Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 7 (July 2025) pp: 13-18
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Spectral decomposition of operator polynomial bundle

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***R**eceived 08 July, 2025; **R**evised 18 July, 2025; **A**ccepted 20 July, 2025 © The author(s) 2025.*
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The spectral theory of operators arose from the considerantions of needs of ordinary differential equations and Cauchy's problem for differential equations [10] with several initial conditions. These investigations led to the study of the spectral theory of polynomial bundle of type

$$A(\lambda) = A_0 + \lambda A_1 B + \lambda^2 A_2 B^2 + \ldots + \lambda^{n-1} A_{n-1} B^{n-1} + \lambda^n B A_n{}^n, \quad (1)$$

where $A_i (i = 0,1,...,n)$ and B are completely continuous operators, acting in Hilbert space H The investigation of this bundle $L(\lambda)$ led to the study of questions of multiple completeness and multiple decomposition on eigen and associated vectors of the polynomial bundle $L(\lambda)$ in Hilbert space. The bundle (1) is known as Keldysh's bundle and it was studied in[1]. In connection with the M.V.Keldysh's considerations it is known that n multiple completeness of eigen and associated vectors of bundle $L(\lambda)$ is true ,when the operators $A_i (i = 0,1,...,n)$ are completely continuous, n B is a self-adjoint completely continuous operator with the restrictions on the location of its spectrum, besides operator B has the finite order and $KerB = \{0\}$ .The completely continuous operator has a finite order if the series of some positive degrees of its eigenvalues converge. The fundamental result of Keldysh [1] was generalized by many authors in many different directions. Here we should note the works of [2], [3], [4], [5] and many others. Theorems about multiple decompositions on eigen and associated vectors with brackets of the Keldysh bundle (1) are proven in works [6], [7] , summation by Abel method o eigen and associated vectors was investigated in [8].

We introduce some definitions [1].

1. λ is an eigenvalue of polynomial bundle (1) if it exists the nonzero element x such, that $A(\lambda)x = x$, x - is an eigenvector, corresponding to an eigenvalue λ .

2. If the number c is an eigenvalue of polynomial bundle (1) ,vector y_0 is corresponding eigenvector,then y_k is called an $k-th$ -associated vector to the

eigenvector y_0 if the following equalities

$$
\begin{aligned}
& y = A(c)y \\
& y_1 = A(c)y_1 + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y \\
& \ldots\ldots\ldots\ldots \\
& y_k = A(c)y_k + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y_{k-1} + \ldots + \frac{1}{k!}\frac{\partial^k A(c)}{\partial c^k}y
\end{aligned}
\qquad (2)
$$

are fulfilled.

The system of linear independent eigen and associated vectors is called a chain of eigen and associated (e.a) vectors of operator $A(\lambda)$ corresponding to eigenvalue c. The number of e.a. vectors in the chain of e.a. vectors is called a chain's length of eigenvector y_0. Totality, all independent e.a. vectors ,corresponding to all eigenvectors with eigenvalue c λ is called the eigenvalue λ multiplicity.

3.M.V. Keldysh built the derivative systems with the help of the formulas

$$
\left[\frac{d^k}{dt^k}e^{\lambda t}\left(x_k + \frac{1}{1!}x_{k-1} + \ldots + \frac{1}{k!}x_0\right)\right](t=0) \quad k = 1, 2, \ldots, s \qquad (3)
$$

4. System of eigen and associated vectors of operator bundle $A(\lambda)$ in space H

forms the n – multiple complete system if any n elements $f_0, f_1, \ldots, f_{n-1}$ of the space H can be approximated with the linear combinations of elements $\{x_i^{(j)}\}_{i=1}^{\infty}, j = 0,1,2,\ldots,n-1$ in accordance with predetermined accuracy and with the same coefficients, not depending on indices of elements $f_0, f_1, \ldots, f_{n-1}$.

The associated vectors , introduced in [1], make possible to approve in many cases about completeless of eigen and associated vectors in space. The derivatives systems, introduced in [1], allow to approve about multible completeless and multible decomposiotions on e.a. vectors of bundle (1) in Hilbert space.

The last is sufficient for the decision of Cashy problem for corresponding operator differensial equation

Eigenvectors of operator $A(\lambda)$ coincide with the first components of the eigenvectors of the operator

$$\tilde{A} = \begin{pmatrix} \tilde{A} = A_0 & A_1 & \dots & A_n \\ B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & B0 \end{pmatrix} \quad (4)$$

ating in direct sum $\tilde{H}$ of $(n+1)$ copies of Hilbert space H .

The operator $\tilde{A}$ is presented in the form

$$\tilde{A}f = \tilde{T}f + i\tilde{S}f$$

where $\tilde{T} = \dfrac{\tilde{A} + \tilde{A}}{2}$ and $\tilde{S} = \dfrac{\tilde{A} - \tilde{A}}{2i}$.

So all operators $A_i(0,1,\dots,n)$ and B -are completely continuous in the space H , operators $\tilde{T}$ and $\tilde{S}$ are completely continuous and self-adjoint in space $\tilde{H}$ [9]. Operators $\tilde{T}$ and $\tilde{S}$ have a sequence of eigenvalues and corresponding eigenvectors, form the orthonormal base of space $\tilde{H}$,correspondingly. We denote the sequence of ortonormal system of eigenvectors of operator $\tilde{T}$ by $(\tilde{e}_1,\dots,\tilde{e}_n,\dots)$, and sequence of eigenvectors of operator $\tilde{S}$ by $\{\tilde{g}_1,\tilde{g}_2,\dots\}$. Similar, if $\tilde{f} = (f_0, f_1,\dots,f_n)$ and $\tilde{g} = (g_0, g_1,\dots,g_n)$,then the inner product of these elements is $\left[\tilde{f},\tilde{g}\right] = (f_0,g_0) + \dots + (f_n,g_n)$

where $(.,.)$ is the inner product in the space $\tilde{H}$. Then each vector $\tilde{f} = (f_0, f_1,\dots,f_n)$ n of the space $\tilde{H}$ may be presented on the elements of bases $(\tilde{e}_1,\dots,\tilde{e}_n,\dots)$ and $(\tilde{g}_1,\tilde{g}_2,\dots)$.

We have

$$T\tilde{f} = \sum_{k=1}^{\infty} \lambda_k [f, \tilde{e}_k]\tilde{e}_k \quad (5)$$

where $\{\lambda_k\}_{k=1}^{\infty}$ is the sequence of eigenvalues of operator $\tilde{T}$.

Further, $\tilde{f} = \sum_{j=1}^{\infty} \left[\tilde{f}, \tilde{g}_j \right] \tilde{g}_j$, (6)

where $\tilde{g}_j = \left(g_{0,j}, g_{1,j}, ..., g_{n,j} \right), j = 1, 2, ...$

is the orthonormal eigenvectors of the operator $\widetilde{S}$,which form the base $\tilde{g}_j, j = 1, 2, ...$

in the space $\tilde{H}$. Substituted inisteal of $\bar{f}$ in (5) expression for $\bar{f}$ from (6), we have

$$\tilde{T}\,\tilde{f} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k \left[\tilde{f}, \tilde{g}_j \right] \left[\tilde{g}_j, \tilde{e}_k \right] \tilde{e}_k \qquad (7)$$

Similar, $\tilde{S}\tilde{f} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu_j \left[\tilde{f}, \tilde{e}_k \right] \left[\tilde{e}_k, \tilde{g}_j \right] \tilde{g}_j$ (8)

Further,

$$\tilde{A}f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k \left[\tilde{f}, \tilde{g}_j \right] \left[\tilde{g}_j, \tilde{e}_k \right] \tilde{e}_k + f \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu_j \left[\tilde{f}, \tilde{e}_k \right] \left[\tilde{e}_k, \tilde{g}_j \right] \tilde{g}_j \qquad (9)$$

Let $\widetilde{f} = (f_0, f_1, ..., f_n)$ be then

$$\sum_{i=0}^{n} A_i f_i = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k \left[\tilde{f}, \tilde{g}_j \right] \left[\tilde{g}_j, \tilde{e}_k \right] e_{0,k} + f \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu_j \left[\tilde{f}, \tilde{e}_k \right] \left[\tilde{e}_k, \tilde{g}_j \right] g_{0,j}$$

$$Bf_0 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k \left[\tilde{f}, \tilde{g}_j \right] \left[\tilde{g}_j, \tilde{e}_k \right] e_{1,k} + f \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu_j \left[\tilde{f}, \tilde{e}_k \right] \left[\tilde{e}_k, \tilde{g}_j \right] g_{1,j}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Bf_{n-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k \left[\tilde{f}, \tilde{g}_j \right] \left[\tilde{g}_j, \tilde{e}_k \right] e_{n,k} + f \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu_j \left[\tilde{f}, \tilde{e}_k \right] \left[\tilde{e}_k, \tilde{g}_j \right] g_{n,j} \qquad (10)$$

If in $\widetilde{f} = (f_0, f_1, ..., f_n)$

we take
$$f_n = B\lambda$$
$$\cdots\cdots\cdots$$
$$f_n = \lambda^n B^{n-1} f$$

that is $\widetilde{f} = \left(f, \lambda Bf, ..., \lambda^{n-1} B^{n-1}, \lambda^n B^{n-1}\right)$ and in (4)

we take , $A_n = B$, then

$$A(\lambda)f = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_k\left[\tilde{f},\tilde{g}_j\right]\left[\tilde{g}_j,\tilde{e}_k\right]e_{0,k} + f\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu_j\left[\tilde{f},\tilde{e}_k\right]\left[\tilde{e}_k,\tilde{g}_j\right]g_{0,j}$$

$$Bf_0 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_k\left[\tilde{f},\tilde{g}_j\right]\left[\tilde{g}_j,\tilde{e}_k\right]e_{1,k} + f\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu_j\left[\tilde{f},\tilde{e}_k\right]\left[\tilde{e}_k,\tilde{g}_j\right]g_{1,}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda^{n-1}B^n f = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_k\left[\tilde{f},\tilde{g}_j\right]\left[\tilde{g}_j,\tilde{e}_k\right]e_{n,k} + f\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu_j\left[\tilde{f},\tilde{e}_k\right]\left[\tilde{e}_k,\tilde{g}_j\right]g_{n,j}$$

where

$$\left[\tilde{f},\tilde{e}_k\right] = \left(f_0, e_{0,k}\right) + \lambda\left(fBf, e_{1,k}\right) + ... + \lambda^n\left(B^n f, e_{n,k}\right)$$

$$\left[\widetilde{f},\widetilde{g}_j\right] = \left(f_{0,} g_{0,j}\right) + \lambda\left(Bf, g_{1,j}\right) + ... + \lambda^n\left(B^n f, g_{n,j}\right) \tag{11}$$

Really, if $\widetilde{f} = (f_0, f_1, ..., f_n)$ is an eigenvector of $\tilde{A}$ with the eigenvalue λ ,then f_0 is the eigenvalue of $A(\lambda)$ with the eigenvalue $1/\lambda$. The spectral decomposition of the operator $\tilde{A}_j$ is the sum of spectral decompositions of its real and imaginary parts.Thus the first line of the system (11) is the spectral decompositions of operator $A(\lambda)$.

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