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Review Paper

A Study of Frieze Groups and their Symmetries

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ABSTRACT: *Frieze Groups is a plane symmetry groups whose translations are isomorphic to Z. Derived from the Greek word which means decorative patter repeated all the way around the building usually seen in Greek architecture. In this paper we shall aim to define what patterns can we form. We shall also see why only seven Frieze groups are possible and not more than that. We shall the different symmetries involved to construct a pattern.*

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I. INTRODUCTION

Frieze is a section of structure located between the roof or top of the structure and the support beams. Originally used in Greek architecture, it is a decorative pattern repeated all the way around the building with each set of line at a particular distance from the previous one. The term has since spread out to many other decorative purposes other than architecture such as fine china, pottery, henna designs etc. Similarly is the case in Group Theory. In Group Theory, frieze Groups is a plane symmetry groups whose translations are isomorphic to Z. Similarly Crystallographic Groups (often called as wallpaper symmetry) are those groups in which the translations are isomorphic to $Z \oplus Z$. However both groups are constructed with certain motions which naturally fulfills other motions

II. MOTIONS ON PLANE SYMMETRIES

The five motions that we need to define are translations, horizontal reflection, vertical reflection, rotations and glide reflections. Frieze patterns depends on the composition of these motions

2.1 Translations

A translation is the linear shift of some figure along the plane. There is only one type of translation in frieze groups whereas Crystallographic groups have two translations. Translations in frieze groups are in some linear path (t) whereas in crystallographic group we have horizontal translations (t) and vertical translations (t').

2.2 Horizontal Reflection

Reflection along a horizontal line such that it preserves distance and symmetry is called horizontal reflection symmetry denoted by h.

2.3 Vertical Reflection

Reflection along a vertical line such that it preserves distance and symmetry is called vertical reflection symmetry denoted by v.

2.4 Rotations

Rotation symmetry is the non trivial rotations around a particular point. It is denoted by r where $0^\circ < r < 360^\circ$. In frieze groups we have only one case of rotation which is of 180° whereas crystallographic groups have many cases thus more diversity. Thus crystallographic groups have more patterns than frieze groups

2.5 Glide Reflection

Glide reflection is the combination of a translation and a horizontal reflection. If we have a reflection on the same axis as a glide reflection, the it is called as the trivial one.

III. CONSTRUCTION OF FRIEZE GROUPS

We will show the different seven patterns of frieze groups using the letter R as it doesn't have any symmetries of its own, thus we can use multiple R's to create any symmetry that we like

3.1 Pattern 1

The first pattern contains only translation(t) and no other symmetry. The group is defined as $G =< t >$ and the motion is shown as follows:-

t^{-1} e t t^2

R R R R

3.2 Pattern 2

The second pattern consists of translations(t) and glide reflections(g) only. The group is defined as $G =< t, g|g^2 = t; g = gt >$ and is generated by two motions together as follows

g^{-2} e g^2 g^4

R R R

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g^{-1} g g^3

3.3 Pattern 3

The third pattern consists of vertical reflection(v) and translations (t). The group is defined as $G =< t, v|v^2 = 1; tv = vt^{-1} >$ and the motion is as follows:-

$t^{-1}v$ t^{-1} v | e tv t t^2v t^2

Я R Я | R Я R Я R

3.4 Pattern 4

The fourth pattern consists of rotations(r) and translations(t). The group is defined as $G =< t, r|r^2 = 1|rt = t^{-1}r >$ and the motion is shown as follows

t^{-2} t^{-1} e t

R R R R

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$t^{-1}r$ r tr t^2r

3.5 Pattern 5

The fifth pattern is generated by vertical reflections(v), rotations(r) , glide reflections $(g)$and translations(t). Here the vertical reflections are automatically formed in this group.s The group is defined as $G =< t, g, r|g^2 = t; r^2 = 1; tg = gt; rt = t^{-1}r >$ and the motion is shown as follows

$g^{-1}r$ e gr g^2

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r g

3.6 Pattern 6

The sixth pattern consists of horizontal reflections (h) and translations (t). The group is defined as $G =< t, h|h^2 = 1; ht = th >$ and the motion is as follows:-

t^{-1} e t t^2

R R R R

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$t^{-1}h$ h th t^2h

3.7 Pattern 7

The seventh pattern consists of horizontal reflections (h), vertical reflections (v) and translations (t). The group is defined as $G =< t, h, v|h^2 = 1; tv = vt^{-1}; ht = th; hv = vh >$

$t^{-1}v$ t^{-1} v e tv t

Я R Я R Я R

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$t^{-1}vh$ $t^{-1}h$ vh h tvh h

IV. WHY ONLY SEVEN FRIEZE PATTERNS?

By definition of a Frieze groups, every motion g, v, r, h is composed with translations and represented by patterns 2, 3, 4 and 6. The remaining patter 5 and 7 are constructed by $< t, g, r >$ and $< t, h, v >$. Now the only remaining possibilities of the different combinations of g, v, r, h, t are $< t, g, v >, < t, v, r >, < t, h, r >$ and $< t, h, v, r >$. These remaining possibilities are isomorphic to the above seven groups.

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Я | R Я R

ꓤ ᴚ $\downarrow g$ ꓤ ᴚ

Я | R Я R

ꓤ ᴚ ꓤ ᴚ

Consider an element of $< t, g, v >: x = gv$. From above it can be clearly seen that $< t, g, v > \cong < t, g, r >$ as $r = gv$. Similarly $< t, v, r > \cong < t, h, v >$ as $g = rv$. In the group $< t, h, r >$, we will consider the element $x = t^{-1}hr$ and we can see that $< t, h, r > \cong < t, h, v >$ as $r = htv$. Similarly $< t, h, v, r > \cong < t, h, v >$, so in this way every possible frieze pattern is isomorphic to one of the seven pattern mentioned above.

V. USE OF FRIEZE GROUPS IN ARCHITECTURE

Frieze groups are not only significant in mathematics but in architecture also. We can see them in different patterns in various historical monuments specially in Greek and ancient Indian structures. We can also develop many new designs using the symmetry and patterns of frieze groups as follows in figure 1. By using the symmetry, we can generate many designs and use them to decorate walls, pottery etc.

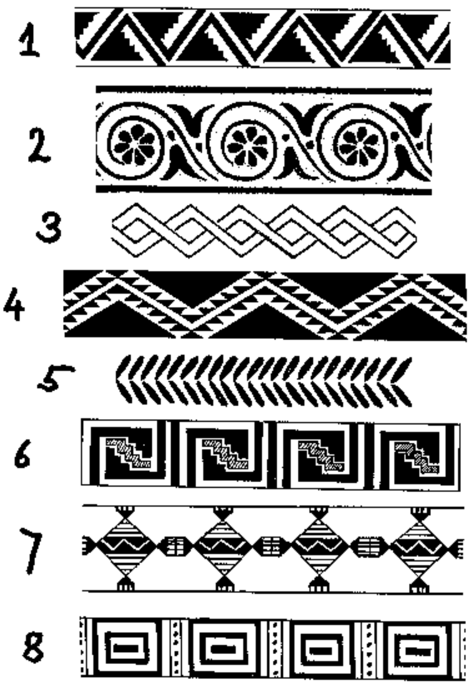

Figure 1

VI. CONCLUSION

After looking at different frieze groups, we can clearly see some unique applications of their patterns. Once we understand their construction it's not hard to notice how vastly they are used in real life. A similar type of group which can be constructed just like frieze are crystallographic groups which are isomorphic to $Z \oplus Z$, however they are not as simple as frieze groups and require difficult construction. This is the beauty of mathematics and how mathematics has been used even for decorative purposes since ancient times. We also saw that after a while, the symmetries of frieze groups starts repeating and thus only 7 frieze patterns exits.

REFERENCES

[1]. J.A. Gallian, Abstract Algebra, 2023: p. 446-467
[2]. D. Schattschneider, Vision of Symmetry, New York: Harry Abrams, 2004
[3]. D.S. Dummit and R.M. Foote, Abstract Algebra. John Wiley & Sons, Inc., 2023