ is called b-metric and $(X, £)$ is called b-metric space.
It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces. Every metric is a b-metric with $t = 1$.
Definition 2.2.[5] Let $(X, £)$ be a b-metric space. Then any sequence $(u_n) \subseteq X$ is said to be convergent to 'u' if $£(u_n, u) \to 0$ as $n \to \infty$.
Definition 2.3.[5] Let $(X_1, £_1)$ and $(X_2, £_2)$ be two b-metric spaces. Let $f : (X_1, £_1) \to (X_2, £_2)$ is called continuous function at a point $u_0 \in X_1$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $£_2(fx, fu_0) < \varepsilon$ whenever $£_1(x, u_0) < \delta$.

Definition 2.4.[5] Let $(X, Y, £)$ be a b-metric space.

(i) A sequence $\{u_n\}$ on $(X, £)$ is said to be Cauchy sequence, if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $£(u_n, u_m) < \epsilon$ for all $n, m \geq N$.

(ii) A b-metric space is said to be complete if every Cauchy sequence is convergent in this space.

The lemma of Jensen Inequality [11], further it will be used in some fixed point theorems.

Lemma 2.5. Assuming that $J: [0,\infty) \to (-\infty,\infty)$ be a convex function. Then, for every $n \in \mathbb{N}$ and $\{x_i\}$, $\{a_i\} \subset [0,\infty)$,$1 \leq i \leq n$, with $\sum_{i=1}^{n} a_i > 0$, we have

$$J\left[\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right] \leq \frac{\sum_{i=1}^{n} a_i J(x_i)}{\sum_{i=1}^{n} a_i}$$

III. Results:

In this research paper, the category of p-contraction related to a family of mappings and (φ,Γ,β)-contraction mappings in b-metric spaces will be explored. Furthermore, we will demonstrate the uniqueness of fixed point and a few fixed point theorems corresponding to these mappings in the broader setting with complete b-metric spaces. Some of the examples are included solely to emphasize on how the findings satisfy the existing results.

Definition 3.1. Consider $(X, £)$ be a b-metric space and $f : X \to X$ be a given mapping then f is called a contraction for all $p \geq 1$ such that

$$£^p(fx, f^2x) + £^p(f^2x, fy) \leq \mu^p[£^p(x, fx) + £^p(fx, y), \forall\, x, y \in X.$$

i.e,

$$£^p(fx, f^2x) + £^p(f^2x, fy) \leq \mu_p[£^p(x, fx) + £^p(fx, y)], \qquad (3.1)$$

Where $\mu^p = \mu_p \in [0,1)$.

Definition 3.2. Consider $(X, £)$ be a b-metric space, $k \in \mathbb{N}, p \geq 1$ and a mapping $f : X \to X$ is said to be p-contraction with respect to $\{M_j\}_{j=1}^{k}$ a family of mappings $M_j : X \times X \to X$, $j \in \mathbb{N}$ if there exist $\mu \in [0,1)$ such that

$$£^p(fx, M_1(fx, fy)) + \sum_{j=1}^{k-1} £^p\left(M_j(fx, fy), M_{j+1}(fx, fy)\right) + £^p(M_k(fx, fy), fy)$$

$$\leq \mu\left[£^p(x, M_1(x, y)) + \sum_{j=1}^{k-1} £^p\left(M_j(x, y), M_{j+1}(x, y)\right) + £^p(M_k(x, y), y)\right] \qquad (3.2)$$

for every $x, y \in X$.

Note that if we take $k = 1$ and $M_1(x, y) = fx$,$\forall\, x, y \in X$ then a p-contraction with respect to $\{M_1\}$ is a mapping f which satisfying (3.1).

In this paper, we take X is a non empty set. A mapping $f : X \to X$, we denote $\{f^n\}$, the sequence of mappings $f^n : X \to X$ which is defined by

$$f^{n+1}x = f(f^n x), f^0 x = x, \forall\, x \in X \text{ and } n \in \mathbb{N}.$$

We denote $Fix(F)$ for the fixed points of f, i.e.,

$$Fix(F) = \{x \in X: fx = x\}.$$

Similarly, by $Fix(M)$, we mean the fixed points of M, i.e.,

$$Fix(M) = \{x \in M: M(x,x) = x\}.$$

Definition 3.3. A mapping $f : X \to X$ is weakly Picard continuous on $(X, £)$ which is a b-metric space if it satisfied the condition:

$$\lim_{n\to\infty} £(f^n x, y) = 0, \forall\, x, y \in X,$$

then there exist a subsequence $\{f^{n_k}x\}$ of $\{f^n x\}$ such that

$$\lim_{k\to\infty} £(f(f^{n_k}x), fy) = 0.$$

Clearly, a continuous mapping is always weakly Picard continuous but its converse need not be true. An illustration is also given for this purpose.

Example 3.4. Let $f: [0,2] \to [0,2]$ is a mapping defined by

$$fx = \begin{cases} \frac{x}{5}, & 0 \le x < 2 \\ \frac{3}{5}, & x = 2\,. \end{cases}$$

Consider $(X, £)$ be a b-metric space with $£(x,y) = |x - y|$ for all $x, y \in X = [0,2]$.
Clearly, we can easily say that f is discontinuous function at $x = 2$. Now, we prove that f is weakly Picard-continuous.
we observe that for all $n \in \mathbb{N}$, we have

$$f^m x = \begin{cases} \frac{x}{5^m}, & 0 \le x < 2 \\ \left(\frac{3}{5}\right)^m, & x = 2 \end{cases}$$

which yields

$$\lim_{m\to\infty} £(f^m x, 0) = 0, x \in X = [0,2].$$

which proved that f is weakly Picard-continuous.

Theorem 3.5. Let $(X, £)$ be a complete b-metric space, $f: X \to X$ be a mapping. Assuming that the following condition are satisfied:

(i) f is a p-contraction with respect to $\{M_j\}_{j=1}^{k}$

(ii) f is weakly Picard continuous.

Then

I. Every Picard sequence in X converges to a unique fixed point x^* in X

II. $x^* \in \bigcap_{i=1}^{k} Fix(M_i)$.

Proof: Let $x_0 \in X$ be any point and the Picard sequence $\{x_n\} \subset X$ defined by

$$x_n = f^n x_0 \text{ for all } n \in \mathbb{N}.$$

By use of (3.2) with $(x, y) = (x_0, x_1)$, we get $£^p(fx_0, M_1(fx_0, fx_1)) + \sum_{j=1}^{k-1} £^p\left(M_j(fx_0, fx_1), M_{j+1}(fx_0, fx_1)\right) + £^p(M_k(fx_0, fx_1), fx_1)$

$$\leq \mu\left[£^p(x_0, M_1(x_0, x_1)) + \sum_{j=1}^{k-1} £^p\left(M_j(x_0, x_1), M_{j+1}(x_0, x_1)\right) + £^p(M_k(x_0, x_1), x_1)\right]$$

i.e.,

$$£^p(x_1, M_1(x_1, x_2)) + \sum_{j=1}^{k-1} £^p\left(M_j(x_1, x_2) + M_{j+1}(x_1, x_2)\right) + £^p(M_k(x_1, x_2), x_2) \leq \mu[£^p(x_0, M_1(x_0, x_1) + \sum_{j=1}^{k-1} £^p(M_j(x_0, x_1), M_{j+1}(x_0, x_1)) + £^p(M_k(x_0, x_1), x_1)] \quad (3.3)$$

Using again of (3.2) with $(x, y) = (x_1, x_2)$, we have

$$£^p(fx_1, M_1(x_1, fx_2)) + \sum_{j=1}^{k-1} £^p\left(M_j(fx_1, fx_2), M_{j+1}(fx_1, fx_2)\right) + £^p(M_k(fx_1, fx_2), fx_2)$$

$$\leq \mu\left[£^p(x_1, M_1(x_1, x_2)) + \sum_{j=1}^{k-1} £^p\left(M_j(x_1, x_2), M_{j+1}(x_1, x_2)\right) + £^p(M_k(x_1, x_2), x_2)\right]$$

this implies that

$$£^p(x_2, M_1(x_2, x_3)) + \sum_{j=1}^{k-1} £^p\left(M_j(x_2, x_3) + M_{j+1}(x_2, x_3)\right) + £^p(M_k(x_2, x_3), x_3) \leq \mu[£^p(x_1, M_1(x_1, x_2) + \sum_{j=1}^{k-1} £^p(M_j(x_1, x_2), M_{j+1}(x_1, x_2)) + £^p(M_k(x_1, x_2), x_2) \quad (3.4)$$

Making use of (3.3) in (3.4), we get

$$£^p(x_2, M_1(x_2, x_3)) + \sum_{j=1}^{k-1} £^p\left(M_j(x_2, x_3) + M_{j+1}(x_2, x_3)\right) + £^p(M_k(x_2, x_3), x_3) \leq \mu^2[£^p(x_0, M_1(x_0, x_1) + \sum_{j=1}^{k-1} £^p(M_j(x_0, x_1), M_{j+1}(x_0, x_1)) + £^p(M_k(x_0, x_1), x_1)] \quad (3.5)$$

Continuing like this, we get this by induction

$$£^p(x_n, M_1(x_n, x_{n+1})) + \sum_{j=1}^{k-1} £^p\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right) + £^p(M_k(x_n, x_{n+1}), x_{n+1})$$
$$\leq \mu^n[£^p(x_0, M_1(x_0, x_1) + \sum_{j=1}^{k-1} £^p(M_j(x_0, x_1), M_{j+1}(x_0, x_1)) + £^p(M_k(x_0, x_1), x_1)]$$

For our suitability,

$$£^p(x_n, M_1(x_n, x_{n+1})) + \sum_{j=1}^{k-1} £^p\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right) + £^p(M_k(x_n, x_{n+1}), x_{n+1} \leq \mu^n \varrho_0, \forall\, n \in \mathbb{N} \quad (3.6)$$

where

$$\varrho_0 = £^p(x_0, M_1(x_0, x_1) + \sum_{j=1}^{k-1} £^p(M_j(x_0, x_1), M_{j+1}(x_0, x_1)) + +£^p(M_k(x_0, x_1), x_1).$$

Now, from the triangle inequality property

$$£(x_n, x_{n+1}) \leq t\big[£(x_n, M_1(x_n, x_{n+1})\big) + £(M_1(x_n, x_{n+1}), M_2(x_n, x_{n+1})) + \cdots + £(M_{k-1}(x_n, x_{n+1}), M_k(x_n, x_{n+1})) + £(M_k(x_n, x_{n+1}), x_{n+1}))],$$

which gives

$$£^p(x_n, x_{n+1}) \leq t^p\big[£(x_n, M_1(x_n, x_{n+1})\big) + £(M_1(x_n, x_{n+1}), M_2(x_n, x_{n+1})) + \cdots +$$
$$+£(M_{k-1}(x_n, x_{n+1}), M_k(x_n, x_{n+1})) + £(M_k(x_n, x_{n+1}), x_{n+1}))]^p \qquad (3.7)$$

Meanwhile, $x \longmapsto x^p$ is a convex function and by using lemma 2.5, we get

$$t^p\big[£(x_n, M_1(x_n, x_{n+1})\big) + £(M_1(x_n, x_{n+1}) + \cdots + £(M_{k-1}(x_n, x_{n+1}), M_k(x_n, x_{n+1}))$$
$$+ £(M_k(x_n, x_{n+1}), x_{n+1}))]^p = t^p\ (k+1)^p$$

$$\left[\frac{£(x_n, M_1(x_n, x_{n+1})}{k+1} + \frac{\sum_{j=1}^{k-1} £\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right)}{k+1} + \frac{£(M_k(x_n, x_{n+1}), x_{n+1}))}{k+1}\right]^p$$
$$\leq \frac{t^p\ (k+1)^p}{k+1}\left[£^p(x_n, M_1(x_n, x_{n+1}) + \sum_{j=1}^{k-1} £^p\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right) + £^p(M_k(x_n, x_{n+1}), x_{n+1}))\right]$$

i.e.,

$$\left[\frac{£(x_n, M_1(x_n, x_{n+1})}{k+1} + \frac{\sum_{j=1}^{k-1} £\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right)}{k+1} + \frac{£(M_k(x_n, x_{n+1}), x_{n+1}))}{k+1}\right]^p$$
$$\leq t^p\ (k+1)^{p-1}\left[£^p(x_n, M_1(x_n, x_{n+1}) + \sum_{j=1}^{k-1} £^p\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right) + £^p(M_k(x_n, x_{n+1}), x_{n+1}))\right]$$

So, (3.7) becomes

$$£^p(x_n, x_{n+1}) \leq\ t^p(k+1)^{p-1}\Bigg[£^p(x_n, M_1(x_n, x_{n+1}) + \sum_{j=1}^{k-1} £^p\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right)$$
$$+ £^p(M_k(x_n, x_{n+1}), x_{n+1}))\Bigg]$$

i.e., $£(x_n, x_{n+1}) \leq\ t(k+1)^{1-\frac{1}{p}}$

$$\left[£^p(x_n, M_1(x_n, x_{n+1}) + \sum_{j=1}^{k-1} £^p\left(M_j(x_n, x_{n+1}), M_{j+1}(x_n, x_{n+1})\right) + £^p(M_k(x_n, x_{n+1}), x_{n+1}))\right]^{\frac{1}{p}}. \qquad (3.8)$$

It is derived from (3.6) and (3.8) that

$$£(x_n, x_{n+1}) \leq\ t(k+1)^{1-\frac{1}{p}} \varrho_0^{\frac{1}{p}} {\mu_p}^n \qquad (3.9)$$

where

$$0 \leq \mu_p = \mu^{\frac{1}{p}} < 1 \qquad (3.10)$$

Now, follow up (3.9),(3.10) and using triangle inequality property for all $n, m \in \mathbb{N}$, $£(x_n, x_{n+m}) \leq t_1[£(x_n, x_{n+1}) + £(x_{n+1}, x_{n+2}) + \cdots + £(x_{n+m-1}, x_{n+m})], t_1 \geq 1$

$$\leq tt_1(k+1)^{1-\frac{1}{p}} \varrho_0^{\frac{1}{p}} [{\mu_p}^n + {\mu_p}^{n+1} + \cdots + {\mu_p}^{n+m-1}]$$

$$= tt_1(k+1)^{1-\frac{1}{p}}\varrho_0^{\frac{1}{p}}[\frac{\mu_p{}^n(1-\mu_p{}^m)}{1-\mu_p}]$$

$$\leq tt_1(k+1)^{1-\frac{1}{p}}\varrho_0^{\frac{1}{p}}[\frac{\mu_p{}^n}{1-\mu_p}] \to 0 \text{ as } n \to \infty,$$

which shows that $\{x_n\}$ is a Cauchy sequence and $(X, £)$ is a complete b-metric space. So, there exist $x^* \in X$ such that

$$\lim_{n\to\infty} £(x_n, x^*) = \lim_{n\to\infty} £(f^n x_0, x^*) = 0, \quad (3.11)$$

Given that f is a weakly Picard continuous mapping. So, there exist a subsequence $\{f^{n_{k_1}}x\}$ of $\{f^n x\}$ such that

$$\lim_{k_1\to\infty} £(f(f^{n_{k_1}}x), fx^*) = £\left(x_{n_{k_1+1}}, fx^*\right) = 0. \quad (3.12)$$

Hence, by seeing (3.11) and (3.12), we obtain that $fx^* = x^*$ i.e., x^* is the fixed point of f.

The uniqueness of fixed point:-

Let us assume, if possible, x^* and x^{**} are two distinct fixed point of f.

By use of (3.2) with $(x, y) = (x^*, x^{**})$, we get

$$£^p(fx^*, M_1(x^*, fx^{**})) + \sum_{j=1}^{k-1} £^p\left(M_j(fx^*, fx^{**}), M_{j+1}(fx^*, fx^{**})\right) + £^p(M_k(fx^*, fx^{**}), fx^{**})$$

$$\leq \mu\left[£^p(x^*, M_1(x^*, x^{**})) + \sum_{j=1}^{k-1} £^p\left(M_j(x^*, x^{**}), M_{j+1}(x^*, x^{**})\right) + £^p(M_k(x^*, x^{**}), x^{**})\right]$$

this implies that

$$£^p(x^*, M_1(x^*, x^{**})) + \sum_{j=1}^{k-1} £^p\left(M_j(x^*, x^{**}), M_{j+1}(x^*, x^{**})\right) + £^p(M_k(x^*, x^{**}), x^{**})$$

$$\leq \mu\left[£^p(x^*, M_1(x^*, x^{**})) + \sum_{j=1}^{k-1} £^p\left(M_j(x^*, x^{**}), M_{j+1}(x^*, x^{**})\right) + £^p(M_k(x^*, x^{**}), x^{**})\right].$$

This holds only if

$$£\big(x^*, M_1(x^*, x^{**}) = £(M_1(x^*, x^{**}), M_2(x^*, x^{**})\big) = \cdots = £\big(M_{k-1}(x^*, x^{**}), M_k(x^*, x^{**})\big)$$

$$= £(M_k(x^*, x^{**}), x^{**}) = 0 \text{ and } 0 \leq \mu < 1,$$

which gives $x^* = x^{**}$. Hence f has a unique fixed point $x^* \in X$. So, part (I) of the theorem is proved.

Now, we will prove the part (II) of the theorem i.e., $x^* \in_{i=1}^{k} \cap \text{Fix}((M_i)$.

We have already proved that $x^*$is the unique fixed point of $f$in X.

Making the use of in place of $(x, y) = (x^*, x^*)$ in (3.2) then we get

$$£^p(fx^*, M_1(x^*, fx^*)) + \sum_{j=1}^{k-1} £^p\left(M_j(fx^*, fx^*), M_{j+1}(fx^*, fx^*)\right) + £^p(M_k(fx^*, fx^*), fx^*)$$

$$\leq \mu\left[£^p(x^*, M_1(x^*, x^*)) + \sum_{j=1}^{k-1} £^p\left(M_j(x^*, x^*), M_{j+1}(x^*, x^*)\right) + £^p(M_k(x^*, x^*), x^*)\right]$$

this implies that

$$£^p(x^*, M_1(x^*, x^*)) + \sum_{j=1}^{k-1} £^p\left(M_j(x^*, x^*), M_{j+1}(x^*, x^*)\right) + £^p(M_k(x^*, x^{**}), x^{**})$$

$$\leq \mu\left[£^p(x^*, M_1(x^*, x^*)) + \sum_{j=1}^{k-1} £^p\left(M_j(x^*, x^*), M_{j+1}(x^*, x^*)\right) + £^p(M_k(x^*, x^*), x^*)\right].$$

This holds only if

$$£\big(x^*, M_1(x^*, x^*) = £(M_1(x^*, x^*), M_2(x^*, x^*)\big) = \cdots = £(M_{k-1}(x^*, x^*), M_k(x^*, x^*)) = £(M_k(x^*, x^*), x^*) = 0$$

and $0 \leq \mu < 1$,

So, in this manner we obtain that

$$x^* = M_1(x^*, x^*) = M_2(x^*, x^*) = \cdots = M_k(x^*, x^*),$$

Hence, $x^* \in \bigcap_{i=1}^{k} Fix(M_i)$.So, the proof of part (II) of the theorem is completed.

Here we discuss some particular cases of above theorem.

Corollary 3.6. Let $(X,£)$ be a complete b-metric space, $p \geq 1, k = 1$ and $M_1 : X \times X \to X$ be a mapping. Consider $f: X \to X$ is a function which satisfied the following condition :

(i) f is weakly Picard continuous

(ii) There exist a constant $\mu \in [0,1)$ such that

$$£^p(fx, M_1(fx,fy)) + £^p(M_1(fx,fy),fy) \leq \mu[£^p(x, M_1(x,y)) + £^p(M_1(x,y),y)], \forall\, x,y \in X. \quad (3.13)$$

Then

I. Every Picard sequence in X converges to a unique fixed point x^* in X.

II. $x^* \in Fix(M_1)$.

By taking $M_1(x,y) = fx, \forall\, x,y \in X$. We reach the following result from the corollary.

Corollary 3.7. Let $(X,£)$ be a complete b-metric space, $p \geq 1$. Consider $f: X \to X$ is a function which satisfied the following condition :

(i) f is weakly Picard continuous

(ii) There exist a constant $\mu \in [0,1)$ such that

$$£^p(fx,f^2x) + £^p(f^2x,fy) \leq \mu^p[£^p(x,fx) + £^p(fx,y)], \forall\, x,y \in X. \quad (3.14)$$

Using $k = 2$ in theorem 3.5., we deduce the following result.

Corollary 3.8. Let $(X,£)$ be a complete b-metric space, $p \geq 1$, and $M_1, M_2 : X \times X \to X$ be a mapping. Consider $f: X \to X$ is a function which satisfied the following condition :

(i) f is weakly Picard continuous

(ii) There exist a constant $\mu \in [0,1)$ such that

$$£^p(fx, M_1(fx,fy)) + £^p\big(M_1(fx,fy), M_2\,(fx,fy)\big) + £^p(M_2(fx,fy),fy)$$
$$\leq \mu[£^p(x, M_1(x,y)) + £^p(M_1(x,y), M_2\,(x,y)) + £^p(M_2(x,y),y)]\,, \forall\, x,y \in X. \quad (3.15)$$

Then

I. Every Picard sequence in X converges to a unique fixed point x^* in X.

II. $x^*\ Fix(M_1) \cap Fix(M_2)$.

If we use continuity in place of weakly Picard continuous then we obtain the following statement from the theorem 3.5.

Corollary 3.9. Let $(X,£)$ be a complete b-metric space, $f: X \to X$ be a mapping. Assuming that the following condition are satisfied:

(i) f is a p-contraction with respect to $\{M_j\}_{j=1}^k$

(ii) f is continuous.

Then

I. Every Picard sequence in X converges to a unique fixed point x^* in X

II. $x^* \in \bigcap_{i=1}^k Fix(M_i)$.

Corollary 3.10. Let $(X, £)$ be a complete b-metric space, $f: X \to X$ be a mapping. Assuming that there exist $\mu \in [0,1)$ such that

$$£(fx, fy) \leq \mu \, £(x, y), \forall\, x, y \in X. \tag{3.16}$$

Then Picard sequence in X converges to a unique fixed point x^* in X.

Proof: Taking $p = k = 1$ and $M_1 : X \times X \to X$ be a mapping such that $M_1\,(x, y) = x$.

So, (3.1) becomes

$$£(fx, fx) + £(fx, fy) \leq \mu[£(x, x) + £(x, y)]$$

i.e.,

$$£(fx, fy) \leq \mu £(x, y), \forall\, x, y \in X.$$

Hence, f satisfied the above Banach contraction condition and f is continuous mapping. By Corollary 2.9., we got our result.

Next here a few examples that highlight the results, we achieve through our efforts.

Example 3.11. Consider $X = \{x_1, x_2, x_3,\}$ and $f: X \to X$ is a mapping defined by

$$fx_1 = x_3, fx_2 = x_2, fx_3 = x_2.$$

$(X, £)$ be a discrete b-metric space. i.e.,

$$£(x_i, x_j) = \begin{cases} 0, & i = j, \\ 1, & i \neq j\,. \end{cases}$$

Note that

$$\frac{£(fx_1, fx_3)}{£(x_1, x_3)} = \frac{£(x_3, x_2)}{£(x_1, x_3)} = 1,$$

which gives that there does not such $\mu \in [0,1)$. So, Banach contraction principle is not valid in the example.

Now, we define a mapping $M_1 : X \times X \to X$ be a mapping such that

$$M_1\,(x_i, x_i) = x_i, M_1\,(x_i, x_j) = M_1\,(x_j, x_i), \quad i, j \in \{1,2,3\}$$

and

$$M_1\,(x_1, x_2) = x_3, M_1\,(x_2, x_3) = M_1\,(x_1, x_3) = x_2.$$

We claim that

$$£\left(fx_i, M_1\left(fx_i, fx_j\right)\right) + £\left(M_1\left(fx_i, fx_j\right), fx_j\right) \leq \frac{2}{3}\left[£\left(x_i, M_1\left(x_i, x_j\right)\right) + £\left(M_1\left(x_i, x_j\right), x_j\right)\right] \text{ for every } x_i, x_j \in X. \tag{3.17}$$

i.e., f provides (3.13) with $\mu = \frac{2}{3}$ and $p = 1$.

Firstly, we take $i = j$,

$$£\left(fx_i, M_1\,(fx_i, fx_i)\right) + £(M_1\,(fx_i, fx_i), fx_i)$$
$$= £(fx_i, fx_i) + £(fx_i, fx_i) = 0,$$

which yield into (3.17).

Then, by symmetry, we have proved that (3.17) is satisfied for $(i, j) \in \{\,(1,2), (2,3), (1,3)\}$.

Case 1:- $(i, j) = (1,2)$. Here, we have

$$\frac{£\left(fx_i, M_1\left(fx_i, fx_j\right)\right) + £\left(M_1\left(fx_i, fx_j\right), fx_j\right)}{£\left(x_i, M_1\left(x_i, x_j\right)\right) + £\left(M_1\left(x_i, x_j\right), x_j\right)}$$

$$= \frac{£\big(fx_1, M_1\,(fx_1, fx_2)\big) + £\big(M_1\,(fx_1, fx_2), fx_2\big)}{£\big(x_1, M_1\,(x_1, x_2)\big) + £\big(M_1\,(x_1, x_2), x_2\big)}$$

$$= \frac{£\big(x_3, M_1\,(x_3, x_2)\big) + £\big(M_1\,(x_3, x_2), x_2\big)}{£\big(x_1, M_1\,(x_1, x_2)\big) + £\big(M_1\,(x_1, x_2), x_2\big)}$$

$$= \frac{£(x_3, x_2) + £(x_2, x_2)}{£(x_1, x_3) + £(x_3, x_2)}$$

$$= \frac{1}{2} < \frac{2}{3}$$

which obtained (3.17).

Case 2:- $(i, j) = (2,3)$, we get

$$\frac{£\Big(fx_i, M_1\,\big(fx_i, fx_j\big)\Big) + £\big(M_1\,\big(fx_i, fx_j\big), fx_j\big)}{£\Big(x_i, M_1\,\big(x_i, x_j\big)\Big) + £\big(M_1\,\big(x_i, x_j\big), x_j\big)}$$

$$= \frac{£\big(fx_2, M_1\,(fx_2, fx_3)\big) + £\big(M_1\,(fx_2, fx_3), fx_3\big)}{£\big(x_2, M_1\,(x_2, x_3)\big) + £\big(M_1\,(x_2, x_3), x_3\big)}$$

$$= \frac{£\big(x_2, M_1\,(x_2, x_2)\big) + £\big(M_1\,(x_2, x_2), x_2\big)}{£\big(x_2, M_1\,(x_2, x_3)\big) + £\big(M_1\,(x_2, x_3), x_3\big)}$$

$$= \frac{£(x_2, x_2) + £(x_2, x_2)}{£(x_2, x_2) + £(x_2, x_3)}$$

$$= 0$$

which result in (3.17) is satisfied.

Case 3:- If $(i, j) = (1,3)$ then we have

$$\frac{£\Big(fx_i, M_1\,\big(fx_i, fx_j\big)\Big) + £\big(M_1\,\big(fx_i, fx_j\big), fx_j\big)}{£\Big(x_i, M_1\,\big(x_i, x_j\big)\Big) + £\big(M_1\,\big(x_i, x_j\big), x_j\big)}$$

$$= \frac{£\big(fx_1, M_1\,(fx_1, fx_3)\big) + £\big(M_1\,(fx_1, fx_3), fx_3\big)}{£\big(x_1, M_1\,(x_1, x_3)\big) + £\big(M_1\,(x_1, x_3), x_3\big)}$$

$$= \frac{£\big(x_3, M_1\,(x_3, x_2)\big) + £\big(M_1\,(x_3, x_2), x_2\big)}{£\big(x_1, M_1\,(x_1, x_3)\big) + £\big(M_1\,(x_1, x_3), x_3\big)}$$

$$= \frac{£(x_3, x_2) + £(x_2, x_2)}{£(x_1, x_2) + £(x_2, x_3)}$$

$$= \frac{1}{2} < \frac{2}{3}$$

which satisfied (3.17).

After analysing all the above cases, we conclude that (3.17) is true for every $(i, j) = \{1,2,3\}$.

As f is continuous function then it is weakly Picard continuous on $(X, £)$ complete b-metric space. As a consequence, Corollary 3.6. is applicable.

$$\text{Fix}(f) = \{x_2\} \text{ and } M_1\,(x_2, x_2) = x_2,$$

which supports corollary 3.6.

Example 3.12. Consider $([0,1], £)$ be a b-metric space with $£(x, y) = |x - y|$ for all $x, y \in [0,1]$ and $f\colon [0,1] \to [0,1]$ is a mapping defined by

$$fx = \begin{cases} 0, & 0 \leq x < 1 \\ \frac{1}{3}, & x = 1\,. \end{cases}$$

We claim that

$$£(fx, f^2x) + £(f^2x, fy) \leq \frac{2}{3}[\,£(x, fx) + \,£(fx, y)], \forall\, x, y \in [0,1]. \qquad (3.18)$$

i.e., f provides (3.14) with $\mu = \frac{2}{3}$ and $p = 1$.

Clearly, we can easily say that f is discontinuous function at $x = 1$. So, f is not a contraction. As a result of which Banach theorem is not applicable.

We will prove that f is weakly Picard-continuous.

Now,

$$f^m x = \begin{cases} 0, & 0 \leq x < 1 \\ \left(\frac{1}{3}\right)^m, & x = 2 \end{cases}$$

which yields into

$$\lim_{m \to \infty} £(f^m x, 0) = 0, x \in [0,1].$$

which provided that f is weakly Picard-continuous.

Condition (i) of Corollary 3.7. is hold.

Here, we discuss following case to achieve condition (ii) of Corollary 3.7 with $\mu = \frac{2}{3}$ and $p = 1$.

Case 1:- $x = y = 1$, then we have

$$\frac{£(fx, f^2x) + £(f^2x, fy)}{£(x, fx) + \,£(fx, y)}$$

$$= \frac{£(f1, f^21) + £(f^21, f1)}{£(1, f1) + \,£(f1,1)}$$

$$= \frac{£\left(\frac{1}{3}, 0\right) + £\left(0, \frac{1}{3}\right)}{£\left(1, \frac{1}{3}\right) + \,£\left(\frac{1}{3}, 1\right)}$$

$$= \frac{1}{2} < \frac{2}{3}$$

which gives as a result of (3.18).

Case 2:- $0 \leq x, y < 1$ then we get

$$£(fx, f^2x) + £(f^2x, fy)$$

$$= \,£(0, f0) + £(f0,0) = 0$$

which yields (3.18).

Case 3:- $0 \leq x < 1, y = 1$ then we have

$$\frac{£(fx, f^2x) + £(f^2x, fy)}{£(x, fx) + \,£(fx, y)}$$

$$= \frac{£(0, f0) + £(f0, f1)}{£(x, 0) + \,£(0,1)}$$

$$= \frac{£(0,0) + £\left(0, \frac{1}{3}\right)}{£(x, 0) + \,£(0,1)}$$

$$= \frac{1}{3(x+1)}$$

$$\leq \frac{2}{3}$$

which satisfied (3.18).

Case 4:- $0 \leq y < 1$, $x = 1$. Here, we have

$$\frac{£(fx, f^2x) + £(f^2x, fy)}{£(x, fx) + £(fx, y)}$$

$$= \frac{£(f1, f^2 1) + £(f^2 1, 0)}{£(1, f1) + £(f1, y)}$$

$$= \frac{£\left(\frac{1}{3}, 0\right) + £(0,0)}{£\left(1,, \frac{1}{3}\right) + £\left(\frac{1}{3}, y\right)}$$

$$= \frac{1}{3(|y-1|)}$$

$$\leq \frac{2}{3}$$

which obtained (3.18).

After examining all the above cases, we conclude that (3.18) is holds or every $x, y \in [0,1]$.

As result of which f holds (3.14) with $\mu = \frac{2}{3}$ and $p = 1$ and we had already proved that f is weakly Picard continuous on $([0,1], £)$ complete b-metric space. As a consequence, Corollary 3.7 hold. Hence, 0 is the unique fixed point of f on $[0,1]$.

Application for fixed point

The following theorem is an application of Theorem 3.5 in which we got the sufficient condition of a mapping to get unique fixed point.

Theorem 3.13. Let $(X, £)$ be usual b-metric space and $g : X \to X$ is a function which satisfy the condition

$$3^{g(a)+g(b)+1} - g(a) - g(b) \leq \gamma(3^{a+b+1} - a - b), \qquad (3.19)$$

for all $a, b \in X = \mathbb{N}$ with $g(a) \neq g(b)$ and $0 \leq \gamma < 1$. Then g give a unique fixed point.

Proof: Firstly, we define a mapping $M_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$M_1(a,b) = \begin{cases} 3^{a+b}, & a \neq b \\ a, & a = b\,. \end{cases}$$

We claim that

$$£\big(ga, M_1(ga, gb)\big) + £(M_1(ga, gb), gb) \leq \gamma(£\big(a, M_1(a,b)\big) + £(M_1(a,b), b)$$

i.e.,

$$|ga - M_1(ga, gb)| + |M_1(ga, gb) - gb| \leq \gamma(|a - M_1(a,b)| + |M_1(a,b) - b|) \qquad (3.20)$$

There are two possible cases discuss.

Case 1:- In this case, we take $ga = gb$ then we have

$$|ga - M_1(ga, gb)| + |M_1(ga, gb) - gb|$$
$$= |ga - M_1(ga, ga)| + |M_1(ga, ga) - ga|$$
$$= |ga - ga| + |ga - ga|$$
$$= 0$$

which satisfies (3.20).

Case 2:- Take $ga \neq gb$ then we get

$$|ga - M_1(ga, gb)| + |M_1(ga, gb) - gb|$$
$$= |ga - 3^{ga+gb}| + |3^{ga+gb} - gb|$$
$$= 3^{ga+gb} - ga + 3^{ga+gb} - gb$$
$$= 3^{ga+gb+1} - ga - gb$$

Making the use of (3.19), then we obtain

$$|ga - M_1(ga, gb)| + |M_1(ga, gb) - gb| \leq \gamma(3^{a+b+1} - a - b) \leq \gamma(3^{a+b} - a + 3^{a+b} - b)$$
$$\leq \gamma(|3^{a+b} - a| + |3^{a+b} - b|)$$
$$\leq \gamma(|a - M_1(a, b)| + |M_1(a, b) - b|)$$

which is the result of (3.20).

Since g is defined on $\mathbb{N}$. So, g is continuous function on $(X, £)$ and g fulfils (3.13) with $p = k = 1$. Hence, g satisfied both the conditions of Corollary 3.6. which yields that g gives a unique fixed point. This completes the proof.

Now, we give an example which supports our theorem 3.13.

Example 3.14. Let us define a mapping $g : \mathbb{N} \to \mathbb{N}$ such that

$$ga = \begin{cases} a - 2, & a \geq 3 \\ 1, & a = 1,2 \end{cases}.$$

And $(\mathbb{N}, £)$ be a usual b-metric space.

We claim that

$$3^{g(a)+g(b)+1} - g(a) - g(b) \leq \frac{1}{9}(3^{a+b+1} - a - b) \quad (3.21)$$

We note that

$$\lim_{a\to\infty} \frac{|ga - g1|}{a - 1} = \lim_{a\to\infty} \frac{a - 3}{a - 1} = 1$$

Which gives that there does not exist any $0 \leq \gamma < 1$. So, g is not a contraction. Hence, Banach theorem is not applicable here.

Without neglecting the generality, due to symmetry of (3.19), there may be possibility of two case which are discuss as follows as:

Case 1:- In this case, $a = 1,\ b \geq 3$ then we get

$$\frac{3^{g(a)+g(b)+1} - g(a) - g(b)}{3^{a+b+1} - a - b}$$
$$= \frac{3^{g(1)+g(b)+1} - g(1) - g(b)}{3^{1+b+1} - 1 - b}$$

$$= \frac{3^b - b + 1}{3^{b+2} - b - 1}$$

$$= \frac{3^{b+2} - 9b + 9}{3^2(3^{b+2} - b - 1)}$$

$$\leq \frac{1}{9}$$

Which yields (3.21).

Similarly, this is hold for $a = 2,\ b \geq 3$.

Case 2:- In this case, $a > b > 2$ then we get

$$\frac{3^{g(a)+g(b)+1} - g(a) - g(b)}{3^{a+b+1} - a - b}$$

$$= \frac{3^{a+b-3} - a - b + 4}{3^{a+b+1} - a - b}$$

$$= \frac{3^{a+b-1} - 9(a + b - 4)}{3^2(3^{a+b+1} - a - b)}$$

$$\leq \frac{1}{9}$$

which is result of (3.21).

Hence, g fulfils (3.19) with $\gamma = \frac{1}{9}$. So, Theorem 3.13 is applicable. Though, 1 is the fixed point of g with satisfying above theorem.

Before defining a new type of contractions, we recall some important properties of Euler Gamma Function,
The Euler Gamma function is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad z > 0. \tag{3.22}$$

An integration by parts, we get

$$\Gamma(z+1) = z\,\Gamma(z),\ z > 0, \tag{3.23}$$

When z is a natural number then we have

$$\Gamma(z+1) = z!\,.$$

Gamma function is In-convex i.e.,

$$\Gamma(\beta z + (1-\beta)x) \leq \Gamma^{\beta}(z)\,\Gamma^{1-\beta}(x),\ z, t > 0 > \text{ and } 0 < \beta < 1. \tag{3.24}$$

We denote Φ be the set of functions $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\varphi(x) \geq a x^b,\ a, b > 0. \tag{3.25}$$

Clearly, we can see that

$$\varphi(x) > 0,\ x > 0. \tag{3.26}$$

Definition 3.15. Let $(X, \pounds)$ be a b-metric space and a mapping $f : X \to X$ is said to be (φ, Γ, β)-contraction if there exist $\beta, \xi \in (0,1)$ and $\varphi \in \Phi$ with the condition

$$\varphi\left(\frac{\Gamma(£(fx,fy)+1)}{\Gamma(£(fx,fy)+\beta)}\right)\leq\ \xi\ \varphi\left(\frac{\Gamma(£(x,y)+1)}{\Gamma(£(x,y)+\beta)}\right) \tag{3.27}$$

for all $x,y\in X$.

Theorem 3.16. Consider $(X,£)$ be a complete b-metric space, $f\colon X\to X$ is a mapping. Assuming that the following condition are hold:

(i) f is a (φ,Γ,β)-contraction

(ii) f is weakly Picard continuous.

Then every Picard sequence in X converges to a unique fixed point x^* in X.

Proof: Let $x_0\in X$ be any point and the Picard sequence $\{x_n\}\subset X$ defined as

$$x_n=f^n x_0 \text{ for all } n\in\mathbb{N}.$$

By use of (3.27) with $(x,y)=(x_0,x_1)$, we have

$$\varphi\left(\frac{\Gamma(£(fx_0,fx_1)+1)}{\Gamma(£(fx_0,fx_1)+\beta)}\right)\leq\ \xi\ \varphi\left(\frac{\Gamma(£(x_0,x_1)+1)}{\Gamma(£(x_0,x_1)+\beta)}\right)$$

i.e.,

$$\varphi\left(\frac{\Gamma(£(x_1,x_2)+1)}{\Gamma(£(x_1,x_2)+\beta)}\right)\leq\ \xi\ \varphi\left(\frac{\Gamma(£(x_0,x_1)+1)}{\Gamma(£(x_0,x_1)+\beta)}\right) \tag{3.28}$$

Further, using in place of $(x,y)=(x_1,x_2)$ in (3.27), we get

$$\varphi\left(\frac{\Gamma(£(fx_1f,x_2)+1)}{\Gamma(£(fx_1,fx_2)+\beta)}\right)\leq\ \xi\ \varphi\left(\frac{\Gamma(£(x_1,x_2)+1)}{\Gamma(£(x_1,x_2)+\beta)}\right)$$

i.e.,

$$\varphi\left(\frac{\Gamma(£(x_2,x_3)+1)}{\Gamma(£(x_2,x_3)+\beta)}\right)\leq\ \xi\ \varphi\left(\frac{\Gamma(£(x_1,x_2)+1)}{\Gamma(£(x_1,x_2)+\beta)}\right) \tag{3.29}$$

Also, it follows from (3.28) and (3.29) that

$$\varphi\left(\frac{\Gamma(£(x_2,x_3)+1)}{\Gamma(£(x_2,x_3)+\beta)}\right)\leq\xi^2\ \varphi\left(\frac{\Gamma(£(x_0,x_1)+1)}{\Gamma(£(x_0,x_1)+\beta)}\right).$$

Using the same approach, by induction method we have

$$\varphi\left(\frac{\Gamma(£(x_n,x_{n+1})+1)}{\Gamma(£(x_n,x_{n+1})+\beta)}\right)\leq\xi^n\ \varphi\left(\frac{\Gamma(£(x_0,x_1)+1)}{\Gamma(£(x_0,x_1)+\beta)}\right),\ n\in\mathbb{N}. \tag{3.30}$$

However, from (3.25), we get

$$\varphi\left(\frac{\Gamma(£(x_n,x_{n+1})+1)}{\Gamma(£(x_n,x_{n+1})+\beta)}\right)\geq a\left(\frac{\Gamma(£(x_n,x_{n+1})+1)}{\Gamma(£(x_n,x_{n+1})+\beta)}\right)^b \tag{3.31}$$

Combining (3.30) and (3.31) implies that

$$a\left(\frac{\Gamma(£(x_n,x_{n+1})+1)}{\Gamma(£(x_n,x_{n+1})+\beta)}\right)^b\leq\xi^n\ \varphi\left(\frac{\Gamma(£(x_0,x_1)+1)}{\Gamma(£(x_0,x_1)+\beta)}\right)$$

i.e.,

$$\frac{\Gamma(£(x_n,x_{n+1})+1)}{\Gamma(£(x_n,x_{n+1})+\beta)}\leq\xi^{\frac{n}{b}}\left[\frac{1}{a}\ \varphi\left(\frac{\Gamma(£(x_0,x_1)+1)}{\Gamma(£(x_0,x_1)+\beta)}\right)\right]^{\frac{1}{b}} \tag{3.32}$$

However, Gamma is In-convex function.so, from (3.24), we have

$$\begin{aligned}\Gamma(£(x_n,x_{n+1})+\beta)&=\Gamma((1-\beta)£(x_n,x_{n+1})+\beta(£(x_n,x_{n+1})+1))\\&\leq\ \Gamma^{1-\beta}(£(x_n,x_{n+1}))\,\Gamma^{\beta}(£(x_n,x_{n+1})+1)\end{aligned} \tag{3.33}$$

By the result of Gamma function (3.23), we get

$$\Gamma(£(x_n,x_{n+1})+1)=£(x_n,x_{n+1})\Gamma\big(£(x_n,x_{n+1})\big)$$

which gives

$$\Gamma^{1-\beta}(£(x_n, x_{n+1}) + 1) = £(x_n, x_{n+1})^{1-\beta}\, \Gamma^{1-\beta}(£(x_n, x_{n+1}))$$

which implies that

$$\Gamma^{1-\beta}\big(£(x_n, x_{n+1})\big) = £(x_n, x_{n+1})^{\beta-1}\, \Gamma^{1-\beta}(£(x_n, x_{n+1}))$$

So, from (3.33), we evaluate that

$$\begin{aligned}\Gamma(£(x_n, x_{n+1}) + \beta) &\leq £(x_n, x_{n+1})^{\beta-1}\, \Gamma^{1-\beta}(£(x_n, x_{n+1}))\;\; \Gamma^{\beta}(£(x_n, x_{n+1}) + 1) \\ &= £(x_n, x_{n+1})^{\beta-1} \Gamma(£(x_n, x_{n+1}) + 1)\end{aligned}$$

i.e.,

$$£(x_n, x_{n+1})^{1-\beta} \leq \frac{\Gamma(£(x_n, x_{n+1})+1)}{\Gamma(£(x_n, x_{n+1})+\beta)} \tag{3.34}$$

Seeing both (3.32) and (3.34), we have

$$£(x_n, x_{n+1})^{1-\beta} \leq \xi^{\frac{n}{b}} \left[\frac{1}{a}\, \varphi\left(\frac{\Gamma(£(x_0, x_1)+1)}{\Gamma(£(x_0, x_1)+\beta)}\right)\right]^{\frac{1}{b}},\; n \in \mathbb{N}$$

i.e.,

$$£(x_n, x_{n+1}) \leq \xi^{\frac{n}{b(1-\beta)}} \left[\frac{1}{a}\, \varphi\left(\frac{\Gamma(£(x_0, x_1) + 1)}{\Gamma(£(x_0, x_1) + \beta)}\right)\right]^{\frac{1}{b(1-\beta)}}$$

which is written as

$$£(x_n, x_{n+1}) \leq \varrho^n \varphi_0,\; n \in \mathbb{N} \tag{3.35}$$

where

$$\varrho = \xi^{\frac{1}{b(1-\beta)}} \in (0,1)$$

and

$$\varphi_0 = \left[\frac{1}{a}\, \varphi\left(\frac{\Gamma(£(x_0, x_1) + 1)}{\Gamma(£(x_0, x_1) + \beta)}\right)\right]^{\frac{1}{b(1-\beta)}}$$

Now, follow up (3.35) and using triangle inequality property for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}£(x_n, x_{n+m}) &\leq t_1[£(x_n, x_{n+1}) + £(x_{n+1}, x_{n+2}) + \cdots + £(x_{n+m-1}, x_{n+m})], t_1 \geq 1 \\ &\leq t_1(\varrho^n + \varrho^{n+1} + \varrho^{n+2} + \cdots + \varrho^{n+m-1})\varphi_0 \\ &= t_1 \frac{\varrho^n(1-\varrho^m)}{1-\varrho} \varphi_0 \\ &\leq t_1 \frac{\varrho^n}{1-\varrho} \varphi_0 \to 0 \text{ as } n \to \infty,\end{aligned}$$

which proves that $\{x_n\}$ is a Cauchy sequence and $(X, £)$ is a complete b-metric space. So, there exist $x^* \in X$ such that

$$\lim_{n\to\infty} £(x_n, x^*) = \lim_{n\to\infty} £(f^n x_0, x^*) = 0, \tag{3.36}$$

Given that f is a weakly Picard continuous mapping. So, there exist a subsequence $\{f^{n_{k_1}} x\}$ of $\{f^n x\}$ such that

$$\lim_{k_1\to\infty} £(f(f^{n_{k_1}} x), f x^*) = £\left(x_{n_{k_1+1}}, f x^*\right) = 0. \tag{3.37}$$

Hence, by seeing both (3.36) and (3.37), we obtain that $fx^* = x^*$ i.e., x^* is the fixed point of f in $(X, £)$.

To prove the uniqueness of fixed point of f, let, if possible, x^* and y^* are two distinct fixed point of f (i.e., $£(x^*, y^*) > 0$).

By use of (3.27) with $(x,y) = (x^*, y^*)$, we get

$$\varphi\left(\frac{\Gamma(£(fx^*, fy^*) + 1)}{\Gamma(£(fx^*, fy^*) + \beta)}\right) \leq \xi\, \varphi\left(\frac{\Gamma(£(x^*, y^*) + 1)}{\Gamma(£(x^*, y^*) + \beta)}\right)$$

i.e.,

$$\varphi\left(\frac{\Gamma(£(x^*,y^*)+1)}{\Gamma(£(x^*,y^*)+\beta)}\right) \leq \xi\, \varphi\left(\frac{\Gamma(£(x^*,y^*)+1)}{\Gamma(£(x^*,y^*)+\beta)}\right) \tag{3.38}$$

Furthermore,

$$\frac{\Gamma(£(x^*, y^*) + 1)}{\Gamma(£(x^*, y^*) + \beta)} > 0$$

So, by (3.26), we get

$$\varphi\left(\frac{\Gamma(£(x^*, y^*) + 1)}{\Gamma(£(x^*, y^*) + \beta)}\right) > 0$$

Hence, from (3.38), we obtain a contradiction with $\xi \in (0,1)$.

Thus, x^* is the unique fixed point of f. This completes the proof.

If we use continuity in place of weakly Picard continuous then we obtain the following Corollary of the Theorem 3.16.

Corollary 3.17. Consider $(X, £)$ be a complete b-metric space, $f: X \to X$ is a mapping. Assuming that the following condition are hold:

(i) f is a (φ, Γ, β)-contraction

(ii) f is continuous.

Then every Picard sequence in X converges to a unique fixed point x^* in X.

We provide below an illustration to achieve our result in theorem 3.16.

Example 3.18. Assume $X = \{x_1, x_2, x_3,\}$ and $f: X \to X$ is a mapping defined by

$$fx_1 = x_1, fx_2 = x_1, fx_3 = x_2.$$

Let $£ : X \times X \to [0, \infty)$ be a function such that

$$£(x_i, x_j) = £(x_j, x_i), £(x_i, x_i) = 0, \quad i, j \in \{1,2,3\}$$

and

$$£(x_1, x_2) = 1, £(x_2, x_3) = 2, £(x_1, x_3) = 3.$$

(B1) and (B2) property of b-metric are hold.

To check (B3) for b-metric:-

$$£(x_1, x_2) = 1 < 5 = £(x_1, x_3) + £(x_3, x_2),$$

$$£(x_2, x_3) = 2 < 4 = £(x_2, x_1) + £(x_1, x_3),$$

Hence, (B3) is also hold with $t = 1$.

As a result, we obtain that $(X, £)$ is b-metric space.

Now, we consider a mapping $\varphi : [0,\infty) \to [0,\infty)$ defined by

$$\varphi(x) = \begin{cases} \frac{1}{2} + \frac{\sqrt{\pi}x}{4}, & 0 \le x \le \frac{2}{\sqrt{\pi}} \\ \frac{\sqrt{\pi}x}{2}, & \frac{2}{\sqrt{\pi}} < x \le \frac{4}{\sqrt{\pi}} \\ \frac{15\sqrt{\pi}x}{32} + \frac{3}{2}, & x \ge \frac{4}{\sqrt{\pi}} \end{cases}$$

It is easy to notice that

$$\varphi(x) \ge \frac{\sqrt{\pi}x}{4}, x \ge 0,$$

Which obtain that $\varphi \in \Phi$ by satisfying (3.25) with $a = \frac{\sqrt{\pi}}{4}$ and $b = 1$.

Furthermore, by (3.27), the definition of φ and the symmetry ,we get

Firstly, $(x,y) = (x_1, x_2)$

$$\frac{\varphi\left(\frac{\Gamma(£(fx_1,fx_2)+1)}{\Gamma(£(fx_1,fx_2)+\frac{1}{2})}\right)}{\varphi\left(\frac{\Gamma(£(x_1,x_2)+1)}{\Gamma(£(x_1,x_2)+\frac{1}{2})}\right)} = \frac{\varphi\left(\frac{\Gamma 1}{\Gamma\frac{1}{2}}\right)}{\varphi\left(\frac{\Gamma 2}{\Gamma\frac{3}{2}}\right)} = \frac{\varphi\left(\frac{2}{\sqrt{\pi}}\right)}{\varphi\left(\frac{4}{\sqrt{\pi}}\right)} = \frac{1}{2}$$

We take $(x,y) = (x_2, x_3)$ in (3.27), we get

$$\frac{\varphi\left(\frac{\Gamma(£(fx_2,fx_3)+1)}{\Gamma(£(fx_2,fx_3)+\frac{1}{2})}\right)}{\varphi\left(\frac{\Gamma(£(x_2,x_3)+1)}{\Gamma(£(x_2,x_3)+\frac{1}{2})}\right)} = \frac{\varphi\left(\frac{\Gamma 2}{\Gamma\frac{3}{2}}\right)}{\varphi\left(\frac{\Gamma 3}{\Gamma\frac{5}{2}}\right)} = \frac{\varphi\left(\frac{4}{\sqrt{\pi}}\right)}{\varphi\left(\frac{16}{3\sqrt{\pi}}\right)} = \frac{1}{8}$$

Using $(x,y) = (x_1, x_3)$ in (3.27), we have

$$\frac{\varphi\left(\frac{\Gamma(£(fx_1,fx_3)+1)}{\Gamma(£(fx_1,fx_3)+\frac{1}{2})}\right)}{\varphi\left(\frac{\Gamma(£(x_1,x_3)+1)}{\Gamma(£(x_1,x_3)+\frac{1}{2})}\right)} = \frac{\varphi\left(\frac{\Gamma 2}{\Gamma\frac{3}{2}}\right)}{\varphi\left(\frac{\Gamma 4}{\Gamma\frac{7}{2}}\right)} = \frac{\varphi\left(\frac{4}{\sqrt{\pi}}\right)}{\varphi\left(\frac{32}{5\sqrt{\pi}}\right)} = \frac{1}{9}$$

Form the above calculations and the symmetric property of £, we obtained that

$$\varphi\left(\frac{\Gamma(£(fx_i,fx_j)+1)}{\Gamma(£(fx_i,fx_j)+\frac{1}{2})}\right) \le \xi\,\varphi\left(\frac{\Gamma(£(x_i,x_j)+1)}{\Gamma(£(x_i,x_j)+\frac{1}{2})}\right)$$

for all $(i,j) \in \{1,2,3\}$ and $\frac{1}{2} \le \xi < 1$.

As a consequence, (3.27) is hold with $\beta = \frac{1}{2}$ and $\frac{1}{2} \le \xi < 1$.

On the other hand, X is finite set then f is continuous function. So, Corollary 3.17 is valid for this example. As we know that every continuous function is weakly Picard continuous. Hence,
Theorem 3.16 is also applicable and the unique fixed point of f is x_1 in X.

IV. Conclusion:-

Banach Contraction Principle is one of famous result in non-linear analysis. It has vital applications in fixed point theory. Meanwhile, this principle is invalid when the mapping is not contraction. Inspired by this, we are introduced two new class of non-contraction self mappings on complete b-metric space $(X, £)$. Firstly, we defined the class of p-contraction with relate to a family of mappings and secondly,(φ, Γ, β)-contraction mappings on complete b- metric spaces. We had been given fixed point theorem for p-contraction with respect to $\{M_j\}_{j=1}^{k}$ a family of mappings $M_j : X \times X \to X$ and weakly Picard continuous mappings on a complete b-metric space $(X, £)$ possesses a unique fixed point. Furthermore, the Picard sequence $\{f^n x_0\}$
converges to this unique fixed point. Additionally, we also proved another fixed point theorem with (φ, Γ, β)-contraction mapping and weakly Picard continuous mappings on a complete b-metric space $(X, £)$ gives the unique fixed point. Some of examples are also given in which Banach contraction principle is invalid but supports our results.

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