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Review Paper

Certain Localization and Compactness in Bergmanand Bargmann-Fock Spaces

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Abstract
The pioneers of [13] study the compactness of operators on Bergman space of the unit ball and on generally weighted Bargmann-Fock spaces in terms of their Berezin transforms and the norms of the operators acting on reproducing kernels. We show how a vanishing Berezin transform combined with certain (integral) growth conditions on an operator T are sufficient to imply that the operator is compact in the Bergman space. We also show that the reproducing kernel for compactness holds for operators satisfying similar growth conditions in (Weighted Bargmann-Fock space). Following [13] we extend the results of Xia and Zheng to the case of Bergman space when $0 < \epsilon < \infty$, and in weighted Bargmann-Fock space.The main results introduced more general new conditions that imply and improved the results of Xia and Zheng by thecase$0 < \epsilon < \infty$.
Key words: *Berezin Transform, Compact Operators, Bergman Space, Fock Space, Toeplitz Operator, Sufficiently Localized Operator.*

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I. Introduction

The Bargmann-Fock space $\mathcal{F}^{1+\epsilon} := \mathcal{F}^{1+\epsilon}(\mathbb{C}^n)$ is the collection of sequences of entire functions f_i on $\mathbb{C}^n$ such that $f_i(\cdot)e^{-\frac{|\cdot|}{2}} \in L^{1+\epsilon}(\mathbb{C}^n, dv)$. Then$\mathcal{F}^2$ is a reproducing kernel Hilbert space with reproducing kernel given by $K_{z_i}(w_i) = e^{\bar{z}_i w_i}$. Here, we denote by k_{z_i} the normalized reproducing kernel at z_i. For a bounded operator T on $\mathcal{F}^{1+\epsilon}$, the Berezin transform of T is the function defined by

$$\tilde{T}(z_i) = \langle Tk_{z_i}, k_{z_i} \rangle_{\mathcal{F}^2}$$

Bauer and the first author in [13] proved that the vanishing of the Berezin transform is sufficient for compactness whenever the operator is in the Toeplitz algebra [1]. However, it is generally very difficult to check whether a given operator T is in the Toeplitz algebra, unless T is itself a Toeplitz operator or a combination of a few Toeplitz operators, and as such one would like a "simpler" sufficient condition to guarantee this.

In [10], Xia and Zheng introduced a class of "sufficiently localized" operators on $\mathcal{F}^2$ which includes the algebraic closure of the Toeplitz operators. These are the operators T acting on $\mathcal{F}^2$ such that there exist constants β and $0 \le \epsilon < \infty$ with

$$\left|\langle Tk_{z_i}, k_{w_i} \rangle_{\mathcal{F}^2}\right| \le \sum_i \frac{1+\epsilon}{(1+|z_i - w_i|)^{(2n+\epsilon)}}. \tag{1.1}$$

It was proved by Xia and Zheng that every bounded operator T from the C^*-algebra generated by sufficiently localized operators whose Berezin transform vanishes at infinity, i.e.,

$$\lim_{|z_i|\to\infty} \sum_i \langle Tk_{z_i}, k_{z_i} \rangle_{\mathcal{F}^2} = 0. \tag{1.2}$$

is compact on $\mathcal{F}^2$. One of their main innovations is providing an easily checkable condition (1.1) which is general enough to imply compactness from the seemingly much weaker condition (1.2).

The aim here is to extend the Xia-Zheng notion of sufficiently localized operators to both a much wider class of weighted Fock spaces (in particular, the class of so-called "generalized Bargmann-Fock spaces" considered in [8]) and to a larger class of operators. So (1.1) easily implies

$$\sup_{z_i \in \mathbb{C}^n} \int_{\mathbb{C}^n} \sum_i \left|\langle Tk_{z_i}, k_{w_i} \rangle_{\mathcal{F}^2}\right| dv(w_i) < \infty$$

and consequently one should look at generalizations of sufficiently localized operators that allow for weaker integral conditions. And the ideas in [10] are essentially frame theoretic (see [5] for a discussion of the ideas in [10]) and therefore one cannot easily extend these ideas to the non-Hilbert space setting. Now, we will provide a simpler, more direct proof of the main result in [10] and which can be extended to other spaces of analytic functions. In particular, we show that our modified main result, holds for the classical Bergman space $A^{1+\epsilon}$ on the ball (and in Section 4 we will discuss the possibility of extending our results to a very wide class of weighted Bergman spaces.)

The extension of the main results in [10] to a larger class of operators and to a wider class of weighted Fock spaces is as follows. Let $d^c = \frac{i}{4}(\bar{\partial} - \partial)$ and let d be the usual exterior derivative. For the rest of the paper let $\phi_i \in (1+\epsilon)^2(\mathbb{C}^n)$ be a real valued function on $\mathbb{C}^n$ such that

$$c\omega_0 < dd^c\phi_i < (1+\epsilon)\omega_0$$

holds uniformly pointwise on $\mathbb{C}^n$ for some positive constants c and $1+\epsilon$ (in the sense of positive (1,1) forms) where $\omega_0 = dd^c|\cdot|^2$ is the standard Euclidean Kähler form. Furthermore, for $0 \le \epsilon \le \infty$, define the generalized Bargmann-Fock space $\mathcal{F}_{\phi_i}^{1+\epsilon}$ to be the space of entire functions f_i on $\mathbb{C}^n$ such that $f_i e^{-\phi_i} \in L^{1+\epsilon}(\mathbb{C}^n, dv)$ (for a detailed properties of $\mathcal{F}_{\phi_i}^{1+\epsilon}$ see [8]). For operators T acting on the reproducing kernels $K(z_i, w_i)$ of $\mathcal{F}_{\phi_i}^2$,, we impose the following conditions. We first assume that

$$\sup_{z_i\in\mathbb{C}^n}\int_{\mathbb{C}^n}\sum_i\left|\left\langle Tk_{z_i}, k_{w_i}\right\rangle_{\mathcal{F}_{\phi_i}^2}\right|dv(w_i) < \infty,\quad \sup_{z_i\in\mathbb{C}^n}\int_{\mathbb{C}^n}\sum_i\left|\left\langle T^*k_{z_i}, k_{w_i}\right\rangle_{\mathcal{F}_{\phi_i}^2}\right|dv(w_i) < \infty. \quad (1.3)$$

which is enough to conclude that the operator T initially defined on the linear span of the reproducing kernels extends to a bounded operator on $\mathcal{F}_{\phi_i}^{1+\epsilon}$ for $0 \le \epsilon \le \infty$ (see Section 3). To show that the operator is compact, we impose the following additional assumptions on T:

$$\lim_{\epsilon\to\infty}\sup_{z_i\in\mathbb{C}^n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle Tk_{z_i}, k_{w_i}\right\rangle_{\mathcal{F}_{\phi_i}^2}\right|dv(w_i) = 0,$$

$$\lim_{\epsilon\to\infty}\sup_{z_i\in\mathbb{C}^n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle T^*k_{z_i}, k_{w_i}\right\rangle_{\mathcal{F}_{\phi_i}^2}\right|dv(w_i) = 0. \quad (1.4)$$

Definition 1.1. We will say that a linear operator T on $\mathcal{F}_{\phi_i}^{1+\epsilon}$ is weakly localized (and for convenience write $T \in \mathcal{A}_{\phi_i}(\mathbb{C}^n)$) if it satisfies the conditions (1.3) and (1.4).

Note that every sufficiently localized operator on $\mathcal{F}^2$ in the sense of Xia and Zheng obviously satisfies (1.3) and (1.4) and is therefore weakly localized in our sense too. Now if $D(z_i, 1+\epsilon)$ is the Euclidean ball with center z_i and radius $1+\epsilon$, and if $\|T\|_e$ denotes the essential norm of a bounded operator T on $\mathcal{F}_{\phi_i}^{1+\epsilon}$ then the following theorem is one of the main results (see [13]):

Theorem 1.2. Let $0 < \epsilon < \infty$ and let T be an operator on $\mathcal{F}_{\phi_i}^{1+\epsilon}$ which belongs to the norm closure of $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$. Then there exists $\epsilon > 0$ (both depending on T) such that

$$\|T\|_e \le (1+\epsilon)\limsup_{|z_i|\to\infty}\sup_{w_i\in D(z_i,1+\epsilon)}\sum_i\left|\left\langle Tk_{z_i}, k_{w_i}\right\rangle\right|$$

In particular, if

$$\lim_{|z_i|\to\infty}\sum_i\left\|Tk_{z_i}\right\|_{\mathcal{F}_{\phi_i}^{1+\epsilon}} = 0$$

then T is compact on $\mathcal{F}_{\phi_i}^{1+\epsilon}$.

Now if $\mathcal{A}(\mathbb{C}^n)$ is the class of sufficiently localized operators on $\mathcal{F}^2$ then note that an application of Proposition 1.4 in [5] in conjunction with Theorem 1.2 immediately proves the following theorem, which provides the previously mentioned generalization of the results in [10] (see Section 3).

Theorem 1.3 (see [13]). Let $0 < \epsilon < \infty$ and let T be an operator on $\mathcal{F}^{1+\epsilon}$ which belongs to the norm closure of $\mathcal{A}(\mathbb{C}^n)$. If $\lim_{|z_i|\to\infty}\left|\left\langle Tk_{z_i}, k_{z_i}\right\rangle_{\mathcal{F}^2}\right| = 0$ then T is compact.

We note that one can easily write the so called "Fock-Sobolev spaces" from [4] as generalized Bargmann-Fock spaces, so that in particular Theorem 1.2 immediately applies to these spaces (see [5] for more details).

To state the main result in the Bergman space setting. Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$ and let the space $A^{1+\epsilon} := A^{1+\epsilon}(\mathbb{B}_n)$ denote the classical Bergman space, i.e., the collection of all holomorphic functions on $\mathbb{B}_n$ such that

$$\|f_i\|_{A^{1+\epsilon}}^{1+\epsilon} := \int_{\mathbb{B}_n}\sum_i|f_i(z_i)|^{1+\epsilon}dv(z_i) < \infty$$

The function $K_{z_i}(w_i) := (1 - \bar{z_i}w_i)^{-(n+1)}$ is the reproducing kernel for A^2 and

$$k_{z_i}(w_i) := \frac{(1-|z_i|^2)^{\frac{n+1}{2}}}{(1-\bar{z_i}w_i)^{(n+1)}}$$

is the normalized reproducing kernel at the point z_i. We also will let $d\lambda$ denote the invariant measure on $\mathbb{B}_n$, i.e.,

$$d\lambda(z_i) = \frac{dv(z_i)}{(1-|z_i|^2)^{n+1}}$$

Now let $0 < \epsilon < \infty$ and let $\epsilon \geq 0$. We are interested in operators T acting on the reproducing kernels of A^2 that satisfy the following conditions. First, we assume that there exists $0 < \delta < min\left\{1+\epsilon, \frac{1+\epsilon}{\epsilon}\right\}$ such that

$$\sup_{z_i \in \mathbb{B}_n} \int_{\mathbb{B}_n} \sum_i \left|\left\langle Tk_{z_i}, k_{w_i}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}} d\lambda(w_i) < \infty,$$

$$\sup_{z_i \in \mathbb{B}_n} \int_{\mathbb{B}_n} \sum_i \left|\left\langle T^* k_{z_i}, k_{w_i}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}} d\lambda(w_i) < \infty. \tag{1.5}$$

These are enough to conclude that the operator T initially defined on the linear span of the reproducing kernels extends to a bounded operator on $A^{1+\epsilon}$ (see the comments following the proof of Proposition 2.5). To treat compactness we make the following additional assumptions on T: there exists $0 < \delta < min\left\{1+\epsilon, \frac{1+\epsilon}{\epsilon}\right\}$ such that

$$\sup_{z_i \in \mathbb{B}_n} \int_{D(z_i, 1+\epsilon)^c} \sum_i \left|\left\langle Tk_{z_i}, k_{w_i}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}} d\lambda(w_i) \to 0,$$

$$\sup_{z_i \in \mathbb{B}_n} \int_{D(z_i, 1+\epsilon)^c} \sum_i \left|\left\langle T^* k_{z_i}, k_{w_i}\right\rangle\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}} d\lambda(w_i) \to 0. \tag{1.6}$$

as $\epsilon \to \infty$.

Definition 1.4. We say that a linear operator T on $A^{1+\epsilon}$ is $1+\epsilon$ weakly localized (which we denote by $T \in \mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$) if it satisfies conditions (1.5) and (1.6).

Note that the condition $0 < \delta < min\left\{1+\epsilon, \frac{1+\epsilon}{\epsilon}\right\}$ implies that both $1 - \frac{2\delta}{(1+\epsilon)(n+1)}$ and $1 - \frac{2\delta\epsilon}{(1+\epsilon)(n+1)}$ are strictly between $\frac{n-1}{n+1}$ and 1. Furthermore, note that when $1+\epsilon = \frac{1+\epsilon}{\epsilon} = 2$, we have that $\frac{n-1}{n+1} < 1 - \frac{\delta}{(n+1)} < 1$ precisely when $0 < \delta < 2$. Thus, in this case we can rewrite condition (1.5) in the following simpler way: there exists $\frac{n-1}{n+1} < a < 1$ where

$$\sup_{z_i \in \mathbb{B}_n} \int_{\mathbb{B}_n} \sum_i \left|\left\langle Tk_{z_i}, k_{w_i}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{a}}{\left\|K_{w_i}\right\|_{A^2}^{a}} d\lambda(w_i) < \infty,$$

$$\sup_{z_i \in \mathbb{B}_n} \int_{\mathbb{B}_n} \sum_i \left|\left\langle T^* k_{z_i}, k_{w_i}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{a}}{\left\|K_{w_i}\right\|_{A^2}^{a}} d\lambda(w_i) < \infty$$

Of course, one can similarly rewrite condition (1.6) when $\epsilon = 1$.

We prove the following result(see [13]).

Theorem 1.5. Let $0 < \epsilon < \infty$ and let T be an operator on $A^{1+\epsilon}$ which belongs to the norm closure of $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. If

$$\lim_{|z_i| \to 1} \sum_i \left\langle Tk_{z_i}, k_{z_i}\right\rangle_{A^2} = 0$$

then T is compact.

It will be clear that the method of proof also will work for the weighted Bergman space $A_\alpha^{1+\epsilon}$, and we leave this.

Note that this result is known in Suárez, [9] in the case of $A^{1+\epsilon}$ when the operator T belongs to the Toeplitz algebra generated by L^∞ symbols (see also [7] for the case of weighted Bergman spaces.) We will prove below that the Toeplitz algebra on $A^{1+\epsilon}$ generated by L^∞ symbols is a subalgebra of the norm closure of $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. In particular, the results of this paper provide a considerably simpler proof of the main results in [7,9] for the $\epsilon \neq 1$ situation (though it should be noted that a similar simplification when $\epsilon = 1$ was provided in [6]).

Now here we provide the extension of the the Xia and Zheng result to the Bergman space on the unit ball $\mathbb{B}_n$, and in particular we prove Theorem 1.5. We prove Theorems 1.2 and 1.3 which provides an extension of the Xia and Zheng result in the case of the generalized Bargmann-Fock spaces. Finally, we will briefly discuss some interesting open problems related to these results.

II. Bergman Space Case

For φ_{z_i} be the Möbius map of $\mathbb{B}_n$ that interchanges 0 and z_i. It is well known that

$$1 - \left|\varphi_{z_i}(w_i)\right|^2 = \frac{(1 - |z_i|^2)(1 - |w_i|^2)}{|1 - \bar{z}_i w_i|^2}$$

and as a consequence we have that

$$\left|\langle k_{z_i}, k_{w_i}\rangle_{A^2}\right| = \frac{1}{\left\|K_{\varphi_{z_i}(w_i)}\right\|_{A^2}}. \quad (2.1)$$

Using the automorphismφ_{z_i}, the pseudohyperbolic and Bergman metrics on $\mathbb{B}_n$ are defined by

$$\rho(z_i, w_i) := \left|\varphi_{z_i}(w_i)\right| \quad \text{and} \quad (2n+\epsilon)(z_i, w_i) := \frac{1}{2}\log\frac{1 + \rho(z_i, w_i)}{1 - \rho(z_i, w_i)}$$

Recall that these metrics are connected by $\rho = \frac{e^{2(2n+\epsilon)} - 1}{e^{2(2n+\epsilon)} + 1} = \tanh(2n+\epsilon)$ and it is well-known that these metrics are invariant under the automorphism group of $\mathbb{B}_n$. We let

$$D(z_i, 1+\epsilon) := \{w_i \in \mathbb{B}_n : (2n+\epsilon)(z_i, w_i) \le 1+\epsilon\} = \{w_i \in \mathbb{B}_n : \rho(z_i, w_i) \le s = \tanh(1+\epsilon)\}$$

denote the hyperbolic disc centered at z_i of radius $1+\epsilon$. Recall also that the orthogonal (Bergman) projection of $L^2(\mathbb{B}_n, dv)$ onto A^2 is given by the integral operator

$$P(f_i)(z_i) := \int_{\mathbb{B}_n} \sum_i \langle K_{w_i}, K_{z_i}\rangle_{A^2} f_i(w_i)\, dv(w_i)$$

Therefore, for all $f_i \in A^2$ we have

$$f_i(z_i) = \int_{\mathbb{B}_n} \sum_i \langle f_i, k_{w_i}\rangle_{A^2} k_{w_i}(z_i)\, d\lambda(w_i). \quad (2.2)$$

As usual an important ingredient in our treatment will be the Rudin-Forelli estimates, see [11] or [6]. Recall the standard Rudin-Forelli estimates:

$$\int_{\mathbb{B}_n} \sum_i \frac{\left|\langle K_{z_i}, K_{w_i}\rangle_{A^2}\right|^{\frac{(1+\epsilon)+s}{2}}}{\left\|K_{z_i}\right\|_{A^2}^{s} \left\|K_{w_i}\right\|_{A^2}^{1+\epsilon}} d\lambda(w_i) \le C = (C)(r,s) < \infty, \quad \forall z_i \in \mathbb{B}_n. \quad (2.3)$$

for all $r > \kappa > s > 0$, where $\kappa = \kappa_n := \frac{2n}{n+1}$. We will use these in the following form: For all $\frac{n-1}{n+1} < a < 1$ we have that

$$\int_{\mathbb{B}_n} \sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{a}}{\left\|K_{w_i}\right\|_{A^2}^{a}} d\lambda(w_i) \le C = C(a) < \infty, \quad \forall z_i \in \mathbb{B}_n. \quad (2.4)$$

To see that this is true in the classical Bergman space setting, for a given $\frac{n-1}{n+1} < a < 1$ set $r = 1 + a$ and $s = 1 - a > 0$. Then $r + s = 2$, and since $a > \frac{n-1}{n+1}$ we have that $r = 1 + a > \frac{2n}{n+1}$. Furthermore since $0 < a < 1$ we have that $0 < s < 1 \le \frac{2n}{n+1}$. By plugging this in (2.3) we obtain (2.4).

We will also need the following uniform version of the Rudin-Forelli estimates.

Lemma 2.1 (see [13]). Let $\frac{n-1}{n+1} < a < 1$. Then

$$\lim_{R\to\infty} \sup_{z_i \in \mathbb{B}_n} \int_{D(z_i,R)^c} \sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{a}}{\left\|K_{w_i}\right\|_{A^2}^{a}} d\lambda(w_i) = 0. \quad (2.5)$$

Proof. Notice first that

$$\int_{D(z_i,R)^c} \sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^{a}}{\left\|K_{w_i}\right\|_{A^2}^{a}} d\lambda(w_i)$$

$$= \int_{D(0,R)^c} \sum_i \left|\left\langle k_{z_i}, k_{\varphi_{z_i}(w_i)}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^a}{\left\|K_{\varphi_{z_i}(w_i)}\right\|_{A^2}^a} d\lambda(w_i)$$

$$= \int_{D(0,R)^c} \sum_i \left|\left\langle k_{z_i}, k_{w_i}\right\rangle_{A^2}\right|^a \frac{\left\|K_{z_i}\right\|_{A^2}^a}{\left\|K_{w_i}\right\|_{A^2}} d\lambda(w_i)$$

$$= \int_{D(0,R)^c} \sum_i \frac{\left|\left\langle K_{z_i}, K_{w_i}\right\rangle_{A^2}^a\right|^a}{\left\|K_{w_i}\right\|_{A^2}^{1+a}} d\lambda(w_i)$$

$$= \int_{D(0,R)^c} \sum_i \frac{dv(w_i)}{|1-\bar{w}_i z_i|^{(n+1)a}(1-|w_i|^2)^{\frac{n+1}{2}(1-a)}}$$

$$= \int_{R'}^1 \int_{\mathbb{S}_n} \sum_i \frac{r^{2n-1} d\xi dr}{|1-z_i r\bar{\xi}|^{(n+1)a}(1-r^2)^{\frac{n+1}{2}(1-a)}}$$

where in the last integral $R = \log \frac{1+R'}{1-R'}$. Notice that $R' \to 1$ when $R \to \infty$ and note that the last integral can be written as

$$\int_{R'}^1 \sum_i I_{(n+1)a-n}(rz_i) \frac{r^{2n-1} d(1+\epsilon)}{(1-r^2)^{\frac{n+1}{2}(1-a)}}$$

where

$$I_c(z_i) := \int_{\mathbb{S}_n} \sum_i \frac{d\xi}{|1-z_i r\bar{\xi}|^{c+n}}$$

By standard estimates (see [11, p. 15] for example), we have that

$$I_{(n+1)a-n}(rz_i) \lesssim \begin{cases} 1, & \text{if } (n+1)a-n<0 \\ \log \dfrac{1}{1-|rz_i|^2}, & \text{if } \quad (n+1)a-n=0 \\ \dfrac{1-|rz_i|^2)^{(n+1)a-n}}{(1-}, & \text{if } \quad (n+1)a-n>0 \end{cases}$$

which gives us that

$$\int_{D(z_i,R)^c} \sum_i \left|\left\langle k_{z_i}, k_{w_i}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^a}{\left\|K_{w_i}\right\|_{A^2}^a} d\lambda(w_i) \lesssim \begin{cases} \displaystyle\int_{R'}^1 \frac{r^{2n-1}}{(1-r^2)^{\frac{n+1}{2}(1-a)}} dr, & \text{if } \quad (n+1)a-n<0 \\ \displaystyle\int_{R'}^1 \log \frac{1}{1-r^2} \frac{r^{2n-1}}{(1-r^2)^{\frac{1}{2}}} dr & \text{if } \quad (n+1)a-n=0 \\ \displaystyle\int_{R'}^1 \frac{r^{2n}-1}{(1-r^2)^{(n+1)a-n+\frac{n+1}{2}(1-a)}} dr, & \text{if } \quad (n+1)a-n>0 \end{cases}$$

Since $a<1$, it is easy to see that all the functions appearing on the right hand side are integrable on $(0,1)$. Therefore, we obtain the desired conclusion by taking the limit as $R \to \infty$ (which is the same as $R' \to 1$).

First, we want to make sure that the class of weakly localized operators is large enough to contain some interesting operators. This is indeed true since every Toeplitz operator with a bounded symbol belongs to this class.

Proposition 2.2 (see [13]). Each Toeplitz operator T_u on $A^{1+\epsilon}$ with a bounded symbol $u(z_i)$ is in $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ for any $0<\epsilon<\infty$.

Proof. Clearly it is enough to show that

$$\sup_{z_i \in \mathbb{B}_n} \int_{D(z_i,1+\epsilon)^c} \sum_i \left|\left\langle T_u k_{z_i}, k_{w_i}\right\rangle_{A^2}\right| \frac{\left\|K_{z_i}\right\|_{A^2}^a}{\left\|K_{w_i}\right\|_{A^2}^a} d\lambda(w_i) \to 0,$$

$$\sup_{z_i \in \mathbb{B}_n} \int_{D(z_i,1+\epsilon)^c} \sum_i \left|\left\langle T_{\bar{u}} k_{z_i}, k_{w_i}\right\rangle\right| \frac{\left\|K_{z_i}\right\|_{A^2}^a}{\left\|K_{w_i}\right\|_{A^2}^a} d\lambda(w_i) \to 0$$

as $\epsilon \to \infty$ for all $\frac{n-1}{n+1} < a < \infty$.

By definition

$$T_u k_{z_i}(w_i) = P\left(u k_{z_i}\right)(w_i) = \int_{\mathbb{B}_n} \sum_i \left\langle K_{x_i}, K_{w_i}\right\rangle_{A^2} u(x_i) k_{z_i}(x_i) dv(x_i).$$

Therefore,

$$\left|\langle T_u k_{z_i}, k_{w_i}\rangle_{A^2}\right| \leq \int_{\mathbb{B}_n} \sum_i \left|\langle k_{w_i}, k_{x_i}\rangle_{A^2}\right| |u(x_i)| \left|\langle k_{z_i}, k_{x_i}\rangle_{A^2}\right| d\lambda(x_i)$$

$$\leq \|u\|_\infty \int_{\mathbb{B}_n} \sum_i \left|\langle k_{w_i}, k_{x_i}\rangle_{A^2} \langle k_{x_i}, k_{z_i}\rangle_{A^2}\right| d\lambda(x_i)$$

Now for $z_i, x_i \in \mathbb{B}_n$, set

$$I_{z_i}(x_i) := \sum_i \left|\langle k_{x_i}, k_{z_i}\rangle_{A^2}\right| \int_{D(z_i,1+\epsilon)^c} \left|\langle k_{w_i}, k_{x_i}\rangle_{A^2}\right| \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{w_i}\|_{A^2}^a} d\lambda(w_i)$$

First note that

$$\int_{D(z_i,1+\epsilon)^c} \sum_i \left|\langle T_u k_{z_i}, k_{w_i}\rangle_{A^2}\right| \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{w_i}\|_{A^2}^a} d\lambda(w_i)$$

$$\leq \|u\|_\infty \int_{D(z_i,1+\epsilon)^c} \sum_i \int_{\mathbb{B}_n} \left|\langle k_{w_i}, k_{x_i}\rangle_{A^2} \langle k_{x_i}, k_{z_i}\rangle_{A^2}\right| d\lambda(x_i) \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{w_i}\|_{A^2}^a} d\lambda(w_i)$$

$$= \|u\|_\infty \int_{\mathbb{B}_n} \int_{D(z_i,1+\epsilon)^c} \sum_i \left|\langle k_{w_i}, k_{x_i}\rangle_{A^2}\right| \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{w_i}\|_{A^2}^a} d\lambda(w_i) \left|\langle k_{x_i}, k_{z_i}\rangle_{A^2}\right| d\lambda(x_i)$$

$$= \|u\|_\infty \int_{\mathbb{B}_n} \sum_i I_{z_i}(x_i) d\lambda(x_i)$$

$$= \|u\|_\infty \sum_i \left(\int_{D\left(z_i,\frac{1+\epsilon}{2}\right)} + \int_{D\left(z_i,\frac{1+\epsilon}{2}\right)^c}\right) I_{z_i}(x_i) d\lambda(x_i).$$

To estimate the first integral notice that for $x_i \in D\left(z_i, \frac{1+\epsilon}{2}\right)$ we have $D(z_i, 1+\epsilon)^c \subset D\left(x_i, \frac{1+\epsilon}{2}\right)^c$. Therefore, the first integral is no greater than

$$\int_{D\left(z_i,\frac{1+\epsilon}{2}\right)} \sum_i \int_{D\left(x_i,\frac{1+\epsilon}{2}\right)^c} \left|\langle k_{w_i}, k_{x_i}\rangle_{A^2}\right| \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{w_i}\|_{A^2}^a} d\lambda(w_i) \left|\langle k_{x_i}, k_{z_i}\rangle_{A^2}\right| d\lambda(x_i).$$

It is easy to see that the last expression is no greater than $C(a)A\left(\frac{1+\epsilon}{2}\right)$, where

$$A(1+\epsilon) = \sup_{z_i \in \mathbb{B}_n} \int_{D(z_i,1+\epsilon)^c} \sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{A^2}\right| \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{w_i}\|_{A^2}^a} d\lambda(w_i)$$

and $C(a)$ is just the bound from the standard Rudin-Forelli estimates (2.4).
Estimating the second integral is simpler. The second integral is clearly no greater than

$$\int_{D\left(z_i,\frac{1+\epsilon}{2}\right)^c} \sum_i \int_{\mathbb{B}_n} \left|\langle k_{w_i}, k_{x_i}\rangle_{A^2}\right| \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{w_i}\|_{A^2}^a} d\lambda(w_i) \left|\langle k_{x_i}, k_{z_i}\rangle_{A^2}\right| d\lambda(x_i)$$

By the standard Rudin-Forelli estimates (2.4) the inner integral is no greater than

$$C(a) \frac{\|K_{z_i}\|_{A^2}^a}{\|K_{x_i}\|_{A^2}^a}$$

where the constant $C(a)$ is independent of z_i and x_i. So, the whole integral is bounded by $C(a)A\left(\frac{1+\epsilon}{2}\right)$. Therefore

$$\sup_{z_i \in \mathbb{B}_n} \int_{D(z_i,1+\epsilon)^c} \sum_i \left|\langle T_u k_{z_i}, k_{w_i}\rangle_{A^2}\right| \frac{\|K_{z_i}\|^a}{\|K_{w_i}\|^a} d\lambda(w_i) \leq \|u\|_\infty \left(C(a)A\left(\frac{1+\epsilon}{2}\right) + C(a)A\left(\frac{1+\epsilon}{2}\right)\right)$$

Applying the uniform Rudin-Forelli estimates (2.5) in Lemma 2.1 completes the proof since $2C(a)\|u\|_\infty A\left(\frac{1+\epsilon}{2}\right) \to 0$ as $\epsilon \to \infty$.

We next show that the class of weakly localized operators forms a $*$-algebra.

Proposition 2.3 (see [13]). If $0 < \epsilon < \infty$ then $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ is an algebra. Furthermore, $\mathcal{A}_2(\mathbb{B}_n)$ is a $*$-algebra.

Proof. It is trivial that $T \in \mathcal{A}_2(\mathbb{B}_n)$ implies $T^* \in \mathcal{A}_2(\mathbb{B}_n)$. It is also easy to see that any linear combination of two operators in $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ must be also in $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. It remains to prove that if $T, S \in \mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$, then $TS \in \mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. To that end, we have that

$$\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle TSk_{z_i},k_{w_i}\right\rangle_{A^2}\right|\frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}d\lambda(w_i)$$

$$=\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle Sk_{z_i},T^*k_{w_i}\right\rangle_{A^2}\right|\frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}d\lambda(w_i)$$

$$=\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\int_{\mathbb{B}_n}\left\langle Sk_{z_i},k_{x_i}\right\rangle_{A^2}\left\langle k_{x_i},T^*k_{w_i}\right\rangle_{A^2}d\lambda(x_i)\right|\frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}d\lambda(w_i)$$

$$\le\int_{\mathbb{B}_n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle k_{x_i},T^*k_{w_i}\right\rangle_{A^2}\right|\frac{d\lambda(w_i)}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}\left|\left\langle Sk_{z_i},k_{x_i}\right\rangle_{A^2}\right|\left\|K_{z_i}\right\|^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}d\lambda(x_i).$$

Proceeding exactly as in the proof of the previous Proposition and using the conditions following from $T,S\in\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ in the place of the local Rudin-Forelli estimates (2.5) (and replacing a with $1-\frac{2\delta}{(1+\epsilon)(n+1)}\Big)$ we obtain that

$$\lim_{\epsilon\to\infty}\sup_{z_i\in\mathbb{B}_n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle TSk_{z_i},k_{w_i}\right\rangle_{A^2}\right|\frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}}d\lambda(w_i)=0$$

The corresponding condition for $(TS)^*$ is proved in exactly the same way.

We next show that every weakly localized operator can be approximated by infinite sums of well localized pieces. To state this property we need to recall the following proposition proved in [6]

Proposition 2.4. There exists an integer $N>0$ such that for any $r\ge 0$ there is a covering $\mathcal{F}_r=\{F_j\}$ of $\mathbb{B}_n$ by disjoint Borel sets satisfying

(1) every point of $\mathbb{B}_n$ belongs to at most N of the sets $G_j:=\{z_i\in\mathbb{B}_n: d(z_i,F_j)\le r\}$,

(2) $\operatorname{diam}_d F_j\le 2r$ for every j.

We use this to prove the following proposition, which is similar to what appears in [6], but exploits condition (1.6).

Proposition 2.5 (see [13]). Let $0<\epsilon<\infty$ and let T be in the norm closure of $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. Then for every $\epsilon>0$ there exists $r>0$ such that for the covering $\mathcal{F}_r=\{F_j\}$ (associated to r) from Proposition 2.4, we have:

$$\left\|TP-\sum_j M_{1_{F_j}}TPM_{1_{G_j}}\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)}<\epsilon$$

Proof. By Proposition 2.3 in conjunction with Proposition 2.4 and a simple approximation argument, we may assume that $T\in\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. Now define

$$S=TP-\sum_j M_{1_{F_j}}TPM_{1_{G_j}}$$

Given ϵ choose r large enough so that

$$\sup_{z_i\in\mathbb{B}_n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle Tk_{z_i},k_{w_i}\right\rangle_{A^2}\right|\frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta\epsilon}{(1+\epsilon)(n+1)}}}d\lambda(w_i)<\epsilon$$

and

$$\sup_{z_i\in\mathbb{B}_n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\left\langle T^*k_{z_i},k_{w_i}\right\rangle_{A^2}\right|\frac{\left\|K_{z_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}}{\left\|K_{w_i}\right\|_{A^2}^{1-\frac{2\delta}{(1+\epsilon)(n+1)}}}d\lambda(w_i)<\epsilon$$

Now for any $z_i\in\mathbb{B}_n$ let $z_i\in F_{j_0}$, so that

$$
\begin{aligned}
|Sf_i(z_i)| &\leq \int_{\mathbb{B}_n} \sum_j \sum_i 1_{F_j(z_i)} 1_{G_j^c}(w_i) \left| \langle T^* K_{z_i}, K_{w_i} \rangle_{A^2} \right| |f_i(w_i)| dv(w_i) \\
&= \int_{G_{j_0}^c} \sum_i \left| \langle T^* K_{z_i}, K_{w_i} \rangle_{A^2} \right| |f_i(w_i)| dv(w_i) \\
&\leq \int_{D(z_i, 1+\epsilon)^c} \sum_i \left| \langle T^* K_{z_i}, K_{w_i} \rangle_{A^2} \right| |f_i(w_i)| dv(w_i)
\end{aligned}
$$

To finish the proof, we will estimate the operator norm of the integral operator on $L^{1+\epsilon}(\mathbb{B}_n, dv)$ with kernel $1_{D(z_i,1+\epsilon)^c}(w_i) \left| \langle T^* K_{z_i}, K_{w_i} \rangle_{A^2} \right|$ by using the classical Schur test. To that end, let $h(w_i) = \|K_{w_i}\|_{A^2}^{\frac{2\delta\epsilon^2}{(1+\epsilon)^2(n+1)}}$ so that

$$
\begin{aligned}
&\int_{\mathbb{B}_n} \sum_i 1_{D(z_i,1+\epsilon)^c}(w_i) \left| \langle T^* K_{z_i}, K_{w_i} \rangle_{A^2} \right| h(w_i)^{\left(\frac{1+\epsilon}{\epsilon}\right)} dv(w_i) \\
&= \int_{D(z_i,1+\epsilon)^c} \sum_i \left| \langle T^* K_{z_i}, K_{w_i} \rangle_{A^2} \right| \|K_{w_i}\|_{A^2}^{\frac{2\delta}{(1+\epsilon)(n+1)}} dv(w_i) \\
&= \int_{D(z_i,1+\epsilon)^c} \sum_i \left| \langle T^* k_{z_i}, k_{w_i} \rangle_{A^2} \right| \|K_{z_i}\| \|K_{w_i}\|_{A^2}^{\frac{2\delta}{(1+\epsilon)(n+1)}-1} d\lambda(w_i) \\
&\leq \sum_i \epsilon \|K_{z_i}\|_{A^2}^{\frac{2\delta}{(1+\epsilon)(n+1)}} = \sum_i \epsilon h(z_i)^{\left(\frac{1+\epsilon}{\epsilon}\right)}
\end{aligned}
$$

Similarly, we have that

$$
\int_{\mathbb{B}_n} \sum_i 1_{D(z_i,1+\epsilon)^c}(w_i) \left| \langle T^* K_{z_i}, K_{w_i} \rangle_{A^2} \right| h(z_i)^{1+\epsilon} dv(z_i) \leq \sum_i \epsilon h(w_i)^{1+\epsilon}
$$

which completes the proof.

It should be noted that a very similar Schur test argument actually proves that condition (1.5) implies that T is bounded on $A^{1+\epsilon}$.

We can now prove one of our main results whose proof uses the ideas in [6, Theorem 4.3] and [5, Lemma 5.3]. First, for any $w_i \in \mathbb{B}_n$ and $0 < \epsilon < \infty$, let $k_{w_i}^{(1+\epsilon)}$ be the "$1+\epsilon$- normalized reproducing kernel" defined by

$$
k_{w_i}^{(1+\epsilon)}(z_i) = \frac{K(z_i, w_i)}{\|K_{w_i}\|^{\frac{2\epsilon}{(1+\epsilon)}}}
$$

Clearly we have that $k_{w_i}^{(2)} = k_{w_i}$ and an easy computation tells us that $\|k_{w_i}^{(1+\epsilon)}\|_{A^{1+\epsilon}} \approx 1$ (where obviously we have equality when $\epsilon = 1$).

Theorem 2.6 (see [13]). Let $0 < \epsilon < \infty$ and let T be in the norm closure of $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. Then there exists $\epsilon \geq 0$ (both depending on T) such that

$$
\|T\|_e \leq (1+\epsilon) \limsup_{|z_i| \to 1^-} \sup_{w_i \in D(z_i, 1+\epsilon)} \sum_i \left| \left\langle T k_{z_i}^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right|
$$

where $\|T\|_e$ is the essential norm of T as a bounded operator on $A^{1+\epsilon}$.

Proof. Since $P: L^{1+\epsilon}(\mathbb{B}_n, dv) \to A^{1+\epsilon}$ is a bounded projection, it is enough to estimate the essential norm of $T = TP$ as an operator on from $A^{1+\epsilon}$ to $L^{1+\epsilon}(\mathbb{B}_n, dv)$.

Clearly if $\|TP\|_{\mathrm{e}} = 0$ then there is nothing to prove, so assume that $\|TP\|_{\mathrm{e}} > 0$. By Proposition 2.5 there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{F_j\}$ associated to r (from Proposition 2.4)

$$
\left\| TP - \sum_j M_{1_{F_j}} TP M_{1_{G_j}} \right\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)} < \frac{1}{2} \|TP\|_{\mathrm{e}}
$$

Since $\sum_{j<m} M_{1_{F_j}} TP M_{1_{G_j}}$ is compact for every $m \in \mathbb{N}$ we have that $\|TP\|_{\mathrm{e}}$ (as an operator from $A^{1+\epsilon}$ to $L^{1+\epsilon}(\mathbb{B}_n, dv)\big)$ can be estimated in the following way:

$$\|TP\|_{\mathrm{e}} \leq \left\| TP - \sum_{j<m} M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)}$$

$$\leq \left\| TP - \sum_{j} M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)} + \|T_m\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)}$$

$$\leq \frac{1}{2}\|TP\|_{\mathrm{e}} + \|T_m\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)}$$

where

$$T_m = \sum_{j \geq m} M_{1_{F_j}} TPM_{1_{G_j}}$$

We will complete the proof by showing that there exists $\epsilon \geq 0$ where

$$\limsup_{m\to\infty} \|T_m\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)} \lesssim (1+\epsilon) \limsup_{|z_i| \to 1^-} \sup_{w_i \in D(z_i, 1+\epsilon)} \sum_i \left| \left\langle Tk_{z_i}^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right| + \frac{1}{4}\|TP\|_{\mathrm{e}}$$

If $f_i \in A^{1+\epsilon}$ is arbitrary of norm no greater than 1 , then

$$\|T_m f_i\|_{A^{1+\epsilon}}^{1+\epsilon} = \sum_{j \geq m} \sum_i \left\| M_{1_{F_j}} TPM_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}^{1+\epsilon}$$

$$= \sum_{j \geq m} \sum_i \frac{\left\| M_{1_{F_j}} TPM_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}^{1+\epsilon}}{\left\| M_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}^{1+\epsilon}} \left\| M_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}^{1+\epsilon} \leq N \sup_{j \geq m} \left\| M_{1_{F_j}} T l_j \right\|_{A^{1+\epsilon}}^{1+\epsilon}$$

where

$$l_j := \sum_i \frac{PM_{1_{G_j}} f_i}{\left\| M_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}}$$

Therefore, we have that

$$\|T_m\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)} \lesssim \sup_{j \geq m \|f_i\|_{A^{1+\epsilon}} \leq 1} \sup \sum_i \left\{ \left\| M_{1_{F_j}} T l_j \right\|_{A^{1+\epsilon}} : l_j = \frac{PM_{1_{G_j}} f_i}{\left\| M_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}} \right\}$$

and hence

$$\limsup_{m\to\infty} \|T_m\|_{A^{1+\epsilon} \to L^{1+\epsilon}(\mathbb{B}_n, dv)} \lesssim \limsup_{j\to\infty} \sup_{\|f_i\|_{A^{1+\epsilon}} \leq 1} \sum_i \left\{ \left\| M_{1_{F_j}} T l_j \right\|_{A^{1+\epsilon}} : l_j = \frac{PM_{1_{G_j}} f_i}{\left\| M_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}} \right\}$$

Now pick a sequence $\{(f_i)_j\}$ in $A^{1+\epsilon}$ with $\|(f_i)_j\|_{A^{1+\epsilon}} \leq 1$ such that

$$\limsup_{j\to\infty} \sup_{\|f_i\| \leq 1} \sum_i \left\{ \left\| M_{1_{F_j}} T g \right\|_{A^{1+\epsilon}} : g = \frac{PM_{1_{G_j}} f_i}{\left\| M_{1_{G_j}} f_i \right\|_{A^{1+\epsilon}}} \right\} - \frac{1}{4}\|TP\|_{\mathrm{e}} \leq \limsup_{j\to\infty} \left\| M_{1_{F_j}} T g_j \right\|_{A^{1+\epsilon}}$$

where

$$g_j = \sum_i \frac{PM_{1_{G_j}} (f_i)_j}{\left\| M_{1_{G_j}} (f_i)_j \right\|_{A^{1+\epsilon}}} = \sum_i \frac{\int_{G_j} \left\langle f_i, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} k_{w_i}^{(1+\epsilon)} d\lambda(w_i)}{\left(\int_{G_j} \left| \left\langle f_i, k_u^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right|^{1+\epsilon} d\lambda(u) \right)^{\frac{1}{1+\epsilon}}}$$

$$= \int_{G_j} \sum_i \tilde{a}_j(w_i) k_{w_i}^{(1+\epsilon)} d\lambda(w_i)$$

where

$$\tilde{a}_j(w_i) = \sum_i \frac{\left\langle f_i, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2}}{\left(\int_{G_j} \left| \left\langle f_i, k_u^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right|^{1+\epsilon} d\lambda(u) \right)^{\frac{1}{1+\epsilon}}}$$

Finally, by the reproducing property and Hölder's inequality, we have that

$$\limsup_{j\to\infty} \left\| M_{1_{F_j}} T g_j \right\|_{A^{1+\epsilon}}^{1+\epsilon}$$

$$\leq \limsup_{j\to\infty} \int_{F_j} \sum_i \left(\int_{G_j} \left| \tilde{a}_j(w_i) \right| \left| T k_{w_i}^{(1+\epsilon)}(z_i) \right| d\lambda(w_i) \right)^{1+\epsilon} dv(z_i)$$

$$= \limsup_{j\to\infty} \int_{F_j} \sum_i \left(\int_{G_j} \left| \tilde{a}_j(w_i) \right| \left| \left\langle T k_{w_i}^{(1+\epsilon)}, k_{z_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right| d\lambda(w_i) \right)^{1+\epsilon} d\lambda(z_i)$$

$$\leq \limsup_{|z_i|\to 1^-} \sup_{w_i \in D(z_i, 3(1+\epsilon))} \sum_i \left| \left\langle T k_{z_i}^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right|^{1+\epsilon} \left(\sup_j \lambda(G_j)^{1+\epsilon} \int_{G_j} \left| \tilde{a}_j(w_i) \right|^{1+\epsilon} d\lambda(w_i) \right)$$

$$\leq C(1+\epsilon) \limsup_{|z_i|\to 1^-} \sup_{w_i \in D(z_i, 3(1+\epsilon))} \sum_i \left| \left\langle T k_{z_i}^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right|^{1+\epsilon}$$

since by Proposition 2.4 we have that $z_i \in F_j$ and $w_i \in G_j$ implies that $d(z_i, w_i) \leq 3(1+\epsilon)$ and $\lambda(G_j) \leq C(1+\epsilon)$ where $C(1+\epsilon)$ is independent of j.

We will finish this section off with a proof of Theorem 1.5. First, for $z_i \in \mathbb{B}_n$, define

$$U_{z_i}^{(1+\epsilon)} f_i(w_i) := f_i(\varphi_{z_i}(w_i)) (k_{z_i}(w_i))^{\frac{2}{1+\epsilon}}$$

which via a simple change of variables argument is clearly an isometry on $A^{1+\epsilon}$. As was shown in [9], an easy computation tells us that there exists a unimodular function $\Phi(\cdot,\cdot)$ on $\mathbb{B}_n \times \mathbb{B}_n$ where

$$\left(U_{z_i}^{(1+\epsilon)}\right)^* k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} = \Phi(z_i, w_i) k_{(\phi_i)_{z_i}(w_i)}^{\left(\frac{1+\epsilon}{\epsilon}\right)}. \tag{2.6}$$

With the help of the operators $U_{z_i}^{(1+\epsilon)}$, we will prove the following general result which in conjunction with Theorem 2.6 proves Theorem 1.5. Note that proof is similar to the proof of [5, Proposition 1.4].

Proposition 2.7 (see [13]). If T is any bounded operator on $A^{1+\epsilon}$ for $0 < \epsilon < \infty$ then the following are equivalent

(a) $\lim_{|z_i|\to 1^-} \sup_{w_i \in D(z_i, 1+\epsilon)} \sum_i \left| \left\langle T k_{z_i}^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right| = 0$ for all $\epsilon \geq 0$,

(b) $\lim_{|z_i|\to 1^-} \sup_{w_i \in D(z_i, 1+\epsilon)} \sum_i \left| \left\langle T k_{z_i}^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right| = 0$ for some $\epsilon \geq 0$,

(c) $\lim_{|z_i|\to 1^-} \sum_i \left| \langle T k_{z_i}, k_{z_i} \rangle_{A^2} \right| = 0$.

Proof. Trivially we have that $(a) \Rightarrow (b)$, and the fact that $(b) \Rightarrow (c)$ follows by definition and setting $z_i = w_i$. We will complete the proof by showing that $(c) \Rightarrow (a)$.

Assume to the contrary that $\left| \langle T k_{z_i}, k_{z_i} \rangle_{A^2} \right|$ vanishes as $|z_i| \to 1^-$but that

$$\limsup_{|z_i|\to 1^-} \sup_{w_i \in D(z_i, 1+\epsilon)} \left| \left\langle T k_{z_i}^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right| \neq 0$$

for some fixed $\epsilon \geq 0$. Thus, there exists sequences $\{(z_i)_m\}, \{(w_i)_m\}$ and some $0 < r_0 < 1$ where $\lim_{m\to\infty} |(z_i)_m| = 1$ and $|(w_i)_m| \leq r_0$ for any $m \in \mathbb{N}$, and where

$$\limsup_{m\to\infty} \sum_i \left| \left\langle T k_{(z_i)_m}^{(1+\epsilon)}, k_{(w_i)_{(z_i)_m}((w_i)_m)}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right\rangle_{A^2} \right| > \epsilon \tag{2.7}$$

for some $\epsilon \geq 0$. Furthermore, passing to a subsequence if necessary, we may assume that $\lim_{m\to\infty} (w_i)_m = w_i \in \mathbb{B}_n$. Note that since $|(w_i)_m| \leq r_0 < 1$ for all m, we trivially have $\lim_{m\to\infty} k_{(w_i)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)} = k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)}$ where the convergence is in the $A^{\left(\frac{1+\epsilon}{\epsilon}\right)}$ norm.

Let $\mathcal{B}(A^{1+\epsilon})$ be the space of bounded operators on $A^{1+\epsilon}$. Since the unit ball in $\mathcal{B}(A^{1+\epsilon})$ is WOT compact, we can (passing to another subsequence if necessary) assume that

$$\hat{T} = \mathrm{WOT} - \lim_{m\to\infty} \sum_i U_{(z_i)_m}^{(1+\epsilon)} T \left(U_{(z_i)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)^*$$

Thus, we have that

$$\begin{aligned}\limsup_{m\to\infty} \sum_i \left|\left\langle T k_{(z_i)_m}^{(1+\epsilon)}, k_{\varphi_{(z_i)_m}((w_i)_m)}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right\rangle_{A^2}\right| &= \limsup_{m\to\infty} \sum_i \left|\left\langle U_{(z_i)_m}^{(1+\epsilon)} T \left(U_{(z_i)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)^* k_0^{(1+\epsilon)}, k_{(w_i)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right\rangle_{A^2}\right| \\ &= \limsup_{m\to\infty} \sum_i \left|\left\langle U_{(z_i)_m}^{(1+\epsilon)} T \left(U_{(z_i)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)^* k_0^{(1+\epsilon)}, k_{w_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right\rangle_{A^2}\right| \\ &= \sum_i \left|\left\langle \hat{T} k_0, k_{w_i}\right\rangle_{A^2}\right|\end{aligned}$$

However, for any $z_i \in \mathbb{B}_n$

$$\begin{aligned}\left|\left\langle \hat{T} k_{z_i}^{(1+\epsilon)}, k_{z_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right\rangle\right| &= \lim_{m\to\infty} \sum_i \left|\left\langle U_{(z_i)_m}^{(1+\epsilon)} T \left(U_{(z_i)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)^* k_{z_i}^{(1+\epsilon)}, k_{z_i}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right\rangle\right| \\ &\approx \lim_{m\to\infty} \sum_i \left|\left\langle T k_{\varphi_{(z_i)_m}(z_i)}^{(1+\epsilon)}, k_{\varphi_{(z_i)_m}(z_i)}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right\rangle_{A^2}\right| = 0\end{aligned}$$

since by assumption $\left|\langle T k_{z_i}, k_{z_i}\rangle\right|$ vanishes as $|z_i| \to 1^-$. Thus, since the Berezin transform is injective on $A^{1+\epsilon}$, we get that $\hat{T} = 0$, which contradicts (2.7) and completes the proof.

III. Generalized Bargmann-Fock Space Case

We will prove Theorems 1.2 and 1.3. Some parts of the proofs are essentially identical to proof of Theorem 1.5 and so we will we only outline the necessary modifications.

For this section, let

$$D(z_i, 1+\epsilon) := \{w_i \in \mathbb{C}^n : |w_i - z_i| < 1+\epsilon\}$$

denote the standard Euclidean disc centered at the point z_i of radius $\epsilon \geq -1$. For $z_i \in \mathbb{C}^n$, we define

$$U_{z_i} f_i(w_i) := f_i(z_i - w_i) k_{z_i}(w_i)$$

which via a simple change of variables argument is clearly an isometry on $\mathcal{F}^{1+\epsilon}$ (though note in general that it is not clear whether U_{z_i} even maps $\mathcal{F}_{\phi_i}^{1+\epsilon}$ into itself). Recall also that the orthogonal projection of $L^2(\mathbb{C}^n, e^{-2\phi_i} dv)$ onto $\mathcal{F}_{\phi_i}^2$ is given by the integral operator

$$P(f_i)(z_i) := \int_{\mathbb{C}^n} \sum_i \langle K_{w_i}, K_{z_i}\rangle_{\mathcal{F}_{\phi_i}^2} f_i(w_i) e^{-2\phi_i(w_i)} dv$$

Therefore, for all $f_i \in \mathcal{F}_{\phi_i}^{1+\epsilon}$ we have

$$f_i(z_i) = \int_{\mathbb{C}^n} \sum_i \langle f_i, \widetilde{k_{w_i}}\rangle_{\mathcal{F}_{\phi_i}^2} \widetilde{k_{w_i}}(z_i) dv(w_i). \tag{3.1}$$

where $\widetilde{k_{w_i}}(z_i) := K_{w_i}(z_i) e^{-\phi_i(w_i)}$. Note that $|K(z_i, z_i)| \approx e^{2\phi_i(z_i)}$ (see [8]) so that

$$\left|\widetilde{k_{w_i}}(z_i)\right| \approx \left|k_{w_i}(z_i)\right|. \tag{3.2}$$

The following analog of Lemma 2.1 is simpler to prove in this case.

Lemma 3.1 [13].

$$\lim_{R\to\infty} \sup_{z_i \in \mathbb{C}^n} \int_{D(z_i,R)^c} \sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{\mathcal{F}_{\phi_i}^2}\right| dv(w_i) = 0. \tag{3.3}$$

To prove this, simply note that there exists $\epsilon > 0$ such that $\sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{\mathcal{F}_{\phi_i}^2}\right| \leq \sum_i e^{-\epsilon|z_i - w_i|}$ for all $z_i, w_i \in \mathbb{C}^n$.

The proof of this is then immediate since

$$\int_{D(z_i,R)^c} \sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{\mathcal{F}_{\phi_i}^2}\right| dv(w_i) \leq \int_{D(0,R)^c} \sum_i e^{-\epsilon|w_i|} dv(w_i)$$

which clearly goes to zero as $R \to \infty$.

As in the Bergman case, $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$ contains all Toeplitz operators with bounded symbols. Also, as was stated in the introduction, any $T \in \mathcal{A}_{\phi_i}(\mathbb{C}^n)$ is automatically bounded on $\mathcal{F}_{\phi_i}^{1+\epsilon}$ for all $0 \leq \epsilon \leq \infty$. To prove this, note that it is enough to prove that T is bounded on $\mathcal{F}_{\phi_i}^1$ and $\mathcal{F}_{\phi_i}^\infty$ by complex interpolation (see [5]). To that end, we only prove that T is bounded on $\mathcal{F}^1_{\phi_i}$ since the proof that T is bounded on $\mathcal{F}^\infty_{\phi_i}$ is similar. If $T \in \mathcal{A}_{\phi_i}(\mathbb{C}^n)$ and $f_i \in \mathcal{F}^1_{\phi_i}$, then the reproducing property gives us that

$$\begin{aligned}|Tf_i(z_i)|e^{-\phi_i(z_i)} &\approx \left|\langle f_i, T^*k_{z_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right| \\ &\lesssim \int_{\mathbb{C}^n}\sum_i |f_i(u)| \left|\langle T^*k_{z_i}, k_u\rangle_{\mathcal{F}^2_{\phi_i}}\right| e^{-\phi_i(u)}dv(u)\end{aligned}$$

Thus, by Fubini's theorem, we have that

$$\|\sum_i Tf_i\|_{\mathcal{F}^1_{\phi_i}} \le \int_{\mathbb{C}^n}\sum_i |f_i(u)| \left(\int_{\mathbb{C}^n}\left|\langle T^*k_{z_i}, k_u\rangle_{\mathcal{F}^2_{\phi_i}}\right| dv(z_i)\right) e^{-\phi_i(u)}dv(u) \lesssim \sum_i \|f_i\|_{\mathcal{F}^1_{\phi_i}}$$

In addition, $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$ satisfies the following two properties:

Proposition 3.2 (see [13]). Each Toeplitz operator T_u on $\mathcal{F}^{1+\epsilon}_{\phi_i}$ with a bounded symbol $u(z_i)$ is weakly localized.

Proof. Since $\sum_i \left|\langle k_{z_i}, k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right| \le \sum_i e^{-\epsilon|z_i-w_i|}$ for some $\epsilon > 0$ we have that

$$\begin{aligned}\sum_i \left|\langle T_u k_{z_i}, k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right| &\lesssim \|u\|_{L^\infty}\int_{\mathbb{C}^n}\sum_i \left|\langle k_{z_i}, k_{x_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|\left|\langle k_{x_i}, k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right| dx_i \\ &\lesssim \|u\|_{L^\infty}\int_{\mathbb{C}^n}\sum_i e^{-\epsilon|z_i-x_i|}e^{-\epsilon|x_i-w_i|}dx_i\end{aligned}$$

Now if $|z_i - w_i| \ge 1+\epsilon$ then by the triangle inequality we have that either $|z_i - x_i| \ge \frac{1+\epsilon}{2}$ or $|x_i - w_i| \ge \frac{1+\epsilon}{2}$ so that

$$\int_{D(z_i,1+\epsilon)^c}\left|\langle T_u k_{z_i}, k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right| dw_i$$
$$\lesssim e^{-\frac{\epsilon(1+\epsilon)}{2}}\|u\|_{L^\infty}\int_{D(z_i,1+\epsilon)^c}\sum_i\int_{\mathbb{C}^n} e^{-\frac{\epsilon}{2}|z_i-x_i|}e^{-\frac{\epsilon}{2}|x_i-w_i|}dx_i dw_i \lesssim e^{-\frac{\epsilon(1+\epsilon)}{2}}\|u\|_{L^\infty}$$

Note that T_u is sufficiently localized even in the sense of Xia and Zheng by [10, Proposition 4.1]. Also note that a slight variation of the above argument shows that the Toeplitz operator $T_\mu \in \mathcal{A}_{\phi_i}(\mathbb{C}^n)$ if μ is a positive Fock-Carleson measure on $\mathbb{C}^n$ (see [8] for precise definitions).

Proposition 3.3. $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$ forms a $*$-algebra.

We will omit the proof of this proposition since it is proved in exactly the same way as it is in the Bergman space case (where the only difference is that one uses (3.1) in conjunction with (3.2) instead of (2.2)).

We next prove that operators in the norm closure of $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$ can also be approximated by infinite sums of well localized pieces. To state this property we need to recall the following proposition proved in [6]

Proposition 3.4. There exists an integer $N > 0$ such that for any $r > 0$ there is a covering $\mathcal{F}_r = \{F_j\}$ of $\mathbb{C}^n$ by disjoint Borel sets satisfying

(1) every point of $\mathbb{C}^n$ belongs to at most N of the sets $G_j := \{z_i \in \mathbb{C}^n : d(z_i, F_j) \le r\}$,

(2) $\mathrm{diam}_d F_j \le 2r$ for every j.

We use this to prove the following proposition, which is similar to what appears in [6], but exploits condition (1.4) (and is proved in a manner that is similar to the proof of [5, Lemma 5.2]). Note that for the rest of this paper, $L^{1+\epsilon}_{\phi_i}$ will refer to the space of measurable functions f_i on $\mathbb{C}^n$ such that $f_i e^{-\phi_i} \in L^{1+\epsilon}(\mathbb{C}^n, dv)$.

Proposition 3.5 (see [13]). Let $0 < \epsilon < \infty$ and let T be in the norm closure of $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$. Then for every $\epsilon > 0$ there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{F_j\}$ (associated to r) from Proposition 3.4

$$\left\|TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}\right\|_{\mathcal{F}^{1+\epsilon}_{\phi_i}\to L^{1+\epsilon}_{\phi_i}} < \epsilon$$

Proof. Again by an easy approximation argument we can assume that $T \in \mathcal{A}_{\phi_i}(\mathbb{C}^n)$.

Furthermore, we first prove the theorem for $\epsilon = 1$.

Define

$$S = TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}$$

Given ϵ choose r large enough so that

$$\sup_{z_i\in\mathbb{C}^n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\langle T^*k_{z_i},k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|dv(w_i)<\epsilon \text{ and } \sup_{z_i\in\mathbb{C}^n}\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\langle Tk_{z_i},k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|dv(w_i)<\epsilon$$

Now for any $z_i\in\mathbb{C}^n$, pick j_0 such that $z_i\in F_{j_0}$. Then we have that

$$|Sf_i(z_i)|\leq\int_{\mathbb{C}^n}\sum_j\sum_i 1_{F_j}(z_i)1_{G_j^c}(w_i)\left|\langle T^*K_{z_i},K_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|f_i(w_i)\mid e^{-2\phi_i(w_i)}dv(w_i)$$

$$=\int_{G_{j_0}^c}\sum_i\left|\langle T^*K_{z_i},K_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right||f_i(w_i)|e^{-2\phi_i(w_i)}dv(w_i)$$

$$\leq\int_{D(z_i,1+\epsilon)^c}\sum_i\left|\langle T^*K_{z_i},K_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right||f_i(w_i)|e^{-2\phi_i(w_i)}dv(w_i)$$

To finish the proof when $\epsilon=1$, we will estimate the operator norm of the integral operator on $L^2_{\phi_i}$ with kernel $1_{D(z_i,1+\epsilon)^c}(w_i)\left|\langle T^*K_{z_i},K_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|$ using the classical Schur test. To that end, let $h(z_i)=e^{\frac{1}{2}\phi_i(z_i)}$ so that

$$\int_{\mathbb{C}^n}\sum_i 1_{D(z_i,1+\epsilon)^c}(w_i)\left|\langle T^*K_{z_i},K_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|h(w_i)^2e^{-2\phi_i(w_i)}dv(w_i)$$

$$\approx\sum_i h(z_i)^2\int_{D(z_i,1+\epsilon)^c}\left|\langle T^*k_{z_i},k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|dv(w_i)\lesssim\epsilon h(z_i)^2.$$

Similarly, we have that

$$\int_{\mathbb{C}^n}\sum_i 1_{D(z_i,1+\epsilon)^c}(w_i)\left|\langle T^*K_{z_i},K_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|h(z_i)^2e^{-2\phi_i(z_i)}dv(z_i)\lesssim\epsilon h(w_i)^2$$

which finishes the proof when $\epsilon=1$.

Now assume that $0<\epsilon<1$. Since T is bounded on $\mathcal{F}^1_{\phi_i}$, we easily get that

$$\left\|\sum_j M_{1_{F_j}}TPM_{1_{G_j}}\right\|_{\mathcal{F}^1_{\phi_i}\to L^1_{\phi_i}}<\infty$$

which by complex interpolation proves the proposition when $0<\epsilon<1$. Finally when $0<\epsilon<\infty$, one can similarly get a trivial $L^1_{\phi_i}\to\mathcal{F}^1_{\phi_i}$ operator norm bound on

$$\left(\sum_j M_{1_{F_j}}TPM_{1_{G_j}}\right)^*=\sum_j PM_{1_{G_j}}T^*PM_{1_{F_j}}$$

since T^* is bounded on $\mathcal{F}^1_{\phi_i}$. Since $\left(\mathcal{F}^{1+\epsilon}_{\phi_i}\right)^*=\mathcal{F}^q_{\phi_i}$ when $0<\epsilon<\infty$ where q is the conjugate exponent of $1+\epsilon$ (see [8]), duality and complex interpolation now proves the proposition when $0<\epsilon<\infty$.

Because of (3.2), the proof of the next result is basically the same as the proof of Theorem 2.6 and therefore we skip it.

Theorem 3.6. Let $0<\epsilon<\infty$ and let T be in the norm closure of $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$. Then there exists $\epsilon\geq 0$ (both depending on T) such that

$$\|T\|_e\leq(1+\epsilon)\limsup_{|z_i|\to\infty}\sup_{w_i\in D(z_i,1+\epsilon)}\sum_i\left|\langle Tk_{z_i},k_{w_i}\rangle_{\mathcal{F}^2_{\phi_i}}\right|$$

where $\|T\|_e$ is the essential norm of T as a bounded operator on $\mathcal{F}^{1+\epsilon}_{\phi_i}$.

As was stated in the beginning of this section, the operator U_{z_i} for $z_i\in\mathbb{C}^n$ is an isometry on $\mathcal{F}^{1+\epsilon}$. Furthermore, since a direct calculation shows that

$$\left|U_{z_i}k_{w_i}(u)\right|\approx\left|k_{z_i-w_i}(u)\right|$$

the proof of Theorem 1.3 now follows immediately by combining Theorem 3.6 with [5, Proposition 1.4].

IV. Concluding Remarks

The following interested and important persisted remarks are mentioned and self competent in [13]:

(a) notice that the proof of Theorem 2.6 did not in any way use the existence of a family of "translation" operators $\left\{U^{(1+\epsilon)}_{z_i}\right\}_{z_i\in\mathbb{B}_n}$ on $A^{1+\epsilon}$ that satisfies

$$\left|\left(U^{(1+\epsilon)}_{z_i}\right)^*k^{((1+\epsilon)')}_{w_i}\right|\approx\left|k^{((1+\epsilon)')}_{(\phi_i)_{z_i}(w_i)}\right|. \tag{4.1}$$

(Hence, one can make a similar remark regarding Theorem 3.6).
(b) A trivial application of Hölder's inequality in conjunction with (a) implies that one can prove the so called "reproducing kernel thesis" for operators in the norm closure of $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ (respectively, $\mathcal{A}_{\phi_i}(\mathbb{C}^n)$) without the use of any "translation" operators.
(c) It would therefore be interesting to know if our results can be proved for the weighted Bergman spaces on the ball that were considered in [3] for example.
(d) It would be interesting to know whether one can use the valid and verified ideas here to modify the results in [6] to include spaces where condition A. 5 on the space of holomorphic functions at hand is not necessarily true (where it is precisely this condition that allows to cook up "translation operators").
(e) It would also be very interesting to know whether "translation" operators are in fact crucial for proving Proposition 2.7 and its generalized Bargmann-Fock space analog (see [5, Proposition 1.4]).
(f) It would be fascinating to know precisely how these translation operators fit into the "Berezin transform implies compactness" philosophy?.
(g) As was noted, the techniques in [10] are essentially frame theoretic, and therefore are rather different than the techniques used here.
(h) The aspects of [10] involves a localization result somewhat similar in spirit to Proposition 3.5 and which essentially involves treating a "sufficiently localized" operator T as a sort of matrix with respect to the frame $\{k_\sigma\}_{\sigma\in\mathbb{Z}^{2n}}$ for $\mathcal{F}^2$.
(k) Also, note that the techniques in [10] were extended in [5] to the generalized Bargmann-Fock space setting to obtain results for $\mathcal{F}^2_{\phi_i}$ that are similar to (some sense) the results obtained here.
(l) Because of these considerable differences in localization schemes, it would be interesting to know if one can combine the localization ideas from this paper with that of [5,10] to obtain new or sharper results on $\mathcal{F}^2_{\phi_i}$ (or new or sharper results on $\mathcal{F}^2$).

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