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Review Paper

Some Theorems on Parametric New Generalised Useful Average Code-Word Length

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ABSTRACT: *The parametric relative information generating function with utilities has been defined in this study. Additionally, we discussed about its specific and limiting cases. The relationship between the parametric new generalised useful information measure $H_\alpha(P,U)$ and the parametric new generalised useful average code-word length $L_\alpha(P,U)$, which we construct in this study, has been discussed. In addition to being novel, the measures outlined in this communication include certain well-known measures that are specific instances of our suggested measures that are already present in the literature on useful information theory.*

KEY WORDS: *Shannon's entropy, codeword length, useful information measure, Kraft inequality, Holder's inequality, Information Generating Function, Discrete Probability Distribution, Utility Distribution.*

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I. INTRODUCTION

Early in the 20th century, as the communications sector grew, a number of researchers looked into the information control of signals. Shannon's 1948 work, which was based on works by Nyquists [13, 14] and Hartley [7], clarified these early attempts into a logical mathematical theory of communication and established the field of study that is now known as information theory. The engineering challenge of information transmission across a noisy channel is the main concept of classical information theory. Shannon's noisy-channel coding theorem, which asserts that dependable communication can be performed over noisy channels as long as the communication rate falls below a specific threshold, known as the channel capacity. The crucial area of coding theory is one of the many deep and wide-ranging applications of information theory, which is a mathematical theory. A recent development in probability and statistics, information theory has a wide range of possible uses in communication systems. There isn't a single definition for the word information theory. This covers a number of useful and affordable techniques for encoding data for transmission as well as the study of uncertainty (information) measures. Information measurements are known to be crucial for real-world information processing applications. A statistical framework based on information entropy, which Shannon [16] established as a measure of information, offers a general method for reviewing information. Additionally to satisfying some desirable axiomatic constraints, the Shannon entropy can be given operational relevance in significant real-world issues, such as coding and telecommunication. The challenge of efficiently coding messages to be delivered over a noiseless channel, where our goal is to maximise the number of messages that can be sent across a channel in a given time, is typically encountered in coding theory.

This paper discusses the relationship between a new parametric generalised useful information measure $H_\alpha(P,U)$ and a new parametric generalised useful average code-word length $L_\alpha(P,U)$. For a discrete noiseless channel, the lower and upper bounds of $L_\alpha(P,U)$ in terms of $H_\alpha(P,U)$ are determined.

II. BASIC CONCEPTS

Let X be a finite source or finite discrete random variable that takes values $x_1, x_2, \dots \dots .., x_n$ with corresponding probability $P = (p_1, p_2, \dots . \dots, p_n), p_i \geq 0\ \forall i = 1,2, \dots \dots .., n$ and $\sum_{i=1}^{n} p_i = 1$ .Shannon [16] defines entropy as the following informational metric.

$$H(P) = -\sum_{i=1}^{n} p_i log p_i \tag{2.1}$$

One suitable way to measure entropy is with (2.1). Assume that $p_1, p_2, \ldots, p_n$ represents the probability that *n* codewords will be sent and that their lengths $l_1, l_2, \ldots, l_n$ meet the Kraft [9] inequality.

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \tag{2.2}$$

Shannon [16] established for uniquely decipherable codes [20, 21] that for all codes meeting (2.2), the lower bound of the mean codeword length,

$$L(P) = \sum_{i=1}^{n} p_i l_i \tag{2.3}$$

includes D, the size of the coding alphabet, and lies between *H(P) and H(P)+1*.

For this method, Belis and Guiasu [1] proposed the following quantitative-qualitative information measure. We refer to

$$H(P,U) = -\sum_{i=1}^{n} u_i p_i log p_i \tag{2.5}$$

as "useful" entropy. The measurement (2.5) can be regarded as sufficiant metric for the typical amount of "valuable" or "useful" data. the Kraft's inequality (2.2), they introduced the following quantity

$$L(P,U) = \frac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i} \tag{2.6}$$

and say it as the code's "useful" mean length. They also came up with a lower bound [22, 23, 25] for (2.6). Longo [10], however, provided some useful explanations of this length and characterised (2.6) as the average transmission cost of the letters with probability and usefulness. He also calculated constraints for the cost function (2.6) in terms of (2.5).

III. PARAMETRIC INFORMATION GENERATING FUNCTION

Along with Golomb [5], Verma [19, 24] also gave the idea of an information generating function of a probability distribution for Kullback-Leiblers's measure of relative information and Shannon's [15] measure of entropy. The information generating function was defined by Golomb [5] as

$$I(t) = -\sum_{i \in N} p_i^t, \ \ t \geq 1 \tag{3.1}$$

where t is a real or complex variable, N is a discrete sample space, and $\{p_i\}$ is a full probability distribution with $i \in N$. Additionally, it should be mentioned that

$$\frac{\partial I(t)}{\partial t} = H(P) = -\sum_{i \in N} p_i log p_i \tag{3.2}$$

where $\{p_i\}$ is the probability associated with the occurrences {Ei}, and H (P) is a Shannon's entropy [15]. The relative information of the events is not taken into consideration by the quantity (3.2), which measures average information. Guiasu and Belis [1] proposed the measure of helpful details

$$H(P,U) = -\sum_{i \in N} u_i p_i log p_i \tag{3.3}$$

where $u_i > 0$ is the utility associated with the i^{th} event that occurs with probability p_i, and $\{u_i\}$is the utility distribution.

Hooda and Bhakar [8] described the following "useful information measures" using mean values:

$$H(P,U) = -\sum_{i \in N} \frac{u_i p_i log p_i}{u_i p_i} \tag{3.4}$$

and

$$H_\alpha(P,U) = \frac{1}{1-\alpha} \log \sum_{i \in N} \frac{u_i p_i^\alpha}{u_i p_i} \tag{3.4.1}$$

The following "useful" information measures have been characterised with new mean values:

$$H_\alpha(P,U) = \frac{1}{1-\alpha} \log[\sum_{i \in N} u_i p_i^\alpha / \sum_{i \in N} u_i p_i] \tag{3.4.2}$$

Additionally, Mahajan and Kumar [11] defined a useful information-generating function as follows:

$$\mathrm{I(P,U,t)} = -\sum_{i\in N}(u_i p_i)^t \Big/ \sum_{i\in N} u_i p_i \tag{3.5}$$

where t is a real or complex variable and $P = \{p_1, p_2, \ldots, p_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ are the probability and utility distributions, respectively. In these section, they examined the characteristics of (3.5) and determined the information-generating function for a certain probability distribution.

Assume that, based on an experiment with utility distribution $U = \{(u_1, u_2, u_3, \ldots \ldots, u_n): u_i > 0\forall i\}$, where N is a discrete sample space, $P = \{(p_1, p_2, \ldots, p_n), 0 \le p_i \le 1, \sum_{i=1}^{n} p_i = 1\}$ is a discrete probability distribution of a set of events $E = \{E_1, E_2, \ldots, E_n\}$ of a discrete infinite sample space N.

We recognise that the following gives the weighted mean of u_i and P_i:

$$\frac{\sum_{i=1}^{n} u_i p_i}{\sum_{i-1}^{n} u_i} \tag{3.6}$$

We obtain a new weighted mean of order $1-\alpha$ as follows if we replace u_i with weights $(u_i p_i)^{\beta_i}$ and p_i of order $\alpha - 1$.

$$M_{\alpha,\beta}(P,U) = \left[\sum_{i=1}^{n}(u_i p_i)^{\beta_i} p_i^{\alpha-1} \Big/ \sum_{i=1}^{n}(u_i p_i)^{\beta_i}\right]^{\frac{1}{\alpha-1}}$$

$$M_{\alpha,\beta}(P,U) = \left[\sum_{i=1}^{n} u_i^{\beta_i} p_i^{\alpha+\beta_i-1} \Big/ \sum_{i=1}^{n}(u_i p_i)^{\beta_i}\right]^{\frac{1}{\alpha-1}};\ \alpha \ge 0, \alpha \ne 1, \beta_i \ge 1 \tag{3.7}$$

The generalised useful information generating function for this is provided by

$$I_{\alpha,\beta}(P,U,t) = [M_{\alpha,\beta_i}(P,U)]^{-t}$$

From (3.7) we get,

$$I_{\alpha,\beta}(P,U,t) = \left[\sum_{i=1}^{n} u_i^{\beta_i} p_i^{\alpha+\beta_i-1} \Big/ \sum_{i=1}^{n}(u_i p_i)^{\beta_i}\right]^{\frac{-t}{\alpha-1}} \tag{3.8}$$

in which t is a complex or real variable.

When we differentiate equation (3.8) with regard to t at t=0, respectively, we obtain

$$H_\alpha^{\beta_i}(P,U) = \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n} u_i^{\beta_i} p_i^{\alpha+\beta_i-1} \Big/ \sum_{i=1}^{n}(u_i p_i)^{\beta_i}\right] \tag{3.9}$$

This is the type β_i and order α generalised useful information measure.

IV. Our Main Work

We define a two parametric new generalized useful information measure in equation (3.9)

$$H_\alpha^{\beta}(P,U) = \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n} u_i^{\beta} p_i^{\alpha+\beta-1} \Big/ \sum_{i=1}^{n}(u_i p_i)^{\beta}\right] \tag{4.1}$$

where $0 < \alpha < 1, \beta_i \ge 1$

If $\beta = 1$ then the equation (4.1) reduce to

$$H_\alpha(P,U) = \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n} u_i p_i^{\alpha} \Big/ \sum_{i=1}^{n}(u_i p_i)\right] \tag{4.2}$$

Particular Cases:

(I) When $\beta = 1, u_i = 12, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$ then the equation (4.1) reduce to

$$H_\alpha(P,U) = \frac{1}{1-\alpha}\log[\sum_{i=1}^{n} p_i^{\alpha}] \tag{4.3}$$

This is the Reyni's [15] entropy.

(II) When $\beta = 1$ *and* $\alpha \to 1$ then the equation (4.1) reduce to

$$H(P,U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} \tag{4.4}$$

This is'useful' information measure for the incomplete distribution due to Bhakar and Hooda [2].

(III) When $\beta = 1, u_i = 12, \dots, n, \sum_{i=1}^{n} p_i = 1$ *and* $\alpha \to 1$ then the equation (4.1) reduce to

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{4.5}$$

This is the Shannon's entropy [16].

(IV) When $\alpha \to 1$ then the measure (4.1) is reduce to useful information measure for the incomplete power distribution p^{β} due to Sharma, Man Mohan, and Mitter [17]. i.e,

$$H^{\beta}(P,U) = -\frac{\sum_{i=1}^{n} (u_i p_i)^{\beta} \log p_i^{\beta}}{\sum_{i=1}^{n} (u_i p_i)^{\beta}} \tag{4.6}$$

(V) When $\alpha \to 1, u_i = 1, \forall i = 1,2,..,n$ *i.e*, when utility aspect is ignored, then the equation (4.1) reduce to measure

$$H^{\beta}(P) = -\frac{\sum_{i=1}^{n} {p_i}^{\beta} \log p_i^{\beta}}{\sum_{i=1}^{n} {p_i}^{\beta}} \tag{4.7}$$

This is a measure of incomplete power probability distribution due to Mitter and Mathur [12].

V. PARAMETRIC NEW GENERALISED USEFUL AVERAGE CODE-WORD LENGTH

Assume a prefix code such that

$$p_i = D^{-l_i} \Rightarrow l_i = -\log_D p_i$$

Then we get

$$p_i^{\alpha} = D^{-\alpha l_i}$$

Substituting in (4.2)

$$H_{\alpha}(P,U) = \frac{1}{1-\alpha}\left[\log_D\left(\sum_{i=1}^{n} u_i D^{-\alpha l_i}\right) - \log_D\left(\sum_{i=1}^{n} u_i D^{-l_i}\right)\right]$$

Now, define a new function for generalized useful average codeword length

$$L_{\alpha}^{\beta} = \frac{\alpha}{1-\alpha}\left[\log_D\left(\frac{\sum_{i=1}^{n} u_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i}\right)\right]$$

$$L_{\alpha}^{\beta} = \frac{\alpha}{1-\alpha}\left[\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)\right] \tag{5.1}$$

Remarks for (5.1)

(I) When $\beta = 1$ then the equation (5.1) reduce to the equation

$$L_{\alpha}(P,U) = \frac{\alpha}{1-\alpha}\left[\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i p_i}\right)\right] \tag{5.2}$$

This is'useful' average codeword length according to Taneja, Hooda, and Tuteja [18].

(II) When $\beta = 1, u_i = 1\ \forall 1,2, \dots, n$ *and* $\sum_{i=1}^{n} p_i = 1$ then from the equation (5.1) reduce to

$$L_{\alpha}(P) = \frac{\alpha}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \tag{5.3}$$

This is exponentiated mean codeword length according to Campbell [4] entropy.

(III) When $\beta = 1\ and\ \alpha \to 1$ then from the equation (5.1) reduce to

$$L(P,U) = \frac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i} \tag{5.4}$$

This is 'useful' codeword length according to Guiasu and Picard [6].

(IV) When $\beta = 1, u_i = 1, \forall 1,2,\ldots,n, \sum_{i=1}^{n} p_i = 1\ and\ \alpha \to 1$ then from (5.1) reduce to

$$L(P) = \sum_{i=1}^{n} p_i l_i \tag{5.5}$$

This is optimal codeword length defined by Shannon [16].

Now we derive the lower and upper bound of (5.1) in terms of (4.2) under the condition

$$\frac{\sum_{i=1}^{n} u_i D^{-l_i}}{\sum_{i=1}^{n} u_i p_i} \le 1 \tag{5.6}$$

It is simple to see that when $\beta = 1, u_i = 1, \forall i = 1,2,\ldots,n.$ that is, when the utility aspect disappears and $\sum_{i=1}^{n} p_i = 1$, then the inequality (5.6) reduces to Kraft's [9] inequality (2.2). A code that satisfies (5.6) would be referred to as a "useful" personal probability code.

Theorem 5.1 If the inequality (5.6) is satisfied by $\{u_i\}_{i=1}^{n}, \{p_i\}_{i=1}^{n}\ and\ \{l_i\}_{i=1}^{n}$ then the parametric generalised "useful" code-word lengths (5.1) also satisfy the inequality

$$L_\alpha(P,U) \ge H_\alpha(P,U),\ \ 0 < \alpha < 1. \tag{5.7}$$

Where $H_\alpha(P,U)$and $L_\alpha(P,U)$ are defined in (4.2) and (5.2) respectively. Furthermore, equality holds good if

$$l_i = -\log_D \left[\frac{p_i^\alpha}{\frac{\sum_{i=1}^{n} u_i p_i^\alpha}{\sum_{i=1}^{n} u_i p_i}} \right] \tag{5.8}$$

Proof: By Holder's Inequality, we have

$$\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \le \sum_{i=1}^{n} x_i y_i \tag{5.9}$$

$$\forall x_i, y_i > 0, i = 1,2,3,\ldots,n\ and\ \frac{1}{p} + \frac{1}{q} = 1, p < 1(\ne 0), q < 0\ or\ q < 1(\ne 0), p < 0.$$

We see that if a positive constant c exists such that

$$x_i^p = c y_i^q \tag{5.10}$$

then the equality holds.

Making the substitution

$$x_i = \frac{u_i^{\frac{\alpha}{\alpha-1}} p_i^{\frac{\alpha}{\alpha-1}}}{\left(\sum_{i=1}^{n} u_i p_i\right)^{\frac{\alpha}{\alpha-1}}} D^{-l_i},\ \ p = \frac{\alpha-1}{\alpha}$$

$$y_i = \frac{u_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha}{1-\alpha}}}{\left(\sum_{i=1}^{n} u_i p_i\right)^{\frac{1}{1-\alpha}}} \qquad \&\ \ q = 1-\alpha$$

Putting the value of $x_i and\ y_i$ in equation (5.6) and after suitable simplification,we get

$$\left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{\sum_{i=1}^{n} u_i p_i^\alpha}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{1-\alpha}} \le \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i}}{\sum_{i=1}^{n} u_i p_i} \tag{5.11}$$

Now, using the inequality (5.6) we get

$$\left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{\sum_{i=1}^{n} u_i p_i^\alpha}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{1-\alpha}} \le 1 \tag{5.12}$$

Equation (5.12), can be written as

$$\left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{\alpha-1}} \leq \left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{\alpha-1}} \tag{5.13}$$

Taking logarithms to both sides with base D then we get,

$$\frac{\alpha}{\alpha-1}\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^{n} u_i p_i}\right) \leq \frac{1}{\alpha-1}\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right) \tag{5.14}$$

Equivalently we can write equation (5.14), as

$$\frac{\alpha}{1-\alpha}\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^{n} u_i p_i}\right) \geq \frac{1}{1-\alpha}\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right) \tag{5.15}$$

This implies

$$L_\alpha(P,U) \geq H_\alpha(P,U)$$

Hece the result for $0 < \alpha < 1, 0 < \beta \leq 1$

Now,
We will show that the equality in (5.7) holds iff

$l_i = -\log_D\left(\frac{p_i^{\alpha}}{\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}}\right), 0 < \alpha < 1$

Equivalently we can write

$$D^{-l_i} = \left(\frac{p_i^{\alpha}}{\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}}\right)$$

Or we can write

$$D^{-l_i} = p_i^{\alpha}\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right)^{-1} \tag{5.16}$$

Taking both side to the power $\frac{\alpha-1}{\alpha}$, to equation (5.16), and after simplification we get,

$$D^{-l_i(\frac{\alpha-1}{\alpha})} = p_i^{(\alpha-1)}\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1-\alpha}{\alpha}} \tag{5.17}$$

Multiply equation (5.17) both sides by $\frac{\sum_{i=1}^{n} u_i p_i}{\sum_{i=1}^{n} u_i p_i}$ and after simplification, we can write

$$\frac{\sum_{i=1}^{n} u_i p_i}{\sum_{i=1}^{n} u_i p_i} D^{-l_i(\frac{\alpha-1}{\alpha})} = \left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{\alpha}} \tag{5.18}$$

Taking logarithms both sides with base D to equation (5.18) and multiply both sides by $\frac{\alpha}{1-\alpha}$, we get,

$$\frac{\alpha}{1-\alpha}\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i}{\sum_{i=1}^{n} u_i p_i} D^{-l_i(\frac{\alpha-1}{\alpha})}\right) = \frac{1}{1-\alpha}\log_D\left(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right)$$

This implies

$$L_\alpha(P,U) = H_\alpha(P,U)$$

Hence the result.

Theorem 5.2 $\mathrm{L}_\alpha(P,U)$ satisfy the inequality

$L_\alpha(P,U) < H_\alpha(P,U) + \beta \; where \; 0 < \alpha < 1, 0 < \beta \leq 1$ for every code lengths $l_1, l_2, \dots \dots, l_n$ which is satisfy the condition (5.6).

Proof: From the theorem 5.1 we have,

$L_\alpha(P,U)=H_\alpha(P,U)$, holds iff $l_i=-\log_D\left(\frac{p_i^\alpha}{\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}}\right)$,$0<\alpha<1$

Or we can write

$$-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)$$

In order to satisfy the inequality, we now select the code-word lengths.

$$-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)\le l_i<-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)+1 \qquad (5.19)$$

Consider the interval

$$\delta_i=\left[-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right),-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)+1\right] \qquad (5.20)$$

There exit one positive integer l_i such that,

$$0<-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)\le l_i<-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)+1 \qquad (5.21)$$

Now, we will show that the sequence $l_1,l_2,\ldots\ldots,l_n$ satisfies generalization of Kraft inequality. From the left inequality of (5.21), we have,

$$-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)\le l_i$$

We can write,

$$D^{-l_i}\le\left(\frac{p_i^\alpha}{\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}}\right)$$

Multiply by u_i both side then summing over i=1,2,...,n both side to the resulted expression, we get,

$$\frac{\sum_{i=1}^n u_i D^{-l_i}}{\sum_{i=1}^n u_i p_i}\le 1$$

Now, the last inequality of (5.21), gives

$$l_i<-\log_D p_i^\alpha+\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)+1$$

$$D^{l_i}\le p_i^{-\alpha}\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)D \qquad (5.22)$$

$$D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}<p_i^{(\alpha-1)}\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)^{\left(\frac{1-\alpha}{\alpha}\right)}D^{\left(\frac{1-\alpha}{\alpha}\right)}$$

$$D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}<p_i^{(\alpha-1)}\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)^{\left(\frac{1-\alpha}{\alpha}\right)}D^{\left(\frac{1-\alpha}{\alpha}\right)}$$

Multiplying both side by $\frac{\sum_{i=1}^n u_i p_i}{\sum_{i=1}^n u_i p_i}$, we get,

$$\frac{\sum_{i=1}^n u_i p_i}{\sum_{i=1}^n u_i p_i}D^{-l_i^{\left(\frac{\alpha-1}{\alpha}\right)}}<\frac{\sum_{i=1}^n u_i p_i}{\sum_{i=1}^n u_i p_i}\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)^{\left(\frac{1-\alpha}{\alpha}\right)}D^{\left(\frac{1-\alpha}{\alpha}\right)}$$

$$\frac{\sum_{i=1}^n u_i p_i}{\sum_{i=1}^n u_i p_i}D^{-l_i^{\left(\frac{\alpha-1}{\alpha}\right)}}<\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right)^{\left(\frac{1}{\alpha}\right)}D^{\left(\frac{1-\alpha}{\alpha}\right)}$$

Taking log on both side

$$\log_D\left(\frac{\sum_{i=1}^n u_i p_i}{\sum_{i=1}^n u_i p_i} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right) < \frac{1}{\alpha}\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right) + \frac{1-\alpha}{\alpha}$$

Multiplying $\frac{\alpha}{1-\alpha}$ on both side

$$\frac{\alpha}{1-\alpha}\log_D\left(\frac{\sum_{i=1}^n u_i p_i}{\sum_{i=1}^n u_i p_i} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right) < \frac{1}{1-\alpha}\log_D\left(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\right) + \beta,\ where\ \beta = \frac{1-\alpha}{\alpha}$$

This implies

$$L_\alpha(P,U) < H_\alpha(P,U) + \beta\ where\ 0 < \alpha < 1, 0 < \beta \leq 1$$

Thus from above two theorems, we have shown that

$$H_\alpha(P,U) \leq L_\alpha(P,U) < H_\alpha(P,U) + \beta\ where\ 0 < \alpha < 1, 0 < \beta \leq 1$$

VI. CONCLUSION

In this study we describe a new parameter generalised "useful" entropy measure, or $H_\alpha(P,U)$. Additionally, we establish two new parametric generalised "useful" code-word mean lengths, $L_\alpha(P,U)$ and $H_\alpha(P,U)$ and then describe $L_\alpha(P,U)$ in terms of $H_\alpha(P,U)$ and showed that

$$H_\alpha(P,U) \leq L_\alpha(P,U) < H_\alpha(P,U) + \beta\ where\ 0 < \alpha < 1, 0 < \beta \leq 1$$

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