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Review Paper

Convergence results for a fixed-point iteration on a subset of Euclidean space

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Abstract
In this paper, we prove convergence of Mann and Ishikawa iterations to the fixed points of continuous functions defined on a linearly ordered closed convex subset of Euclidean space $\mathbb{R}^m$. Numerical examples are given to illustrate the feasibility of the results presented. Our results extend corresponding results in literature.
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I. Introduction

Let E a closed and convex subset of m - dimensional Euclidean space $\mathbb{R}^m$ and $T: E \longrightarrow \mathbb{R}^m$ a mapping. A point $p \in E$ is called a fixed point of T if it remains invariant under the action of mapping T. A fixed-point problem (FPP, for short) for a continuous mapping T is to find $x \in E$ such that

$$Tx = x \tag{1.1}$$

Denote by $F(T)$ the solution set of FPP (1.1).
Recall that, a mapping T is called L-Lipschitzian if there exist a constant $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\| \ \forall x, y \in E. \tag{1.2}$$

where $\|x\| = \max\{|x_i|: i = 1, \cdots, n\}$. If $L < 1$ and T is a self-map, then the mapping T is a contraction. For a contraction mapping the Banach Contraction Principle guarantees the existence of a fixed point of T in E. If it is not possible to find a fixed point analytically, then it could be approximated with the help of a Picard iteration process, perhaps the simplest one. It reads as follows:

$$x_0 \in E, x_n = T^n(x_0) \text{ for all } n = 1,2, \cdots$$

If $L = 1$ and $T: E \longrightarrow E$, then the mapping satisfying (1.2) is called a nonexpansive mapping. For such mappings, the sequence of successive approximations given above may fail to converge even if the set of fixed points is singleton. The situation could become more complicated when $L > 1$ even with a self mapping on E. To handle such situations, different iterative processes have been developed and used to approximate fixed points of nonlinear mappings on suitable domains. For example, see [2,3,8] and references mentioned therein. Following are the two well-known fixed-point iteration methods, namely, the Mann iteration process and the Ishikawa iteration process which can be used to compute fixed points of a given map.

Definition 1.1 (Mann Iteration Process [4]). Let $\{\alpha_n\}$ be a real sequence with the following properties:
(1) $0 \leq \alpha_n \leq 1$
(2) $\lim_{n\to\infty} \alpha_n = 0$
(3) $\sum_{n=1}^{\infty} \alpha_n = +\infty$.

The Mann iteration process is defined as follows:

$$x_1 \in E, x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n)$$

Definition 1.2 (Ishikawa Iteration Process [7]). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences with the following properties:
(1) $0 \leq \alpha_n, \beta_n \leq 1$
(2) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$
(3) $\sum_{n=1}^{\infty} \alpha_n = +\infty$.

Ishikawa iteration process is defined by

$$x_1 \in E, x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(y_n), \text{ and}$$
$$y_n = (1-\beta_n)x_n + \beta_n f(x_n)$$

There are several results on convergence of the Mann and the Ishikawa iteration processes to the fixed point of given maps. We remark that the two processes exhibit different behavior for different classes of nonlinear maps even though they may look similar, see Rhoades [10]. In its original form, the Ishikawa iteration process does not include Mann process as a special case. This is because the original condition on the real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ was $0 \leq \alpha_n \leq \beta_n \leq 1$. However, in an attempt to have an Ishikawa type iteration process which does include the Mann process as a special case, several authors have modified the inequality condition to read $0 \leq \alpha_n, \beta_n \leq 1$.

In 1991, Borwein and Borwein [1] proved that the Mann iteration converges to a fixed point of a continuous self-mapping$T$ defined on a closed and bounded interval of $\mathbb{R}$.

In 1996, Herceg and Krejic [5] proved the following convergence result for a non-self-mapping defined on a closed and bounded interval of $\mathbb{R}$.

Theorem 1.3. Let $T: [a, b] \longrightarrow \mathbb{R}$ be a Lipschitz mapping with constant $L > 0$. Let $\{\alpha_n\}$ be a real sequence with the following properties:
(1) $0 \leq \alpha_n \leq (L+1)^{-1}$
(2) $\lim_{n\to\infty} \alpha_n = 0$
(3) $\sum_{n=1}^{\infty} \alpha_n = +\infty$,
and $Ta \in [a, b], Tb \in [a, b]$. Then the sequence $\{x_n\}$ defined as follows:

$$x_1 \in \{a, b\}, x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$$

is convergent and its limit is a fixed point of T.

In 1997, Huang [6] studied the Mann iteration process for Lipschitz functions on an m dimensional rectangle. To prove their main result, they used the partial order induced by the positive cone

$$\mathbb{R}_+^m = \{x = (x_1, x_2, \dots, x_m)^T \in \mathbb{R}^m : x_i \geq 0 \forall i = 1, \cdots, m\}$$

and component wise limits of a sequence $\{x_n\}$ in $\mathbb{R}^m$ defined as follows:

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} (x_n)_i = x_i \forall i = 1, \cdots, m.$$

Their main theorem is the following:
Theorem 1.4. [6] Let $E = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be a bounded interval of $\mathbb{R}^n$. Let $f: E \longrightarrow \mathbb{R}^n$ be a Liptschitz function on E with the partial order induced by $\mathbb{R}^n_+$. Let $\{\alpha_n\}$ be a real sequence with the properties:
(1) $0 \leq \alpha_n \leq (L+1)^{-1}$
(2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$
(3) $f((x_1, x_2, \cdots, x_n)^T) \in E$, for any $j \in \mathbb{N}, x_j = a_j$ or b_j.

Define $\{x_n\}$ as follows:

$$x_1 = a = (a_1, \cdots, a_n)^T, x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n)$$

Then the sequence $\{x_n\}$ thus obtained converges to a fixed point of f.
In 2006, Qing and Qihou [9] showed that the boundedness of a sequence $\{x_n\}$, generated by the Mann and Ishikawa iteration processes for a continuous self mapping defined on a closed (not necessarily bounded) interval of $\mathbb{R}$, is necessary and sufficient condition for the convergence of $\{x_n\}$ to a fixed point of the mapping. Their result extends the result of [1].

In this paper, we take up the problem of approximating the fixed point of continuous functions defined on a closed subset of $\mathbb{R}^n$, with partial ordering induced by the positive cone of $\mathbb{R}^n$, through the convergence of a sequence generated by Mann and Ishikawa iteration processes. Our results extend several comparable results in the existing literature, see for example, the results in [1], [6] and [9].

II. Main Results

Throughout this section, we consider the partial ordering on $\mathbb{R}^n$ induced by nonnegative orthant $\mathbb{R}^n_+$. That is, for any $x, y \in \mathbb{R}^n; x \leq y$ if and only if $y - x \in \mathbb{R}^n_+$.

We start with the following fixed-point theorem.
Lemma 2.1. Let E be linearly ordered closed and convex subset of $\mathbb{R}^n$, which may be unbounded and $f: E \longrightarrow \mathbb{R}^n$ a continuous mapping with $f(E) \subseteq$ E. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be real sequences in [0,1] with $0 \leq \tau_n + \beta_n \leq 1$ and $0 \leq \gamma_n + \alpha_n \leq 1$ satisfying the following properties
(i) $\sum_{n=1}^{\infty} \beta_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty,$.
(ii) $\sum_{n=1}^{\infty} \alpha_n = +\infty, \lim_{n\to\infty} \alpha_n = 0$.

Let $\{x_n\}$ be a sequence generated as follows:$x_1 \in E$

$$z_n = (1-\mu_n)x_n + \mu_n f(x_n)$$

$$y_n = (1-\tau_n-\beta_n)z_n + \tau_n z_n + \beta_n f(z_n)(2.1)$$

$$x_{n+1} = (1-\gamma_n-\alpha_n)y_n + \gamma_n y_n + \alpha_n f(y_n)$$

If $\{x_n\}$ converges, it converges to a fixed point of f.
Proof. Suppose that $x_n \to a$ as $n \to \infty$. We show that $f(a) = a$. The proof is by contradiction and it is as follows: Suppose $f(a) \neq a$. Since f is continuous, the sequence $\{f(x_n)\}$ is bounded. From condition (i), we have that $z_n = (1-\mu_n)x_n + \mu_n f(x_n) \longrightarrow a$ (since $\{f(x_n)\}$ is bounded and $x_n \to a$). This implies, by the continuity of f, that $\{f(z_n)\}$ is bounded. Thus $y_n = (1-\tau_n-\beta_n)z_n + \tau_n z_n + \beta_n f(z_n) \longrightarrow a$ (since $z_n \to a$ and $\{f(z_n)\}$ is bounded). This again implies, by the continuity of f, that $\{f(y_n)\}$ is bounded.

From (2.1), we have the following:

$$\begin{aligned} x_{n+1} - y_n &= \alpha_n(f(y_n) - y_n) \\ z_n - x_n &= \mu_n(f(x_n) - x_n) \\ y_n - z_n &= \beta_n(f(z_n) - z_n) \end{aligned}$$

So that

$$(x\_(n+1)-x\_n = x\_(n+1)-y\_n+y\_n-z\_n+z\_n-x\_n$$
$$= \alpha\_n\,(f(y\_n)-y\_n)+\mu\_n\,(f(x\_n)-x\_n)+\beta_n(f(z_n)-z_n)(2.2)$$

Put $p_k = f(y_k)-y_k, q_k = f(z_k)-z_k$ and $r_k = f(x_k)-x_k$. Then from continuity of f, it follows that

$$\lim_{k\to\infty} p_k = \lim_{k\to\infty}(f(y_k)-y_k) = f(a)-a = p(\text{ say }) \neq 0$$
$$\lim_{k\to\infty} q_k = \lim_{k\to\infty}(f(z_k)-z_k) = f(a)-a = q(\text{ say }) \neq 0$$
$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}(f(x_k)-x_k) = f(a)-a = r(\text{ say }) \neq 0$$

Now by (2.2),

$$x_n - x_1 = x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_2 - x_1$$
$$= \sum_{k=1}^{n-1} \big(\alpha_k(f(y_k)-y_k) + \mu_k(f(x_k)-x_k) + \beta_k(f(z_k)-z_k)\big)$$

which implies that

$$x_n - x_1 = \sum_{k=1}^{n-1} (\alpha_k p_k + \mu_k r_k + \beta_k q_k) = \sum_{k=1}^{n-1} \alpha_k p_k + \sum_{k=1}^{n-1} \mu_k r_k + \sum_{k=1}^{n-1} \beta_k q_k$$

Thus

$$x_n = x_1 + \sum_{k=1}^{n-1} \alpha_k p_k + \sum_{k=1}^{n-1} \mu_k r_k + \sum_{k=1}^{n-1} \beta_k q_k$$

Since $q_k \longrightarrow q \neq 0$ and $\sum_{k=1}^{\infty} \beta_k < \infty$, it follows that $\sum_{k=1}^{\infty} \beta_k q_k < \infty$. Similarly, since $r_k \longrightarrow r \neq 0$ and $\sum_{k=1}^{\infty} \mu_k < \infty$, it follows that $\sum_{k=1}^{\infty} \mu_k r_k < \infty$. However, since $p_k \longrightarrow p \neq 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$, we obtain that $\{x_n\}$ is divergent, a contradiction. Hence $f(a) = a$.

Our main result is the following.
Theorem 2.2. Let E be linearly ordered closed and convex subset of $\mathbb{R}^n$, which may be unbounded and $f: E \longrightarrow \mathbb{R}^n$ a continuous mapping with $f(E) \subseteq E$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be real sequences in [0,1] with $0 \leq \tau_n + \beta_n \leq 1$ and $0 \leq \gamma_n + \alpha_n \leq 1$ satisfying (i) -(ii) in Lemma 2.1. Let a sequence $\{x_n\}$ be generated by the iterative scheme (2.1). If $\{x_n\}$ is bounded, then it converges.

Proof. Let $\{x_n\}$ be a bounded sequence. We show that $\{x_n\}$ is convergent. The proof is by contradiction. Suppose $\{x_n\}$ is not convergent, let

$$a_i = \liminf_{n\to\infty} (x_n)_i \text{ and}$$
$$b_i = \limsup_{n\to\infty} (x_n)_i \forall i = 1, \cdots, n$$

Then $a_i < b_i$ for each $i = 1, \cdots, n$, which implies that $a < b$, where $a, b \in \mathbb{R}^n$.
Claim: If $a < m < b$, then $f(m) = m$. That is, $a_i < m_i < b_i$ implies that $f_j(m_i) = m_i \forall i = 1, \cdots, n$ and for each $j = 1, \cdots n$.
Proof of claim: Suppose $f(m) \neq m$. Without loss of generality, let $f(m) - m > 0$. Since f is continuous, there exists $\delta > 0, \delta < \|b-a\|$ such that

$$f(x) - x > 0 \text{ for } \|x - m\| \leq \delta.$$

Since $\{x_n\}$ is bounded, there exists $M_1 > 0$ such that $\|x_n\| \leq M_1$. From continuity of f, it follows that there exists $M_2 > 0$ such that $\|f(x_n)\| \leq M_2$, that is, the sequence $\{f(x_n)\}$ is bounded. Thus, the sequences $\{z_n\}, \{f(z_n)\}, \{y_n\}$ and $\{f(y_n)\}$ are all bounded. Note that

$$\begin{aligned}\|z_n - x_n\| &= \|\mu_n(f(x_n) - x_n)\| = \mu_n\|f(x_n) - x_n\| \\ \|y_n - z_n\| &= \|\beta_n(f(z_n) - z_n)\| = \beta_n\|f(y_n) - y_n\| \\ \|x_{n+1} - y_n\| &= \|\alpha_n(f(y_n) - y_n)\| = \alpha_n\|f(y_n) - y_n\| \\ \|x_{n+1} - x_n\| &= \|x_{n+1} - y_n + y_n - z_n + z_n - x_n\| \\ &\le \|x_{n+1} - y_n\| + \|y_n - z_n\| + \|z_n - x_n\| \\ \|y_n - x_n\| &= \|y_n - z_n + z_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|\end{aligned}$$

From condition (i) - (ii), it follows that

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|y_n - z_n\| = \lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$$

which implies

$$\lim_{n\to\infty}\|y_n - x_n\| = 0 \text{ and } \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$$

From definition of $\|.\|, weobtainthat$

$$\lim_{n\to\infty}\left|(x_{n+1})_j - (x_n)_j\right| = \lim_{n\to\infty}\left|(y_n)_j - (x_n)_j\right| = \lim_{n\to\infty}\left|(z_n)_j - (x_n)_j\right| = 0 \quad \text{for each } j \in \{1,2,\dots,n\}$$

Choose $\epsilon_j = \frac{\delta_j}{2}$, there exists $N_j \in \mathbb{N}$ such that

$$\left|(x_{n+1})_j - (x_n)_j\right| < \frac{\delta_j}{2}, \left|(z_n)_j - (x_n)_j\right| < \frac{\delta_j}{2} \text{ and } \left|(y_n)_j - (x_n)_j\right| < \frac{\delta_j}{2} \text{ for all } n > N_j. \tag{2.3}$$

Since $m_j < b_j = \limsup_{n\to\infty}(x_n)_j$, there exists $k_1 \in \mathbb{N}, k_1 > N_j$ such that $m_j < x_{n_{k_1}}$. That is, there exists a subsequence $\{x_{n_{k_1}}\}$ that satisfies the inequality $m_j < \left(x_{n_{k_1}}\right)_j$ for each $j = 1, \cdots, n$.

Set $n_{k_1} = k$. Then we have $(x_k) > m$, that is, $(x_k)_j > m_j \forall j = 1, \cdots, n$. Thus $m_j < (x_k)_j < b_j \forall j = 1, \cdots, n$.
Now we consider the following two cases:
Case 1: If $(x_k)_j > m_j + \frac{\delta_j}{2}, \forall j = 1, \cdots, n$, then we have $(x_{k+1})_j > (x_k)_j - \frac{\delta_j}{2} \ge m_j$, using (2.3), and hence $(x_{k+1})_j > m_j$.

Case 2: If $m_j < (x_k)_j < m_j + \frac{\delta_j}{2}, \forall j = 1, \cdots, n$, then, from (2.3), we have

$$m_j - \frac{\delta_j}{2} < (y_k)_j < m_j + \delta_j$$

and

$$m_j - \frac{\delta_j}{2} < (z_k)_j < m_j + \delta_j$$

$\forall j = 1, \cdots, n$. So that

$$\left|(x_k)_j - m_j\right| < \frac{\delta_j}{2} < \delta_j$$

implies that

$$\left|(y_k)_j - m_j\right| < \delta_j$$

and

$$|(z_k)_j - m_j| < \delta_j$$

Thus

$$f((x_k)_j) - (x_k)_j > 0$$

$$f((y_k)_j) - (y_k)_j > 0. (2.4)$$

$$and\, f((z\_k\,)\_j\,) - (z\_k\,)\_j > 0.$$

From (2.2) we have

$$((x\_(k+1)\,)\_j = (x\_k\,)\_j + \alpha\_k\,((f(y\_k\,))\_j - (y\_k\,)\_j\,) + \mu\_k\,((f(x\_k\,))\_j - (x\_k\,)\_j\,) + \beta_k\left((f(z_k))_j - (z_k)_j\right)(2.5)$$

Hence, from (2.4) and (2.5), we have

$$\begin{aligned}(x_{k+1})_j = (x_k)_j &+ \alpha_k\left((f(y_k))_j - (y_k)_j\right)\\ &+\mu_k\left((f(x_k))_j - (x_k)_j\right) + \beta_k\left((f(z_k))_j - (z_k)_j\right)\\ &> (x_k)_j\end{aligned}$$

Thus $(x_{k+1})_j > (x_k)_j > m_j$. From the Case 1 and the Case 2, we conclude that $(x_{k+1})_j > m_j \forall j = 1, \cdots, n$ which implies that $x_{k+1} > m$. Similarly, we have

$$x_{k+1} > m, x_{k+2} > m, x_{k+3} > m, \cdots$$

So there exists a subsequence $\{x_\nu\}$ such that $x_\nu > m, \forall \nu > k = n_{k_1}$. That is, $x_{n_{k_1}+1} > m, x_{n_{k_1+2}} > m, \cdots$. As $a = \liminf_{n\to\infty} x_n$, there exists a subsequence $\{x_{n_k}\}$ which converges to a. This implies that there exists $N \in \mathbb{N}$ such that for all $n_k > N$, we have $a = \lim_{N\to\infty} x_{n_k}$. Hence for all $\nu > n_k, x_\nu > m$, we have $a = \liminf_{n\to\infty} x_{n_k} \geq m$. This is a contradiction since $a < m$. Thus $f(m) = m$.

Next, we consider the following two cases of the sequence $\{x_n\}$.
Case I: Since $\{x_n\}$ is bounded, there exists M such that $a < x_M < b$ and $f(x_M) = x_M$. Thus

$$\begin{aligned} z_M &= (1-\mu_M)x_M + \mu_M f(x_M) = x_M\\ y_M &= (1-\tau_M-\beta_M)z_M + \tau_M z_M + \beta_M f(z_M)\\ &= (1-\tau_M-\beta_M)x_M + \tau_M x_M + \beta_M f(x_M) = x_M\\ x_{n+1} &= (1-\gamma_M-\alpha_M)y_M + \gamma_M y_M + \alpha_M f(y_M)\\ &= (1-\gamma_M-\alpha_M)x_M + \gamma_M x_M + \alpha_M f(x_M) = x_M\end{aligned}$$

Analogously, $x_M = x_{M+1} = x_{M+2} = x_{M+3} = \cdots$ which implies that $x_n \longrightarrow x_M$. Also, since there exists a subsequence $x_{n_k} \longrightarrow a$, it follows that $x_M = a$ and $x_n \longrightarrow a$, a contradiction.
Case II: Suppose that $x_n \leq a$ or $x_n \geq b$ for all n. Choose $\epsilon = \frac{b-a}{2} > 0$. By $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, there exists $N \in \mathbb{N}$ such that $\|x_{n+1} - x_n\| < \frac{b-a}{2}$, for $n > N$. So it is always that $x_n \leq a$ for $n > N$ or it is always that $x_n \geq b$ for $n > N$. Now if $x_n \leq a$ for $n > N$, then $b = \lim_{l\to\infty} x_{n_l} \leq a$, which is a contradiction to $a < b$. On the other hand, if $x_n \geq b$ for $n > N$, then $a = \lim_{n\to\infty} \geq b$, again a contradiction with $a < b$. Thus, from Case I and Case II, we conclude that $\{x_n\}$ converges. This completes the proof.

Theorem 2.3. Let E be linearly ordered closed and convex subset of $\mathbb{R}^n$, which may be unbounded and $f: E \longrightarrow \mathbb{R}^n$ a continuous mapping with $f(E) \subseteq E$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be real sequences in [0,1] with $0 \leq \tau_n + \beta_n \leq 1$ and $0 \leq \gamma_n + \alpha_n \leq 1$ satisfying (i) -(ii) in Lemma 2.1. The sequence $\{x_n\}$ generated by the iterative scheme (2.1) converges to a fixed point of f if and only if $\{x_n\}$ is bounded.

Proof. ($\Longrightarrow$) Let $\{x_n\}$ be bounded. Then by Theorem 2.2, $\{x_n\}$ converges. Let $x_n \longrightarrow a, n \longrightarrow \infty$. Then $f(a) = a$, by Theorem 2.1. That is, $\{x_n\}$ converges to a fixed point of f.
($\Longleftarrow$) Let $\{x_n\}$ converge to a fixed point of f but is not bounded. Then then for all $m \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $x_n > m \forall n > N$. This implies $\{x_n\}$ does not converge which contradicts our assumption. Thus $\{x_n\}$ is bounded.

Theorem 2.4. Let E be linearly ordered closed and convex subset of $\mathbb{R}^n$, which may be unbounded and $f: E \longrightarrow \mathbb{R}^n$ a continuous mapping. Let $\{x_n\}$ be the Mann iterative sequence for the mapping f. If the parameters $\{\alpha_n\}$ satisfy the conditions $(i) - (iii)$ stated in the Definition 1.1, then $\{x_n\}$ converges to a fixed point of f if and only if $\{x_n\}$ is bounded.

Proof. In Theorem 2.3, put $\beta_n = 0$ to get the result.

III. Numerical Example

In this section, we present an example to illustrate the convergence result presented above. The result presented here were generated with MATLAB programming.

Example 3.1. Let $f: E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $f(x, y) = \left(\frac{1}{2}x^2 + 3y^2 + 0.1, 0.5x^3 + 0.2y + 0.1\right)^T$ for all $(x, y) \in E$, where E is linearly ordered closed and convex set. Choose $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{2n}$ with initial data set at $(x_0, y_0) = (-0.7, -0.5)$. The results in this example were generated using MATLAB programming with tolerance $e = 10^{-3}$.

	Ishikawa		Mann	
n	(x_n, y_n)	$\lVert f(x_n, y_n) - (x_n, y_n)\rVert$	$(x_n y_n)$	$\lVert f(x_n, y_n) - (x_n, y_n)\rVert$
1.0	(0.140707, 0.078704)	0.040327	(0.375000, 0.091500)	0.268464
2.0	(0.136789, 0.098835)	0.022290	(0.285215,0.008284)	0.178472
3.0	(0.138173, 0.106489)	0.017006	(0.237103,0.043275)	0.126004
4.0	(0.139869, 0.110627)	0.014473	(0.211259,0.061286)	0.095574
5.0	(0.141385, 0.113255)	0.012927	(0.195724,0.072423)	0.076154
6.0	(0.142683, 0.115090)	0.011845	(0.185585,0.080058)	0.062825
7.0	(0.143793, 0.116451)	0.011025	(0.178565,0.085651)	0.053177
200.0	(0.158843, 0.126619)	0.001999	(0.156209,0.124470)	0.003395
201.0	(0.158852, 0.126622)	0.001994	(0.156221,0.124481)	0.003388
605.0	(0.160381, 0.127166)	0.001047	(0.158593,0.126219)	0.002048
608.0	(0.160386, 0.127168)	0.001044	(0.158602,0.126224)	0.002043
609.0	(0.160388, 0.127168)	0.001043	(0.158604,0.126226)	0.002042
658.0	(0.000000, 0.000000)	0.000000	(0.158739,0.126302)	0.001966
1014.0	(0.000000, 0.000000)	0.000000	(0.159416,0.126658)	0.001581
1015.0	(0.000000, 0.000000)	0.000000	(0.159417,0.126658)	0.001580
1301.0	(0.000000, 0.000000)	0.000000	(0.159749,0.126817)	0.001388
1303.0	(0.000000, 0.000000)	0.000000	(0.159751,0.126818)	0.001387
1454.0	(0.000000, 0.000000)	0.000000	(0.159885,0.126880)	0.001308
2000.0	(0.000000, 0.000000)	0.000000	(0.160236,0.127034)	0.001102

2385.0	(0.000000, 0.000000)	0.000000	(0.160407,0.127105)	0.001000
2386.0	(0.000000, 0.000000)	0.000000	(0.160407,0.127105)	0.001000
2387.0	(0.000000, 0.000000)	0.000000	(0.000000,0.000000)	0.000000

Table 1: Table showing numerical results

IV. Conclusion

From the above table, one can conclude that Mann and Ishikawa iterative sequences converge to the fixed point $(0.160407, 0.127105)$ of the function $f(x,y) = \left(\frac{1}{2}x^2 + 3y^2 + \frac{1}{10}, \frac{1}{2}x^3 + \frac{1}{5}y + \frac{1}{10}\right)$ in $\mathbb{R}^2$. The Ishikawa iterative process converges to the accuracy of 10^{-6} after 655 iterations while the Mann iteration process converges after 2386 iterations. Thus the Ishikawa iterative process converges faster than that of Mann.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors worked on the manuscript and approved the final manuscript.

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