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Research Paper

A Step on $(2-\epsilon)$-Bergman Kernel for Bounded Domains in $\mathbb{C}^n$

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Abstract
Following the way of J. Ning, H. Zhang, and X. Zhou [23], we show some properties of the $(2-\epsilon)$-Bergman kernels by applying $L^{2-\epsilon}$ extension theorem. We also show that for any bounded domain in $\mathbb{C}^n$, it is pseudoconvex if and only if its $(2-\epsilon)$-Bergman kernel is an exhaustion function, for any $0<\epsilon<2$. As an application, we give a negative answer to a conjecture of Tsuji.

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I. Introduction

The authors in [16] proved the Ohsawa-TakegoshiL^2 extension theorem, which turns out to be useful in several complex variables and complex geometry. [2] proved the $L^{2/m}$ version of Ohsawa-Takegoshi theorem for $m \in \mathbb{N}$. Recently, [9] obtained optimal estimate for $L^{2-\epsilon}(0 \le \epsilon < 2)$ extension as an application of their solution of a sharp L^2 extension problem.

Here, we study the $(2-\epsilon)$-Bergman kernels for bounded domains in $\mathbb{C}^n$, and apply $L^{2-\epsilon}$ extension theorem to give some properties of $(2-\epsilon)$-Bergman kernels (see [23]).

We can introduced a$(2-\epsilon)$-Bergman kernel as follows:

Definition 1.1. For a domain $\Omega \subseteq \mathbb{C}^n$ and $0 \le \epsilon < 2$, the $(2-\epsilon)$-Bergmann kernel $K_{\Omega,2-\epsilon}$ is denoted by

$$K_{\Omega,2-\epsilon}(z^2-1) = \sup_{f_j \in A^{2-\epsilon}(\Omega)} \sum_j \frac{|f_j(z^2-1)|^{2-\epsilon}}{\int_\Omega |f_j|^{2-\epsilon}}$$

where

$$A^{2-\epsilon}(\Omega) = \left\{ f_j \in \mathcal{O}(\Omega) : \int_\Omega \sum_j |f_j|^{2-\epsilon} < +\infty \right\}$$

Where the integral on Lebesgue measure.

According to the extreme property, the usual Bergman kernel is just 2-Bergman kernel for the case $\epsilon = 0$ and $j = 1$ in the above definition, which has been studied for years.

Let S be a closed complex subvariety of a domain $U \subset \mathbb{C}^n$. It's known that one has the same Bergman kernels on U and $U \setminus S$, since for any $f_j \in A^2(U \setminus S)$, one can holomorphically extend the sequence of functionsf_j to U. That is to say, one can not distinguish U and $U \setminus S$ by the Bergman kernel.

However, the $(2-\epsilon)$-Bergman kernel may give some distinction. We will show that for a bounded domain, it is pseudoconvex if and only if its $(2-\epsilon)$-Bergman kernel is an exhaustion function for any $0<\epsilon<2$. Besides, the $(2-\epsilon)$-Bergman kernel is interesting per se. We'll also give estimate about the boundary behavior of the $(2-\epsilon)$-Bergman kernel for a bounded pseudoconvex domain. Lastly, we'll answer negatively a conjecture of [20].

II. The $(2-\epsilon)$-Bergman kernel

Note that when $\epsilon = 0$ and $j = 1$, the $(2-\epsilon)$-Bergmann kernel is just the usual Bergman kennel. For simplicity, we write K_Ω for $K_{\Omega,2}$. The $(2-\epsilon)$-Bergmann kernel has some properties similar to the usual Bergman kernel, for example, it is easy to see that $K_{\Omega_1,2-\epsilon}(z^2-1) \geq K_{\Omega_2,2-\epsilon}(z^2-1)$ for $(z^2-1) \in \Omega_1$ and two domains $\Omega_1 \subseteq \Omega_2$, and the $(2-\epsilon)$-Bergmann kernels are plurisubharmonic.

We will study some more properties of $K_{\Omega,2-\epsilon}$.

Proposition 2.1 (see [23]). Let $\Omega_1 \subset \mathbb{C}^n$ be simply connected domain and $\Omega_2 \subset \mathbb{C}^n$ be a domain. Then for any $\phi_j : \Omega_1 \to \Omega_2$ biholomorphism, we have $K_{\Omega_1,2-\epsilon}(z^2-1) = K_{\Omega_2,2-\epsilon}(\phi_j(z^2-1))|J\phi_j(z^2-1)|^2$, where $J\phi_j$ is the determinant of Jacobian of ϕ_j. In particular, if $(2-\epsilon) = \frac{2}{m}$, where $m \in \mathbb{N}$, there is no need for the condition that Ω_1 is simply connected.

Proof. As Ω_1 is simply connected and $J\phi_j$ is nonvanishing, we can choose a single valued holomorphic function of $\log J\phi_j$.

Then

$$\Phi: A^{2-\epsilon}(\Omega_2) \to A^{2-\epsilon}(\Omega_1)$$
$$f_j \mapsto f_j \circ \phi_j e^{\frac{2}{2-\epsilon}\log(J\phi_j)}$$

is isometric, since

$$\int_{\Omega_2} \sum_j |f_j|^{2-\epsilon} = \int_{\Omega_1} \sum_j |f_j \circ \phi_j|^{2-\epsilon} |J\phi_j|^2 = \int_{\Omega_1} \sum_j \left| f_j \circ \phi_j e^{\frac{2}{2-\epsilon}\log(J\phi_j)} \right|^{2-\epsilon}.$$

When $(2-\epsilon) = \frac{2}{m}, m \in \mathbb{N}$, we take

$$\Phi: A^{2-\epsilon}(\Omega_2) \to A^{2-\epsilon}(\Omega_1)$$
$$f_j \mapsto f_j \circ \phi_j (J\phi_j)^m,$$

in this case, the simply connected condition is not needed any more.

By definition,

$$K_{\Omega_2,2-\epsilon}(\phi_j(z^2-1)) = \sup_{f_j \in A^{2-\epsilon}(\Omega_2)} \sum_j \frac{|f_j(\phi_j(z^2-1))|^{2-\epsilon}}{\int_{\Omega_2} |f_j|^{2-\epsilon}}$$
$$= \sup_{f_j \in A^{2-\epsilon}(\Omega_2)} \sum_j \frac{|f_j(\phi_j(z^2-1))|^{2-\epsilon}}{\int_{\Omega_1} |f_j \circ \phi_j|^{2-\epsilon} |J\phi_j|^2}$$
$$= \sum_j \frac{1}{|J\phi_j(z^2-1)|^2} \sup_{f_j \in A^{2-\epsilon}(\Omega_2)} \frac{\left| f_j \circ \phi_j (z^2-1) e^{\frac{2}{2-\epsilon}\log(J\phi_j(z^2-1))} \right|^{2-\epsilon}}{\int_{\Omega_1} \left| f_j \circ \phi_j e^{\frac{2}{2-\epsilon}\log(J\phi_j)} \right|^{2-\epsilon}}$$
$$= \sum_j \frac{K_{\Omega_1,2-\epsilon}(z^2-1)}{|J\phi_j(z^2-1)|^2}$$

It's easy to see that, if $J\phi_j$ is constant, then the above proposition is still true without the assumption that Ω_1 is simply connected. For example, if the domain Ω is a G-invariant domain w.r.t. a linear action of a semisimple Lie group G, then the $(2-\epsilon)$-Bergmann kernel is G-invariant.

The condition that Ω_1 is simply connected is necessary for some $0 < \epsilon < 2$ (see Remark 2.3).

Similar to the usual Bergman kernel, the following proposition holds for the $(2-\epsilon)$-Bergman kernel.

Proposition 2.2 (see [23]). Suppose that $\Omega_j \subset \mathbb{C}^n$ are bounded domains and $\Omega_j \subset \Omega_{j+1}$ for $j \geq 1, \cup_{j=1}^{\infty} \Omega_j = \Omega$, where Ω is a bounded domain in $\mathbb{C}^n$. Then for $0 \leq \epsilon < 2$,

$$\lim_{j\to\infty} K_{\Omega_j,2-\epsilon}(z^2-1) = K_{\Omega,2-\epsilon}(z^2-1)$$

and the convergence is uniform on compact subsets of Ω.

Proof. As $K_{\Omega_j,2-\epsilon}(z^2-1)$ is decreasing,

$$\lim_{j\to\infty} K_{\Omega_j,2-\epsilon}(z^2-1)$$

exists and $\geq K_{\Omega,2-\epsilon}(z^2-1)$.

For fixed $(z^2-1)\in\Omega$, we may assume $(z^2-1)\in\Omega_{j_0}$. There is $f_{j_0}\in\mathcal{O}(\Omega_{j_0})$ such that

$$\int_{\Omega_j}|f_{j_0}|^{2-\epsilon}=1$$

and

$$|f_{j_0}(z^2-1)|^{2-\epsilon}=K_{\Omega_j,2-\epsilon}(z^2-1)$$

for each $j\geq j_0$.

By the Montel theorem, there is a subsequence of $(j_0)_k$ such that

$$\lim_{k\to\infty}f_{(j_0)_k}$$

is uniformly convergent to $f_j\in\mathcal{O}(\Omega)$.

It is easy to check that

$$\int_\Omega\sum_j\quad|f_j|^{2-\epsilon}\leq 1$$

By the definition, we have

$$K_{\Omega,2-\epsilon}(z^2-1)\geq\sum_j\quad|f_j(z^2-1)|^{2-\epsilon}=\lim_{j_0\to\infty}K_{\Omega_j,2-\epsilon}(z^2-1)$$

As $K_{\Omega,2-\epsilon}(z^2-1)$ is continuous and $K_{\Omega_j,2-\epsilon}(z^2-1)$ is decreasing, it follows that $K_{\Omega_j,2-\epsilon}(z^2-1)$ converges uniformly to $K_{\Omega,2-\epsilon}(z^2-1)$ on compact subsets of Ω.

Theorem 2.3 (see [23]). Let Ω be one of the classical domains (see [11], [12], [13]):

$\mathfrak{R}_1:=\{(Z^2-1)\in M(m,n): I^{(m)}-(Z^2-1)(\overline{Z^2-1})'>0\}$

$\mathfrak{R}_2:=\{(Z^2-1)\in M(n,n): I^{(n)}-(Z^2-1)(\overline{Z^2-1})'>0, (Z^2-1)=(Z^2-1)'\}$,

$\mathfrak{R}_3:=\{(Z^2-1)\in M(n,n): I^{(n)}-(Z^2-1)(\overline{Z^2-1})'>0, (Z^2-1)=-(Z^2-1)'\}$,

$\mathfrak{R}_4:=\{(Z^2-1)\in M(1,n): |(Z^2-1)(Z^2-1)'|+1-2(\overline{Z^2-1})(Z^2-1)'>0, |(Z^2-1)(Z^2-1)'|<1\}$.

Then

$$K_{\Omega,1+\epsilon}(Z^2-1)=K_{\Omega,2}(Z^2-1)$$

for $(Z^2-1)\in\Omega$ for $\epsilon\geq 0$.

Proof. For $(Z^2-1)\in\Omega$ and $|t|\leq 1$, we have $t(Z^2-1)\in\Omega$.

For any $f_j\in\mathcal{O}(\Omega)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_j\quad\left|f_j\left(e^{i\theta}(Z^2-1)\right)\right|^{1+\epsilon}d\theta\geq\sum_j\quad|f_j(0)|^{1+\epsilon}$$

Then by the Fubini Theorem,

$$\int_\Omega|f_j|^{1+\epsilon}=\frac{1}{2\pi}\int_0^{2\pi}\int_\Omega\sum_j\quad\left|f_j\left(e^{i\theta}(Z^2-1)\right)\right|^{1+\epsilon}dV_{(Z^2-1)}d\theta$$

$$=\int_\Omega dV_{(Z^2-1)}\frac{1}{2\pi}\int_0^{2\pi}\sum_j\quad\left|f_j\left(e^{i\theta}(Z^2-1)\right)\right|^{1+\epsilon}d\theta\geq\sum_j\quad|f_j(0)|^{1+\epsilon}\mathrm{Vol}(\Omega)$$

we have

$$K_{\Omega,1+\epsilon}(0)=\frac{1}{\mathrm{Vol}(\Omega)}$$

As Ω is homogenous, it is well known that Ω is also simply connected, combining with the above proposition, we have $K_{\Omega,1+\epsilon}(Z^2-1)=K_{\Omega,2}(Z^2-1)$ for $(Z^2-1)\in\Omega$.

Remark 2.1. The above result is true for any complete circular and bounded homogeneous domain. It's known that any bounded symmetric domain is such a domain.

For a general bounded homogenous domain Ω, we have $K_{\Omega,1+\epsilon}(z^2-1)\geq K_{\Omega,2}(z^2-1)$. It is well known that $K_\Omega(z^2-1,w)$ is zero free and Ω is simply connected, we can define a holomorphic function $\log K_\Omega(z^2-1,w)$ for $(z^2-1)\in\Omega$ and fixed $w\in\Omega$. Then $e^{\frac{2}{1+\epsilon}\log K_\Omega(z^2-1,w)}\in A^{1+\epsilon}(\Omega)$, and it is easy to get $K_{\Omega,1+\epsilon}(z^2-1)\geq K_{\Omega,2}(z^2-1)$.

It seems to be strange that the $(1+\epsilon)$-Bergmann kernel may be independent of $(1+\epsilon)$ for some domains. From the following theorem, we can deduce that, in general, $K_{\Omega,1+\epsilon}$ is dependent on $(1+\epsilon)$.

Lemma 2.4 (see [23]). For $\Omega \subset \mathbb{C}^n$, we have

$$K_{\Omega,\frac{2-\epsilon}{m}}(z^2-1) \geq K_{\Omega,2-\epsilon}(z^2-1)$$

for any $0<\epsilon<2$ and $m \in \mathbb{N}$.

Proof. If $f_j \in A^{2-\epsilon}(\Omega)$, then

$$f_j^m \in A^{\frac{2-\epsilon}{m}}(\Omega)$$

and

$$\int_\Omega \sum_j \quad |f_j|^{2-\epsilon} = \int_\Omega \sum_j \quad |f_j^m|^{\frac{2-\epsilon}{m}}$$

By the definition of $(2-\epsilon)$-Bergman kernel, we have

$$K_{\Omega,\frac{2-\epsilon}{m}}(z^2-1) \geq K_{\Omega,2-\epsilon}(z^2-1)$$

The next theorem needs the $L^{2-\epsilon}$ extension theorem. We state it in the following. For the proof, see [2] or [9].

Theorem 2.5. (see [2] or [9]) Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, L be a complex affine line in $\mathbb{C}^n$, and $\Omega \cap L \neq \emptyset$. For $0 \leq \epsilon < 2$, then for any $f_j \in A^{2-\epsilon}(\Omega \cap L)$, there is $F_j \in A^{2-\epsilon}(\Omega)$, such that $F_j\big|_{\Omega\cap L} = f_j$ and

$$\int_\Omega \sum_j \quad |F_j|^{2-\epsilon} \leq C \int_{\Omega\cap L} \sum_j \quad |f_j|^{2-\epsilon}$$

where C is a constant depending only on $\operatorname{diam}\Omega$ and n.

Theorem 2.6 (see [23]). Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $0<\epsilon<2$ and $l = \max\left\{s \in \mathbb{N}_+ : s < \frac{2}{2-\epsilon}\right\}$. Then we have

$$K_{\Omega,2-\epsilon}(z^2-1) \geq \frac{1+\epsilon}{\delta(z^2-1)^{(2-\epsilon)l}}$$

where $\delta(z^2-1) = \inf_{w\in\partial\Omega} d(z^2-1,w)$ and $(1+\epsilon)$ is a constant positive number.

Proof. For any complex line L, after a unitary transform, we may assume $L = \{(z^2-1)_2 = \cdots = (z^2-1)_n = 0\}$.

Let $(z^2-1)^0 = ((z^2-1)_1^0, 0, \ldots, 0) \in \partial\Omega \cap L$, take

$$f_j = \frac{1}{((z^2-1)_1 - (z^2-1)_1^0)^l} \in A^{2-\epsilon}(\Omega \cap L)$$

From the $L^{2-\epsilon}$ extension theorem 2.5 , we get $F_j \in A^{2-\epsilon}(\Omega)$ such that $F_j\big|_{\Omega\cap L} = f_j$, and

$$\int_\Omega \sum_j \quad |F_j|^{2-\epsilon} \leq C \int_{\Omega\cap L} \sum_j \quad |f_j|^{2-\epsilon} \leq \frac{1}{1+\epsilon}$$

for some constant $\epsilon \geq 0, (1+\epsilon)$ depends only on $\operatorname{diam}\Omega$ and n.

Then

$$K_{\Omega,2-\epsilon}(z^2-1)\big|_{\Omega\cap L} \geq \frac{1+\epsilon}{|(z^2-1)_1 - (z^2-1)_1^0|^{(2-\epsilon)l}}$$

As we can choose arbitrary complex line and boundary points, we get

$$K_{\Omega,2-\epsilon}(z^2-1) \geq \frac{1+\epsilon}{\delta(z^2-1)^{(2-\epsilon)l}}$$

According to the above theorem and the fact that the $(2-\epsilon)$-Bergman kernel is plurisubharmonic, we can easily get the following interesting theorem.

Theorem 2.7. For any bounded domain Ω in $\mathbb{C}^n, \Omega$ is pseudoconvex if and only if $K_{\Omega,2-\epsilon}(z^2-1)$ is an exhaustion function for $0<\epsilon<2$.

Remark 2.2. The condition that Ω is bounded is necessary. If we consider $\Omega = \mathbb{C} \setminus \Delta$, then $K_{\Omega,2-\epsilon}(z^2-1)$ is bounded near ∞ for $0 \leq \epsilon < 2$.

Theorem 2.8 (see [23]). Let $\Delta^* = \{(z^2-1) \in \mathbb{C}: 0 < |z^2-1| < 1\}$ and $0 \le \epsilon < 2$, then we have $K_{\Delta^*,1+\epsilon}(z^2-1) = O\left(\frac{1}{|z^2-1|^{1+\epsilon}}\right)$.

Proof. For any $f_j \in A^{1+\epsilon}(\Delta^*)$, we have $f_j(z^2-1) = \sum_{n=-\infty}^{\infty} \sum_j \quad a_n^j (z^2-1)^n$, then $g_j(z^2-1) := \sum_{n=0}^{\infty} \sum_j \quad a_n^j (z^2-1)^n$ is holomorphic on $\Delta = \{(z^2-1) \in \mathbb{C}: |z^2-1| < 1\}$.

In the present proof, we denote by $\|f_j\|_{1+\epsilon} = \left(\int_{\Delta^*} \sum_j \quad |f_j|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}}$ for $f_j \in A^{1+\epsilon}(\Delta^*)$.

Obviously, $\int_{\Delta_\tau^*} \sum_j \quad |g_j(z^2-1)|^{1+\epsilon} < \infty$, where $\Delta_\tau^* = \{(z^2-1) \in \mathbb{C}: 0 < |z^2-1| < \tau\}$ and $0 < \tau < 1$.

It's easy to see that

$$\int_{\Delta^*} \left|\frac{1}{z^2-1}\right|^{1+\epsilon} dxdy = \int_0^1 \int_0^{2\pi} r^{-\epsilon} d\theta dr = \frac{2\pi}{1-\epsilon}$$

From

$$\|g_j + h_j\|_{1+\epsilon} \le \|g_j\|_{1+\epsilon} + \|h_j\|_{1+\epsilon}$$

we get

$$h_j(z^2-1) := \sum_{n=-\infty}^{-2} \sum_j \quad a_n^j (z^2-1)^n \in A^{1+\epsilon}(\Delta_\tau^*).$$

We want to prove $h_j = 0$.

$$\int_{\Delta_\tau^*} \sum_j \quad |h_j(z^2-1)|^{1+\epsilon} dxdy = \int_{\mathbb{C}\setminus\Delta_{\frac{1}{\tau}}} \sum_j \quad \left|h_j\left(\frac{1}{z^2-1}\right)\right|^{1+\epsilon} \frac{dxdy}{|z^2-1|^4}$$

$$= \int_{\frac{1}{\tau}}^{\infty} \frac{1}{r^3} dr \int_0^{2\pi} \sum_j \quad \left|h_j\left(\frac{e^{i\theta}}{r}\right)\right|^{1+\epsilon} d\theta$$

Let $\tilde{h}_j(z^2-1) = h_j\left(\frac{1}{z^2-1}\right)$, then $\tilde{h}_j$ is holomorphic on $\mathbb{C} \setminus \Delta_{\frac{1}{\tau}}$ and

$$\tilde{h}_j(z^2-1) = \sum_{n=2}^{\infty} \sum_j \quad a_{-n}^j (z^2-1)^n$$

If $\tilde{h}_j$ is not 0 , then there is $n_0 > 1$ such that $a_{-n_0}^j \neq 0$ and $a_{-n}^j = 0$ for $1 < n < n_0$. Write $\tilde{h}_j(z^2-1) = (z^2-1)^{n_0}(f_j)_1(z^2-1)$, where $(f_j)_1(z^2-1) = \sum_{n=n_0}^{\infty} \sum_j \quad a_{-n}^j (z^2-1)^{n-n_0}$.

By the submean property

$$\int_0^{2\pi} \sum_j \quad \left|(f_j)_1\left(\frac{e^{i\theta}}{r}\right)\right|^{1+\epsilon} d\theta \ge 2\pi \sum_j \quad \left|a_{-n_0}^j\right|^{1+\epsilon}$$

and $n_0(1+\epsilon) - 3 > -1$, it follows that

$$\int_{\Delta_\tau^*} \sum_j \quad |h_j(z^2-1)|^{1+\epsilon} dxdy \ge 2\pi \sum_j \quad \left|a_{-n_0}^j\right|^{1+\epsilon} \int_{\frac{1}{\tau}}^{\infty} r^{n_0(1+\epsilon)-3} dr = \infty.$$

Therefore, $h_j = 0$. That is to say, for any $f_j \in A^{1+\epsilon}(\Delta^*)$, we have $f_j(z^2-1) = \sum_{n=-1}^{\infty} \sum_j \quad a_n^j (z^2-1)^n$.

Note that

$$K_{\Delta^*,1+\epsilon}(z^2-1) \ge \frac{\frac{1}{|(z^2-1)(z^2-1)^{(1+\epsilon)}}}{\int_{\Delta^*} \left|\frac{1}{z^2-1}\right|^{1+\epsilon}} \ge \frac{1-\epsilon}{2\pi|z^2-1|^{1+\epsilon}} \tag{1}$$

Since

$$|z^2-1|^{1+\epsilon} K_{\Delta^*,1+\epsilon}(z^2-1) = |z^2-1|^{1+\epsilon} \sup_{f_j \in A^{1+\epsilon}(\Delta)} \sum_j \quad \frac{\left|\frac{a^j}{z^2-1} + f_j(z^2-1)\right|^{1+\epsilon}}{\int_{\Delta^*} \left|\frac{a^j}{z^2-1} + f_j(z^2-1)\right|^{1+\epsilon} dxdy}$$

$$= \sup_{f_j \in A^{1+\epsilon}(\Delta)} \sum_j \quad \frac{|a^j + (z^2-1)f_j(z^2-1)|^{1+\epsilon}}{\int_{\Delta^*} \left|\frac{a^j}{z^2-1} + f_j(z^2-1)\right|^{1+\epsilon} dxdy}. \tag{2}$$

From (1), for (z^2-1) near 0 , we may take $a^j = 1$. For $f_j \in A^{1+\epsilon}(\Delta)$

(a) If $\|f_j\|_{1+\epsilon}^{1+\epsilon} > 2^{(1+\epsilon)}\frac{2\pi}{1-\epsilon}$, then $\left\|f_j(z^2-1)+\frac{1}{z^2-1}\right\|_{1+\epsilon} \geq \|f_j(z^2-1)\|_{1+\epsilon} - \left\|\frac{1}{z^2-1}\right\|_{1+\epsilon} > \frac{1}{2}\|f_j(z^2-1)\|_{1+\epsilon}$, so

$$\sum_j \frac{|1+(z^2-1)f_j(z^2-1)|^{1+\epsilon}}{\int_{\Delta^*}\left|\frac{1}{z^2-1}+f_j(z^2-1)\right|^{1+\epsilon}dxdy} < \sum_j \frac{2^{(1+\epsilon)}\left(1+|(z^2-1)f_j(z^2-1)|^{1+\epsilon}\right)}{(1/2^{(1+\epsilon)})\int_{\Delta^*}|f_j|^{1+\epsilon}}$$
$$< 2^{2(1+\epsilon)}\left(\frac{1-\epsilon}{2^{(2+\epsilon)}\pi} + |z^2-1|^{1+\epsilon}K_{\Delta,1+\epsilon}(z^2-1)\right).$$

(b) If $\|f_j\|_{1+\epsilon}^{1+\epsilon} \leq 2^{(1+\epsilon)}\frac{2\pi}{1-\epsilon}$, then $|f_j(z^2-1)| \leq C$ for all (z^2-1) near 0 , where C is a positive constant independent on f_j.

Since

$$\int_{\Delta^*}\sum_j \left|\frac{1}{z^2-1}+f_j(z^2-1)\right|^{1+\epsilon}dxdy = \int_0^1 r^{-\epsilon}dr\int_0^{2\pi}\sum_j \left|1+re^{i\theta}f_j\left(re^{i\theta}\right)\right|^{1+\epsilon}d\theta$$
$$\geq 2\pi\int_0^1 r^{-\epsilon}dr = \frac{2\pi}{1-\epsilon}$$

then

$$\sum_j \frac{|1+(z^2-1)f_j(z^2-1)|^{1+\epsilon}}{\int_{\Delta^*}\left|\frac{1}{z^2-1}+f_j(z^2-1)\right|^{1+\epsilon}dxdy} < \frac{(1-\epsilon)(1+|z^2-1|C)^{1+\epsilon}}{2\pi}$$

According to (a) and (b), we get that $|z^2-1|^{1+\epsilon}K_{\Delta^*,1+\epsilon}(z^2-1)$ is bounded near 0 .

From the above theorem, we know the lower bounds of Theorem 2.6 is optimal.

Remark 2.3 (see [23]). Let $D = \{(z^2-1) \in \mathbb{C}: |z^2-1| > 1\}$, for $0 < \epsilon < 2$, there is $(1+\epsilon) = (1+\epsilon)^2 > 0$ such that

$$K_{D,1+\epsilon}(z^2-1) \leq \frac{1+\epsilon}{|z^2-1|^{2(1+\epsilon)}}$$

for $|z^2-1| \gg 1$.

Let $\varphi: \Delta^* \to D, (z^2-1) \mapsto 1/(z^2-1)$. For $0 < \epsilon < 2$,

$$K_{\Delta^*,\frac{4}{3}+\epsilon}(z^2-1) \neq K_{D,\frac{4}{3}+\epsilon}(1/(z^2-1))\frac{1}{|z^2-1|^4}$$

Proof of the Remark:

For any $f_j \in A^{\frac{4}{3}+\epsilon}(D)$, we have

$$f_j(z^2-1) = \sum_{n=-1}^{\infty}\sum_j a_n^j(z^2-1)^n + \sum_{n=2}^{\infty}\sum_j b_n^j(z^2-1)^{-n}$$

Let $(f_j)_1(z^2-1) = \sum_{n=-1}^{\infty}\sum_j a_n^j(z^2-1)^n$ and $(f_j)_2(z^2-1) = \sum_{n=2}^{\infty}\sum_j b_n^j(z^2-1)^{-n}$.

It is easy to check that there is $r \gg 1$ such that $\int_{\{|z^2-1|>r\}}\sum_j \left|(f_j)_2\right|^{\frac{4}{3}+\epsilon} < \infty$ holds.

Hence $\int_{\{|z^2-1|>r\}}\sum_j \left|(f_j)_1\right|^{\frac{4}{3}+\epsilon} < \infty$.

If $(f_j)_1$ is not 0 , we may choose k to be the integer such that $a_n^j = 0$ for $n < k, a_k^j \neq 0$, then

$$\int_{\{|z^2-1|>r\}}\sum_j \left|(f_j)_1\right|^{\frac{4}{3}+\epsilon} = \int_{\{|z^2-1|>r\}}\left|\sum_{n=k}^{\infty}\sum_j a_n^j(z^2-1)^n\right|^{\frac{4}{3}+\epsilon}$$
$$= \int_r^{\infty}\sum_j \rho_j d\rho_j\int_0^{2\pi}(\rho_j)^{k\left(\frac{4}{3}+\epsilon\right)}\left|\sum_{n=k}^{\infty}a_n^j(z^2-1)^{n-k}\right|^{\frac{4}{3}+\epsilon} \geq \sum_j 2\pi\left|a_k^j\right|^{\frac{4}{3}+\epsilon}\int_r^{\infty}(\rho_j)^{1+k\left(\frac{4}{3}+\epsilon\right)} = \infty.$$

Therefore, $(f_j)_1 = 0$.

We get $K_{D,\frac{4}{3}+\epsilon}(z^2-1) \leq \frac{1+\epsilon}{|z^2-1|^{2\left(\frac{4}{3}+\epsilon\right)}}$ for $|z^2-1| \gg 1$.

By the above theorem, $K_{\Delta^*,\frac{4}{3}+\epsilon}(z^2-1)=O\left(\frac{1}{|z^2-1|^{\frac{4}{3}+\epsilon}}\right)$.

As

$$K_{D,\frac{4}{3}+\epsilon}(1/(z^2-1))\frac{1}{|z^2-1|^4}\leq\frac{1+\epsilon}{|z^2-1|^{4-2(\frac{4}{3}+\epsilon)}}$$

for $|z^2-1|\ll 1$,

if $\epsilon>0$, then

$$K_{\Delta^*,\frac{4}{3}+\epsilon}(z^2-1)\neq K_{D,\frac{4}{3}+\epsilon}(z^2-1)\frac{1}{|z^2-1|^4}$$

We have finished the proof of the remark.

III. A conjecture of H. Tsuji

We first recall a definition for complex manifolds, see H. Tsuji [20].

Definition 3.1. Let M be a complex manifold with the canonical line bundle K_M, for every positive integer m, we set

$$(Z^2-1)_m:=\left\{\sigma_j\in\Gamma\big(M,\mathcal{O}_M(mK_M)\big)\left|\int_M\sum_j\quad(\sigma_j\wedge\bar{\sigma}_j)^{\frac{1}{m}}\right|<+\infty\right\}$$

and

$$K_{M,m}:=\sup\left\{|\sigma_j|^{\frac{2}{m}};\sigma_j\in\Gamma\big(M,\mathcal{O}_M(mK_M)\big)\left|\int_M\sum_j\quad(\sigma_j\wedge\bar{\sigma}_j)^{\frac{1}{m}}\right|\leq 1\right\}$$

where the sup denotes the pointwise supremum.

Then let

$$K_{M,\infty}:=\lim_{m\to\infty}\sup K_{M,m}$$

and $(h_j)_{(1+\epsilon)a^j\ n,\ M}:=$ the lower envelope of $\frac{1}{K_{M,\infty}}$.

Lemma 3.1 (see [23]). For $\Omega\subset\mathbb{C}^n$, we have

$$\sup_{m\in\mathbb{N}}K_{\Omega,\frac{2}{m}}(z^2-1)=\sup_{0\leq\epsilon<2}K_{\Omega,2+\epsilon}(z^2-1).$$

Proof. By Lemma 2.4, we have

$$\sup_{m\in\mathbb{N}}K_{\Omega,\frac{2}{m}}(z^2-1)=\sup_{0\leq\epsilon<2\cap\mathbb{Q}}K_{\Omega,2+\epsilon}(z^2-1).$$

If $f_j\in\mathcal{O}(\Omega)$ and $\int_\Omega\sum_j\quad|f_j|^{2+\epsilon}<\infty$, then

$$\lim_{q\to 2+\epsilon,q<2+\epsilon}\int_\Omega\sum_j\quad|f_j|^q=\int_\Omega\sum_j\quad|f_j|^{2+\epsilon}$$

So

$$\sup_{0\leq\epsilon<2\cap\mathbb{Q}}K_{\Omega,2+\epsilon}(z^2-1)=\sup_{0\leq\epsilon<2}K_{\Omega,2+\epsilon}(z^2-1)$$

and the lemma follows.

For $\Delta^*=\{(z^2-1)\in\mathbb{C}:0<|z^2-1|<1\}$, since the canonical bundle K_{Δ^*} is trivial, so when we consider $K_{\Delta^*,\infty}$ and $(h_j)^{-1}_{(1+\epsilon)a^j\ n,\ \Delta^*}$, we can omit the form dt.

H. Tsuji [20] proposed the following conjecture (see Conjecture 2.16 in [20]):

$$(h_j)^{-1}_{(1+\epsilon)a^j\ n,\Delta^*}=O\left(\frac{1}{|z^2-1|^2(\log\ |z^2-1|)^2}\right)$$

holds.

However, we get the following theorem:

Theorem 3.2 (see [23]). One has

$$(h_j)^{-1}_{(1+\epsilon)a^j\ n,\Delta^*}(z^2-1)\geq K_{\Delta^*,\infty}(z^2-1)\geq\frac{1}{2\pi e}\frac{1}{|z^2-1|^2|\log\ |z^2-1||}$$

for $0<|z^2-1|<e^{-1}$.

Proof. Since

$$\int_{\Delta^*}\left|\frac{1}{z^2-1}\right|^{2+\epsilon}=\frac{2\pi}{-\epsilon}$$

by Lemma 2.4 and Lemma 3.1, we get

$$K_{\Delta^*,\infty}(z^2-1)=\lim_{m\to\infty}\sup K_{\Delta^*,m}(z^2-1)=\sup_{m\geq 1}K_{\Delta^*,m}(z^2-1)$$
$$=\sup_{0\leq\epsilon<2}K_{\Delta^*,2+\epsilon}(z^2-1)\geq\sup_{0\leq\epsilon<2}\frac{-\epsilon}{2\pi}\frac{1}{|z^2-1|^{2+\epsilon}}$$

For $0<|z^2-1|<e^{-1}$, let

$$\epsilon=\frac{1}{\log\ |z^2-1|}\in[1,2]$$

therefore

$$\frac{-\epsilon}{2\pi}\frac{1}{|z^2-1|^{2+\epsilon}}=\frac{1}{2\pi e}\frac{1}{|z^2-1|^2|\log\ |z^2-1||}$$

so

$$K_{\Delta^*,\infty}(z^2-1)\geq\frac{1}{2\pi e}\frac{1}{|z^2-1|^2|\log\ |z^2-1||}$$

Hence

$$(h_j)^{-1}_{(1+\epsilon)a^j\ n,\Delta^*}(z^2-1)\geq K_{\Delta^*,\infty}(z^2-1)\geq\frac{1}{2\pi e}\frac{1}{|z^2-1|^2|\log\ |z^2-1\ |}$$

From the above theorem, we know that $(h_j)^{-1}_{(1+\epsilon)a^j\ n,\Delta^*}$ is not integrable near 0 .

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