Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 5 (May 2025) pp: 39-45
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Normal operator analogue in Banach space

Rakhshanda Dzhabarzadeh
Institute of Mathematics and Mechanics, Azerbaijan

***R**eceived 06 May., 2025; **R**evised 15 May., 2025; **A**ccepted 17 May., 2025 © The author(s) 2025.*
P**ublished with open access at **www.questjournas.org

In this article we introduce the new class of operators in Banach space with the main properties of normal operators in Hilbert space.
Earlier, In [1] it was introduced and researched the class of self-adjoint operators in Banach space.

Previously, we give some definitions and notions.

If for some $y \in B$ at all $x \in D_A \subset B$ and any real numbers t *the following equality*

$$\|Ax+ty\|^2+\|x-tz\|^2=\|Ax-ty\|^2+\|x+tz\|^2 \tag{1}$$

is true , then we have the operation when the element $y \in B$ is matched the element $z \in B$ [1].

Theorem 1. If operator A is defined on everywhere dense set i*n* B , then the matching, defined above, is definitely defined.
Proof.[1] Let be the statement of Theorem 1 is incorrect, then for some element y there are such two elements z_1 and z_2. that the following ratios

$$\|Ax+ty\|^2+\|x-tz_1\|^2=\|Ax-ty\|^2+\|x+tz_1\|^2 \tag{2}$$

and

$$\|Ax+ty\|^2+\|x-tz_2\|^2=\|Ax-ty\|^2+\|x+tz_2\|^2 \tag{3}$$

are true at all $x \in D_A \subset B$ and real numbers t .

Subtracting from (2) the ratio (3), we have

$$\|x+tz_1\|^2-\|x-tz_2\|^2=\|x+tz_1\|^2-\|x+tz_2\|^2 \tag{4}$$

Because (4) is correct for all x , then, assuming $x=tz_1$ we have,

$$-4\|z_1\|^2=\|z_1-z_2\|^2 \ -\|z_1+z_2\| \tag{5}$$

and, assuming $x=tz_2$, we have

$$-4\|z_2\|^2 = \|z_1 - z_2\|^2 - \|z_1 + z_2\| \qquad (6)$$

Compare (5) and (6), then we get

$$\|z_1\| = \|z_2\| \qquad (7)$$

Further, from (5), taking into account (7),

$$\|z_1 - z_2\|^2 = -4\|z_1\|^2 + \|z_1 + z_2\|^2 \le -4\|z_1\|^2 + \left(\|z_1\| + \|z_2\|\right)^2$$ and

$$\|z_1 - z_2\| = 0$$

or

$$z_1 = z_2$$

Definition 1.

This operation is called an adjoint operator to the operator A and it is denoted A^* .

Definition 2.

If $A = A^*$,and $\| AAx + ty\|^2 + \| x - tAy\| = \| Ax - ty\|^2 + \| x + tAy\|^2$ (8)

then operator A ts called a self-adjoint operator in Banach space.

The expression $\| Ax + y\|^2 + \| x - Ay\|^2 = \| Ax - y\|^2 + \| x + Ay\|^2$ (9)

is an equvalent to the definition (8).

Definition 3. If A^* is am adjoint operator to the operator A , accord to the Definition 1 , then operator A is called *the normal operator in Banach space if* $AA^* = A^*A$ and

$$\|Ax + ty\|^2 + \|x - tA^*y\|^2 = \|Ax - ty\|^2 + \|x + +tA^*y\|^2 \quad (10)$$

The definition of adjoint operator , given by the formula (10), is an equivalent of the definition of adjoint operator, given by the formula

$$\|Ax + y\|^2 + \|x - A^*y\|^2 = \|Ax - y\|^2 + \|x + A^*y\|^2 \quad (11)$$

Later, when proving subsequent theorems, we use any of the definitions for the sake of convenience.

Theorem 2 . Norms of operators A and its adjoint operator A^* ,defined with help of formula (10) in Banach space, coincide.

Proof. The formulas

$$\| AA^*x + ty\|^2 + \|A\ x - tAy\|^2 = \| AA^*x - ty\|^2 + \|A\ x + tAy\|^2 \qquad (12)$$

$$\| AA^*x + ty\|^2 + \| A^*x - tA^*y\|^2 = \| AA^*x - ty\|^2 + \| A^*x + tA^*y\|^2 \quad (13)$$

are true. Subtract equality (13) from (12) , then we get

$$[(1+t)^2 - (1-t)^2]\| A^*x\|^2 = [(1+t)^2 - (1-t)^3]\|Ax\|^2$$

or $\left\|A^{*}x\right\|=\left\|Ax\right\|$

Last means the norms of operator A and its adjoint operator A^{*} coincide.
Theorem 2 is proven.

Theorem 3.The whole $n-$ degree of a normal operator in Banach space is a normal operator.

Proof. We carry out the proof by the method of mathematical induction. Let be first $n=2$ .So operator A is a normal, then

$$\left\|A(Ax)+ty\right\|^{2}+\left\|Ax-tA^{*}y\right\|^{2}=\left\|A(Ax)-ty\right\|^{2}+\left\|Ax+tA^{*}y\right\|^{2} \qquad (14)$$

and also $\left\|Ax-tA^{*}y\right\|^{2}+\left\|x+t(A^{*})^{2}y\right\|^{2}=\left\|Ax+tA^{*}y\right\|^{2}+\left\|x-t(A^{*})^{2}y\right\|^{2}$ (15)

Substituted into (14) expression for $\left\|AAx-tA^{*}y\right\|^{2}$ from (15), we get

$$\left\|A(Ax)+ty\right\|^{2}+\left\|Ax+tA^{*}y\right\|^{2}-\left\|x+tA^{2}y\right\|^{2}+\left\|x-tA^{*}y\right\|^{2}=\left\|A^{2}x-ty\right\|^{2}+\left\|Ax+tA^{*}y\right\|^{2}$$
$$\left\|A^{2}x+ty\right\|^{2}+\left\|x-tA^{*}y\right\|^{2}=\left\|A^{2}x-ty\right\|^{2}+\left\|x+tA^{*}y\right\|^{2} \qquad (16)$$

Further, we suppose that our statement is fair for $k=m$, then we proof the validity of this statement for $k=m+1$ by the manner, described above.

Let the bounded inverse operator A^{-1} to the operator A exists . Put in formula

$$\left\|Ax-ty\right\|^{2}+\left\|x+tA^{*}y\right\|^{2}=\left\|Ax+ty\right\|^{2}+\left\|x-tA^{*}y\right\|^{2}$$

$Ax=z,$ and $A^{*}y=r$,then

$$\left\|z-t(A^{*})^{-1}r\right\|^{2}+\left\|A^{-1}z=tr\right\|^{2}=\left\|A^{-1}z+tr\right\|^{2}+\left\|z-t(A^{*})^{-1}r\right\|^{2}$$

Thus any whole degree of normal operator is the normal operator.
Theorem 3 is proven.

Theorem 4. The product of two normal commuting operators A and B in Banach space is the normal operator,or

$$\left\|ABx+ty\right\|^{2}+\left\|x-tB^{*}A^{*}y\right\|=\left\|AB\ x-ty\right\|^{2}+\left\|\ x+tB^{*}A^{*}y\right\|^{2}$$

Proof. From the conditions of the Theorem 4

we have $\left\|Ax-ty\right\|^{2}+\left\|x+tA^{*}y\right\|^{2}=\left\|Ax+ty\right\|^{2}+\left\|x-tA^{*}y\right\|^{2}$

$\left\|Bx-ty\right\|^{2}+\left\|x+tB^{*}y\right\|^{2}=\left\|Bx+ty\right\|^{2}+\left\|x-tB^{*}y\right\|^{2}$ and $AB=BA$

Besides, sequentially using the normality andalso commutation of these operators A and B, we have

$$\left\|ABx+ty\right\|^{2}+\left\|Bx-tA^{*}y\right\|=\left\|AB\ x-ty\right\|^{2}+\left\|\ B\,x+tA^{*}y\right\|^{2} \qquad (17)$$

$$\left\|Bx-tA^{*}y\right\|^{2}\left\|x+tB^{*}A^{*}y\right\|=\left\|B\ x+tA^{*}y\right\|^{2}+\left\|\ x-tB^{*}A^{*}y\right\|^{2} \qquad (18)$$

Taking into account (18) in (17) and commutation of operators A and B, we get, that $(AB)^*$ is an adjoint operator to the operator AB .Besides, this product AB is a normal operator.

$$\|ABx+ty\|^2+\|x-tB^*A^*y\|=\|AB\ x-ty\|^2+\|\ x+tB^*A^*y\|^2.$$

The operator

$(AB)^*$ is also the adjoint operator to the operator AB

$$\|ABx+ty\|^2+\|x-tB^*A^*y\|=\|AB\ x-ty\|^{ba2}+\|\ x+tB^*A^*y\|^2$$

Further, $(AB)^*AB=AB(AB)^*$

Because these operators A and B are commuting, their product AB is a normal operator. Theorem 4 is proven.

The statement of Theorem 3 follows from the results of the Theorem4.

Theorem 5. The eigenvectors corresponding to eigenvalues with different modules of the normal in Banach space operator A are orthogonal in the described below sense. (Elements x, y of Banach space B are orthogonal if for all $t(-\infty\le t\le\infty)$. $\|x+ty\|=\|x-ty\|$

Proof. The proof of this Theorem 5 is similar to the proof of the similar statement for the self-adjoint analoque operator in Banach space.[1].

L*et be* λ and μ two eigenvalues of normal in Banach space operator A with different modules ,

$$Ax=\lambda x$$

$$Ay=\mu y$$ *and* for all real *numbers* t :

$$\|Ax+ty\|^2+\|x-tA^*y\|^2=\|Ax-ty\|^2+\|x-tA^*y\|^2$$

$$\|\lambda x+ty\|^2+\|x-t\bar{\mu}y\|^2=\|\lambda x-ty\|^2+\|x+t\bar{\mu}y\|^2$$

or, if insteadof y we put λy , then

$$\|\lambda x+\lambda ty\|^2+\|x-t\lambda\mu y\|^2=\|\lambda x-\lambda ty\|^2+\|x+t\lambda y\|^2$$

$$|\lambda|^2\left\{\|x+ty\|^2-\|x-ty\|^2\right\}=\|x+t\lambda\mu y\|^2-\|x-t\lambda\mu y\|^2$$

Similarly, $|\mu\ |^2\left\{\|x+ty\|^2-\|x-ty\|^2\right\}=\|ty+\lambda\mu x\|^2-\|ty-\lambda\mu x\|^2$

Then, taking into account last equality, we get

$$\frac{|\lambda|^2}{|\mu|^2}=\frac{\|x+t\lambda\mu y\|^2-\|x-t\lambda\mu y\|^2}{\|ty+\lambda\mu x\|^2-\|ty-\lambda\mu x\|^2}=\frac{\|x+t\lambda^2\mu^2 y\|^2-\|x-\lambda^2\mu^2 ty\|^2}{\|t\lambda\mu y+\lambda\mu x\|^2-\|t\lambda\mu y-\lambda\mu x\|^2}=$$

$$=\frac{\|x+t\lambda^2\mu^2 y\|^2-\|x-\lambda^2\mu^2 ty\|^2}{|\lambda^2||\mu^2|\left(\|ty+x\|^2-\|ty-x\|^2\right)}=\frac{\|x+t\lambda^2\mu^2 y\|^2-\|x-t\lambda^2\mu^2 y\|^2}{|\lambda^2|\left\{\|ty+\lambda\mu x\|^2-\|ty-\lambda\mu x\|^2\right\}}$$

for λ , μ and n the following evalution for $\frac{|\lambda|^2}{|\mu|^2}$

$$\frac{|\lambda|^2}{|\mu|^2} = \frac{\|x + t\lambda^{n+1}\mu^{n+1}y\|^2 - \|x - t\lambda^{n+1}\mu^{n+1}y\|^2}{|\lambda^{2n}|\left(\|ty + \lambda\mu x\|^2 - \|ty - \lambda\mu x\|^2\right)} =$$

$$= \frac{\left(\|x + t\lambda^{n+1}\mu^{n+1}y\| - \|x - t\lambda^{n+1}\mu^{n+1}y\|\right)\left(\|x + t\lambda^{n+1}\mu^{n+1}y\| + \|\lambda - t\lambda^{n+1}\mu^{n+1}y\|\right)}{|\lambda^{2n}|\left(\|ty + \lambda\mu x\|^2 - \|ty - \lambda\mu x\|^2\right)}$$

are fair,

$$\frac{|\lambda|^2}{|\mu|^2} \le \frac{2|t||\lambda|^{n+1}|\mu|^{n+1}\left(2\|x\| + 2|t||\lambda|^{n+1}|\mu|^{n+1}\|y\|\right)}{|\lambda^{2n}|\left|\|ty + \lambda\mu x\|^2 - \|ty - \lambda\mu x\|^2\right|} \qquad (19)$$

$$\frac{|\lambda|^2}{|\mu|^2} \le \frac{2\|x\|\left(2\|x\| + 2|t||\lambda|^{n+1}|\mu|^{n+1}\|y\|\right)}{|\lambda^{2n}|\left(\|ty + \lambda\mu x\|^2 - \|ty - \lambda\mu x\|^2\right)} \qquad (20)$$

Expression $\frac{|\lambda|^2}{|\mu|^2}$ may be finite, then and only then when at all real numbers t

$\|x + ty\| = \|x - ty\|$

Really , for some real t denominator in $\frac{|\lambda|^2}{|\mu|^2}$ is not equal to zero, then at $|\mu| < |\lambda| < 1$

$|\lambda| < 1$ in (19) ,and $|\mu| < |\lambda| \,(|\lambda| > 1)$ in (20) expression $\frac{|\lambda|^2}{|\mu|^2}$ is less any given little number. Last

statement is not possible because λ , μ are finite numbers. Consequently, the right parts of unequalities (19) abd (20) cannot be e in any proximity to zero . Last means that the statement of the Theorem 5 is proven.

Theorem 6 . If $\|e\| = 1$ and A is a normal operator in B, then, $\|A\ (e)\|^2 \le \|A^2 e\|$

Proof. In [1] the case of self-adjoint operator acting in Banach space is considered.

Let be A is a normal operator in Banach space. We have the following equalities

$$\|AA^*e + e\|^2 + \|Ae - Aee\|^2 = \|AA^*e - e\|^2 + \|Ae + Aee\|^2$$

or $\|AA^*e + e\|^2 = \|AA^*e - e\|^2 + \|Ae + Aee\|^2$

Further

$$4\|Ae\|^2 = (\|AA^*e + e\| + \|AA^*e - e\|)(\|(AA^*e + e\| - \|(AA^*e - e\|)\|AA^*e\| \le 4\|A^*Ae\|\|e\| = 4\|A^2e\|$$

So the norms of normal operator A and its adjoint operator A^* on any element of Banach space coincide,then, that isthus, $\|A^*(Ae)\| = \|A(Ae)\| = \|A^2e\|$

Theorem 6 is proven.

Definition 4. Maximum unit vector f of bounded operator A in Banach space is called such unit vector f for which the maximum value of operator A on unit vectors is reached $\|A\ f\| = |M|$,where $\|A\ \| = |M|$.

Theorem 7. In Banach space each normal completely continuous operator A has a maximum vector.
Proof. The proof is similar to the proof of similar statement in Hilbert space.

We choose a convergent sequence, $y_n = Ax_n$, where $\|x_n\| = 1$, $n = 1, 2, \ldots$ *such that* $\lim \|Ax_n\| = \|A\|$.

We substitute that vector $z = \frac{1}{M} y$ is the desired maximum vector. Because, operator A is bounded

$$Az = \lim_{n\to\infty} A\left(\frac{y_n}{M}\right) = \lim_{n\to\infty} A\left(\frac{Ax_n}{M}\right) \tag{21}$$

Vectors $\frac{Ax_n}{M}$ belong to the unit sphere , consequently, their legth $A\left(\frac{Ax_n}{M}\right)$ is not more than $|M|$. Taking into account the statement of the Theorem 6 we get, that

$$|M| > | \ \left\|\left(\frac{Ax_n}{M}\right)\right\| = \frac{1}{|M|}\|A^2 x_n\| \geq \frac{1}{|M|}\|Ax_n\|^2 \to |M| \tag{22}$$

Compare (22) and (21), we have, that Z is maximum vector of operator A .
Theorem 7 is proven.

Theorem 8. If the operator A^2 has eigenvector with the eigenvalue M^2 , then operator A has the eigenvector with the eigenvalue $+M$ and $-M$.

Proof. The $A^2\, e = M^2\, e$ may be written in the form
$(A + ME)(A - ME)e = 0$ (E – unit operator).

If $(A - ME)e = z$, then Z is eigenvector of operator A with eigenvalue $-M$. If $A^2\, e = M^2\, e$ is written in the form $(A - ME)(A + ME)e = 0$,then operator A in Banach space has the eigenvalue $+M$ .Thus at conditions of the Theorem 8 operator A has the eigenvalues $+M$ and $-M$.

Remarks. Theorem 6 and Theorem 8 are the generalizations of similar results for self-adjoint operators in Hilbert space [2](pp.195) .

The example of normal bounded analoque operator in direct sum l_1^n of n copies of Banach space l_1, defined by the matrice

$$A=\begin{pmatrix} 0 & 0 & 0 & ... & 1 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ . & . & . & ... & 0 \\ 0 & 0 & 0 & ..1 & 0 \end{pmatrix}.$$

$$x=(x_1,...,x_n),\ y=(y_1,...,y_n)$$

The adjoint operator to the operator A has the form

$$A^*=\begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ 0 & 0 & 0 & 1.. & 0 \\ . & . & . & ... & . \\ 1 & 0 & 0 & ... & 0 \end{pmatrix}.$$

It is not difficult to state that $AA^* = A^*A$ amd (11) is fulfilled.

Really, $Ax+ty=\left(x_n+ty_1, x_1+ty_2,..., x_{n-1}+ty_n\right)$

$x+tA^*y=(x_1+ty_2, x_2+ty_3,..., x_n+ty_1)$, and, consequently,

$\left\|Ax+ty\right\|^2=\left\|x+tA^*y\right\|^2$. Similar $\left\|Ax-ty\right\|^2=\left\|x-tA^*y\right\|^2$.

In conclusion I would like to thank my collegue N.Q. Vagabov for his advice to continue my research from [1], and tointroduce the class of normal operators in Banach space. I also thanked him for interesting discussion of results.

References

[1]. Dzhabarzadeh R.M. Some specreal properties of analoques self-adjoint oprtators in Banach space . Spectral theory of operators,Baku, Elm, 1977, pp.100-109 (in Russian)

[2]. Q.E. Shilov. Mathematical analysis. State Publishing of physical and mathematical literature. B-71 (in Russian)

[3]. N.I. Akhiezer and I.M.Glazman The theory of linear operators in Hilbert space. Science Publishng. Chief reduction of physical and mathematical literature. Moscow 1966 (in Russian)