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Review Paper

Weighted Composition Operators of Topological Structure on Weighted Banach Spaces

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Abstract
We follow the validity that [24] dealt with considering the topological structure problem concerning the set of composition operators restricted and differ by a compact operator is known as path connected but alternatively not usually a component. So for weighted both of composition operators on Banach spaces endowed with sup-norm the set of weighted composition operators is really path connected but in the second is not a component.

Keywords: *Topological structure, composition operator, weighted composition operator, weighted Banach space with sup-norm.*

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I. Introduction

For $H(\mathbb{D})$ be the space of all holomorphic functions on the unit disc $\mathbb{D}$ and $\mathcal{S}(\mathbb{D})$ the set of all holomorphic self-maps of $\mathbb{D}$. For two series of functions $\sum_m \psi^m \in H(\mathbb{D})$ and $\sum_m \varphi^m \in \mathcal{S}(\mathbb{D})$, a weighted composition operator $W_{\sum_m \psi^m, \sum_m \varphi^m}$ is defined by $\sum_m W_{\sum_m \psi^m, \sum_m \varphi^m}(\sum_m f_m) = \sum_m \psi^m \cdot (\sum_m f_m \circ \sum_m \varphi^m), \sum_m f_m \in H(\mathbb{D})$. When the series of functions $\sum_m \psi^m$ is identically 1, the operator $W_{\$,}$ reduces to the composition operator $C_{\sum_m \varphi^m -}$ These operators $C_{\sum_m \varphi^m}$ and $W_{\dot{y}, \sum_m \varphi^m}$ have been studied on various function spaces (see [9,21]). Among them, the study of topological structure problem for spaces of bounded (weighted) composition operators with operator norm topology. For Hardy, H^∞, Bloch and Banach spaces see [10,16,22], [11,15,19], [13], and [6]. For the study of topological structure problem for both spaces of composition operators and weighted composition operators on weighted Banach spaces with sup-norm generated by a radial weight (see [24]). We show an over view of confirmative on [24].

For a radial weight on $\mathbb{D}$ we mean a positive function v_m on **D** with $v_m(z) = v_m(|z|), z \in \mathbb{D}$, where $v_m(r)$ is continuous and increasing on $[0,1)$ and $v_m(r) \to \infty$ as $r \to 1^-$. For a weight v_m on D, we define the following weighted Banach spaces of H^∞:

$$H_{v_m}(\mathbb{D}) = \left\{\sum_m f_m \in H(\mathbb{D}): \| \sum_m f_m \|_{v_m} := \sup_{z \in D} \sum_m \frac{|\sum_m f_m(z)|}{v_m(z)} < \infty\right\}$$

and

$$H^0_{v_m}(\mathbb{D}) = \left\{\sum_m f_m \in H(\mathbb{D}): \lim_{|z| \to 1^-} \sum_m \frac{|\sum_m f_m(z)|}{v_m(z)} = 0\right\},$$

endowed with the norm $\|\cdot\|_{v_m}$, so, for a series of functions $g_m: \mathbb{D} \to [0, \infty), \lim_{|z| \to 1^-} \sum_m g_m(z) := \lim_{r \to 1^-} \sup_{|z| > r} \sum_m g_m(z)$. Likely known as weighted Bergman spaces of infinite order. We denote X_{v_m} to be either of the spaces $H_e(\mathbb{D})$ or $H^0_{v_m}(\mathbb{D})$.

To characterize topological properties of spaces X_{v_m} or linear operators between them in term of weights, see [1, 2] we use the so-called associated weights. For a given weight v_m on $\mathbb{D}$, its associated weight is defined by (see [3, Definition 1.1])

$$\tilde{v_m}(z) = \sup \sum_m \left\{|\sum_m f_m(z)|: \sum_m f_m \in H_{v_m}(\mathbb{D}), \| \sum_m f_m \|_{v_m} \leq 1\right\}$$

Note that $\tilde{v_m}(z) = \tilde{v_m}(|z|)$ and $0 < \widetilde{v_m}(z) \leq v_m(z)$ for all $z \in \mathbb{D}, \widetilde{v_m}(r)$ is increasing and log-convex on $[0,1)$ (i.e. the function $\log \tilde{v_m}(e^x)$ is convex on $(-\infty, 0)$), and $H_{v_m}(\mathbf{D}) = H_{\tilde{e}}(\mathbb{D})$ isometrically. Moreover, in [14, Lemma 2.2] it was shown that for a log-convex weight v_m on $\mathbb{D}$, there is some constant M such that $\bar{v_m}(z) \leq v_m(z) \leq M\tilde{v_m}(z), z \in \mathbb{D}$. Thus, we use log-convex weights. Next, for a log-convex weight v_m, by [4, Theorem 2.3] the following condition from [17, p. 310] and [18, Definition 2. 1]

$$\limsup_{k\to\infty} \sum_m \frac{v_m(1-2^{-k-1})}{v_m(1-2^{-k})} < \infty \tag{1.1}$$

is equivalent to the continuity of all compositions operators $C_{\sum_m \varphi^m}, \sum_m \varphi^m \in S(\mathbb{D})$, on X_{v_m}. Consequently, for log-convex weights satisfying (1.1) and only for them, the spaces $\mathcal{C}(H_{v_m}(\mathbf{D}))$ and $\mathcal{C}(H_{v_m}^0(\mathbb{D}))$ of all bounded composition operators on $H_{v_m}(\mathbb{D})$ and, respectively, $H_{v_m}^0(\mathbb{D})$ coincide and equal to the space $\{C_{\sum_m \varphi^m}: \sum_m \varphi^m \in S(\mathbb{D})\}$ of all composition operators. For some conditions of various types that are equivalent to (1.1) see [1, Lemma 2.6]. In particular, (1.1) $\Leftrightarrow v_m(r) = O(v_m(r^2))$ as $r \to 1^-$.

We consider the topological structure problem for the spaces of (weighted) composition operators on spaces X_{v_m} given by log-convex weights v_m satisfying (1.1). Let $\mathcal{V}$ denote the set of all such weights. The standard weights $(v_m)_\alpha(z) = (1-|z|^2)^{-\alpha}, \alpha > 0$, belong to $\mathcal{V}$. We suppose that $v_m \in \mathcal{V}$.

We recall some auxiliary results on spaces X_{v_m} and (weighted) composition operators on them.

We study the topological structure of the space $\mathcal{C}(X_{v_m})$ of all composition operators on X_{v_m}. We prove that the set $[C_{\sum_m \varphi^m}]$ of all composition operators that differ from the given one $C_{\sum_m \varphi^m}$ by a compact operator is path connected in $\mathcal{C}(X_{v_m})$. A component in $\mathcal{C}(X_{v_m})$ is not in general the set of such type [19] showed these results for the space H^∞ and we now extend them to the family of all Bergman spaces of infinite order given by weights from $\mathcal{V}$. We show that the condition that completely characterizes isolated composition operators $C_{\sum_m \varphi^m}$ in the setting of the space H^∞ (see [19, Corollary 9] and [12, Theorem 4.1]) is necessary for $C_{\sum_m \varphi^m}$ to be isolated in all spaces $\mathcal{C}(X_{v_m})$ with v_m in $\mathcal{V}$.

The space $\mathcal{C}_{w_m}(X_{v_m})$ consists of all bounded nonzero weighted composition operators on X_{v_m}. And [6] investigated the topological structure problem for the space $\mathcal{C}_{w_m}^0(X_{v_m})$ of all bounded weighted composition operators on X_{v_m}. Now every operator $W_{\sum_m \psi^m, \sum_m \varphi^m}$ in $\mathcal{C}_{w_m}^0(X_{v_m})$ and the zero operator 0 are always connected by the path $W_{t\sum_m \psi^m, \sum_m \varphi^m}, t \in [0,1]$. This implies that $\mathcal{C}_{w_m}^0(X_{v_m})$ is path connected. Some results be considered again in $\mathcal{C}_{w_m}(X_{v_m})$. [6] and some arguments used cannot be applied to $\mathcal{C}_{w_m}(X_{v_m})$ (see [6]). We show some new ideas to prove that the set $\mathcal{C}_{w_m,0}(X_{v_m})$ of all nonzero compact weighted composition operators is path connected in $\mathcal{C}_{w_m}(X_{v_m})$; but not a path component. We also give a simple sufficient condition to ensure that two operators in $\mathcal{C}_{w_m}(X_{v_m})$ belong to the same path component of this space. These results clarify and improve the corresponding ones in [6, Theorems 3.2 and 4.2]. We describe two path connected sets of the same type in $\mathcal{C}_{w_m}(X_{v_m})$, one of which is a path component, while another is not.

We state some results concerning properties of functions in the spaces X_e and (weighted) composition operators and their differences on these spaces.

We defined the pseudo-hyperbolic distance by

$$\rho(z,\zeta) = \left|\frac{z-\zeta}{1-\bar{z}\zeta}\right|, \quad z,\zeta \in \mathbb{D}$$

For a function $\sum_m \varphi^m \in H(\mathbb{D})$, put

$$\| \sum_m \varphi^m \|_\infty = \sup_{z\in D} \sum_m |\sum_m \varphi^m(z)| \text{ and } M(\sum_m \varphi^m, r) = \sup_{|z|\leq r} \sum_m |\sum_m \varphi^m(z)|, r \in (0,1)$$

We denote by $|E|$ the Lebesgue measure of E on the unit circle ∂D.

Lemma 2.1 (see [24]). There is a constant $C > 0$, dependent only on v_m, such that for every $\sum_m f_m \in H_{v_m}$(D) and $z, \zeta \in \mathbb{D}$,

$$\left|\sum_m f_m'(z)\right| \leq C \sum_m \frac{v_m(z)}{1-|z|} \| \sum_m f_m \|_{v_m} \tag{2.1}$$

and

$$\left|\sum_m f_m(z) - \sum_m f_m(\zeta)\right| \leq C \sum_m \| \sum_m f_m \|_{v_m} \rho(z,\zeta) \,\mathrm{m}\ \{v_m(z), v_m(\zeta)\} \tag{2.2}$$

Proof. In [1, Theorem 2.8] it was proved that for every weight $v_m \in \mathcal{V}$, the differentiation operator D is bounded from $H_{v_m}(\mathbb{D})$ to $H_{(v_m)_1}(\mathbb{D})$ with $(v_m)_1(r) = v_m(r)/(1-r)$, which implies (2.1).

The inequality (2.2) was obtained in [5, Lemma 1].

The next result follows from [4, Proposition 2.1 and Theorems 2.3 and 3.3].

Proposition 2.2. Let $\sum_m \varphi^m \in \mathcal{S}(\mathrm{D})$.
(a) The operator $C_{\sum_m \varphi^m}$ is bounded on X_{v_m}. Moreover,
$$\|C_{\sum_m \varphi^m}\| \leq \sup_{x\in D} \sum\nolimits_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} < \infty$$
(b) The operator $C_{\sum_m \varphi^m}$ is compact on X_{v_m} if and only if
$$\lim_{|z|\to 1^-} \sum\nolimits_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} = 0$$
This proposition implies that
$$\mathcal{C}\big(H_{v_m}(\mathbb{D})\big) = \mathcal{C}\big(H^0_{v_m}(\mathbb{D})\big) = \Big\{C_{\sum_m \varphi^m} : \sum\nolimits_m \varphi^m \in \mathcal{S}(\mathbb{D})\Big\}$$
and the sets $\mathcal{C}_0\big(H_{v_m}(\mathbf{D})\big)$ and $\mathcal{C}_0\big(H^0_{v_m}(\mathbb{D})\big)$ of all compact composition operators on $H_e(\mathbb{D})$ and $H^0_{v_m}(\mathbb{D})$, respectively, coincide; more precisely
$$\mathcal{C}_0\big(H_{v_m}(\mathbf{D})\big) = \mathcal{C}_0\big(H^0_{v_m}(\mathbf{D})\big) = \Big\{C_{\sum_m \varphi^m} : \sum\nolimits_m \varphi^m \in \mathcal{S}(\mathbf{D}), v_m(\sum\nolimits_m \varphi^m(z)) = o(v_m(z)), |z| \to 1^-\Big\}$$
Thus, all results and arguments will be stated simultaneously for both spaces $\mathcal{C}\big(H_{v_m}(\mathbb{D})\big)$ and $\mathcal{C}\big(H^0_{v_m}(\mathbb{D})\big)$.
Compactness of differences of two composition operators between weighted Banach spaces with sup-norm was characterized in [5, Corollary 7 and Theorem 9]. To state these results for composition operators from X_{v_m} into itself with $v_m \in \mathcal{V}$, we need the following (see [24]).

Remark 2.3. Let $\sum_m \varphi^m \in \mathcal{S}(\mathbb{D})$ and $g_m : \mathbb{D} \to [0,\infty)$. As usual, we put
$$\lim_{|\sum_m \varphi^m(z)|\to 1^-} \sum\nolimits_m g_m(z) := \begin{cases} \lim\limits_{r\to 1^-} \sup\limits_{|\sum_m \varphi^m(z)|>r} \sum\nolimits_m g_m(z) & \text{if } \| \sum\nolimits_m \varphi^m \|_\infty = 1 \\ 0 & \text{if } \| \sum\nolimits_m \varphi^m \|_\infty < 1. \end{cases}$$
Then
$$\lim\nolimits_{ll\to 1^-} \sum\nolimits_m g_m(z) = 0 \quad \textit{implies that } lim_{|\sum_m f_m(z)|\to 1^-} \sum\nolimits_m g_m(z) = 0. \tag{2.3}$$
Indeed, it is enough to check this statement for $\sum_m \varphi^m$ with $\| \sum_m \varphi^m \|_\infty = 1$. Given $r \in (0,1)$, letting $\tilde{r} := M(\sum_m \varphi^m, r)$, we get that
$$\sup_{|\sum_m \varphi^m(2)|>\bar{r}} \sum\nolimits_m g_m(z) \leq \sup_{|z|>r} \sum\nolimits_m g_m(z) \text{ and } \bar{r} \to 1^- \text{as } r \to 1^-,$$
Which implies (see [24]).

Proposition 2.4. Let $\sum_m \varphi^m, \sum_m \phi^m \in \mathcal{S}(\mathbb{D})$. Then the following statements are equivalent.
(i)The difference $C_{\sum_m \varphi^m} - C_{\sum_m \phi^m}$ is compact on $H_{v_m}(\mathrm{D})$.
(ii)The difference $C_{\sum_m \varphi^m} - C_{\sum_m \phi^m}$ is compact on $H^0_{v_m}(\mathbb{D})$.
(iii)
$$\lim_{|z|\to 1^-} \sum\nolimits_m \frac{v_m\big(\sum_m \varphi^m(z)\big)}{v_m(z)} \rho(\sum\nolimits_m \varphi^m(z), \sum\nolimits_m \phi^m(z)) = \lim_{|z|\to 1^-} \sum\nolimits_m \frac{v_m\big(\sum_m \phi^m(z)\big)}{v_m(z)} \rho(\sum\nolimits_m \varphi^m(z), \sum\nolimits_m \phi^m(z)) = 0$$

Proof. (i) $\Longrightarrow$ (ii) is obvious.
(ii) $\Longrightarrow$ (iii). Suppose that $C_{\sum_m \varphi^m} - C_{\sum_m \phi^m}$ is compact on $H^0_{v_m}(\mathbb{D})$. If $\| \sum_m \varphi^m \|_\infty = \| \sum_m \phi^m \|_\infty = 1$, then the assertion follows from [5, Theorem 9]. If $\| \sum_m \varphi^m \|_\infty < 1$ (similarly to the case $\| \sum_m \phi^m \|_\infty < 1$), then $C_{\sum_m \varphi^m}$ is compact on $H^0_{v_m}(\mathrm{D})$. Hence, $C_{\sum_m \phi^m}$ is also compact on $H^0_{v_m}(\mathbf{D})$, which and Proposition 2.2(b) imply that
$$\lim_{|z|\to 1^-} \sum\nolimits_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} = \lim_{|z|\to 1^-} \sum\nolimits_m \frac{v_m(\sum_m \phi^m(z))}{v_m(z)} = 0$$
Using this and the fact that $\rho(\sum_m \varphi^m(z), \sum_m \phi^m(z)) \leq 1, z \in \mathbb{D}$, we get (iii).
(iii) $\Longrightarrow$ (i). By Remark 2.3, (iii) implies that
$$\lim\nolimits_{(z)|\to 1^-} \sum\nolimits_m \frac{v_m\big(\sum_m \varphi^m(z)\big)}{v_m(z)} \rho(\sum\nolimits_m \varphi^m(z), \sum\nolimits_m \phi^m(z)) = \lim\nolimits_{|\sum_m \phi^m(z)|\to 1^-} \sum\nolimits_m \frac{v_m\big(\sum_m \phi^m(z)\big)}{v_m(z)} \rho(\sum\nolimits_m \varphi^m(z), \sum\nolimits_m \phi^m(z)) = 0.$$

It remains to use [5, Corollary 7] to obtain (i).
Boundedness and compactness of the operator $W_{\sum_m \psi^m, \sum_m \varphi^m}$ between weighted Banach spaces with sup-norm were characterized in [8, Propositions 3.1 and 3.2, Corollaries 4.3 and 4.5], from which we get the following result.

Proposition 2.5. Let $\sum_m \varphi^m \in S(\mathbb{D})$ and $\sum_m \psi^m \in H(\mathbb{D})$. Then the next two assertions hold:
(a) The operator $W_{\sum_m \psi^m, \sum_m \varphi^m}: X_{v_m} \to X_{v_m}$ is bounded if and only if $\sum_m \psi^m \in X_{v_m}$ and

$$\sup_{z\in D} \sum_m \frac{|\sum_m \psi^m(z)| v_m(\sum_m \varphi^m(z))}{v_m(z)} < \infty.$$

(b) The opemtor $W_{\sum_m \psi^m, \sum_m \varphi^m}: X_{v_m} \to X_{v_m}$ is compact if and only if $\sum_m \psi^m \in X_{v_m}$ and

$$\lim_{|\sum_m \varphi^m(z)|\to 1^-} \sum_m \frac{|\sum_m \psi^m(z)| v_m(\sum_m \varphi^m(z))}{v_m(z)} = 0 \text{ for } X_{v_m} = H_{v_m}(\mathbf{D})$$

or

$$\lim_{|z|\to 1^-} \sum_m \frac{|\sum_m \psi^m(z)| v_m(\sum_m \varphi^m(z))}{v_m(z)} = 0 \text{ for } X_{v_m} = H^0_{v_m}(\mathbf{D})$$

From this proposition it follows that $\mathcal{C}_{w_m}(H^0_{v_m}(\mathbb{D}))$ and $\mathcal{C}_{w_m,0}(H^0_{v_m}(\mathbf{D}))$ are. proper subsets of $\mathcal{C}_{w_m}(H_{v_m}(\mathbf{D}))$ and, respectively, $\mathcal{C}_{w_m,0}(H_{v_m}(\mathbf{D}))$. Some arguments will be presented separately for spaces $\mathcal{C}_{w_m}(H_{v_m}(\mathbb{D}))$ and $\mathcal{C}_{w_m}(H^0_{v_m}(\mathbf{D}))$.

We also need the following lemma.

Lemma 2.6. Let $[z,\zeta]$ denote the closed interval connecting points z and ζ in D. Then $\rho(\xi,\eta) \le \rho(z,\zeta)$ for all $\xi,\eta \in [z,\zeta]$. Proof. Without loss of generality we may assume that the points lie in the interval in the following order: $z \to \xi \to \eta \to \zeta$. We have the next obvious relations:

$$\begin{aligned} |\xi-\eta| &= |z-\zeta| - (|z-\xi| + |\zeta-\eta|) \\ |1-\bar{\xi}\eta| &\ge |1-\bar{z}\zeta| - |\bar{z}\zeta - \bar{\xi}\eta| \\ &\ge |1-\bar{z}\zeta| - (|\zeta||\bar{z}-\bar{\xi}| + |\bar{\xi}\,\|\,\zeta-\eta|) \\ &\ge |1-\bar{z}\zeta| - (|z-\xi| + |\zeta-\eta|), \end{aligned}$$

and

$$|z-\zeta| < |1-\bar{z}\zeta|$$

Then

$$\begin{aligned} \rho(\xi,\eta) = \frac{|\xi-\eta|}{|1-\bar{\xi}\eta|} &\le \frac{|z-\zeta| - (|z-\xi| + |\zeta-\eta|)}{|1-\bar{z}\zeta| - (|z-\xi| + |\zeta-\eta|)} \\ &\le \frac{|z-\zeta|}{|1-\bar{z}\zeta|} = \rho(z,\zeta). \end{aligned}$$

We consider the topological structure problem for the space $\mathcal{C}(X_{v_m})$ of all composition operators on X_{v_m} under the operator norm topology. We will write $C_{\sum_m \varphi^m} \sim C_{\sum_m \phi^m}$ in $\mathcal{C}(X_{v_m})$ if these operators are in the same path component of $\mathcal{C}(X_{v_m})$. Two composition operators $C_{\sum_m \varphi^m}$ and $C_{\sum_m \phi^m}$ are said to be compactly equivalent in $\mathcal{C}(X_{v_m})$ if their difference $C_{\sum_m \varphi^m} - C_{\sum_m \phi^m}$ is compact on X_{v_m}. Obviously, this relation is an equivalence one. Denote by $[C_{\sum_m \varphi^m}]$ the equivalence class of all composition operators that are equivalent to the given operator $C_{\sum_m \varphi^m}$. Note that the set $\mathcal{C}_0(X_{v_m})$ of all compact composition operators on X_{v_m} coincide with the class $[C_0]$ of all operators from $\mathcal{C}(X_{v_m})$ that are equivalent to the operator $C_0: \sum_m f_m \mapsto \sum_m f_m(0)$. We have the following (see [24])

Theorem 3.1. Each equivalence class $[C_{\sum_m \varphi^m}]$ is path connected in the space $\mathcal{C}(X_{v_m})$
Proof. Let $\sum_m \varphi^m \in \mathcal{S}(\mathbb{D})$ and $C_{\sum_m \phi^m}$ be an arbitrary operator from $[C_{\sum_m \varphi^m}]$. Then $C_{\sum_m \varphi^m} - C_{\sum_m \phi^m}$ is compact on X_{v_m} and, by Proposition 2.4, we have

$$\begin{aligned} &\lim_{|z|\to 1^-} \sum_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} \rho\left(\sum_m \varphi^m(z), \sum_m \phi^m(z)\right) \\ &\qquad = \lim_{|z|\to 1^-} \sum_m \frac{v_m(\sum_m \phi^m(z))}{v_m(z)} \rho\left(\sum_m \varphi^m(z), \sum_m \phi^m(z)\right) = 0. \qquad (3.1) \end{aligned}$$

For each $t \in [0,1]$, put $\sum_m \varphi_t^m(z) = (1-t)\sum_m \varphi^m(z) + t\sum_m \phi^m(z), z \in \mathbb{D}$. Clearly, $\sum_m \varphi_t^m \in \mathcal{S}(\mathbf{D})$ for all $t \in [0,1]$ and, by Proposition 2.2(a), the corresponding operators $C_{\sum_m \varphi_t^m}, t \in [0,1]$, are bounded on X_{v_m}. Moreover, all differences $C_{\sum_m \varphi^m} - C_{\sum_m \varphi_t^m}$ are compact on X_{v_m}. Indeed, $|\sum_m \varphi_t^m(z)| \leq \max\{|\sum_m \varphi^m(z)|, |\sum_m \phi^m(z)|\}$ and, hence, $v_m(\sum_m \varphi_t^m(z)) \leq \max\{v_m(\sum_m \varphi^m(z)), v_m(\sum_m \phi^m(z))\}$ for all $z \in \mathbb{D}$ and $t \in [0,1]$.

Next, by Lemma 2.6, $\rho(\sum_m \varphi^m(z), \sum_m \varphi_t^m(z)) \leq \rho(\sum_m \varphi^m(z), \sum_m \phi^m(z))$. From the above inequalities and (3.1) it follows that

$$\lim_{|z|\to 1^-} \sum_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} \rho\left(\sum_m \varphi^m(z), \sum_m \varphi_t^m(z)\right)$$
$$\leq \lim_{|z|\to 1^-} \sum_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} \rho(\sum_m \varphi^m(z), \sum_m \phi^m(z)) = 0$$

and

$$\lim_{|z|\to 1^-} \sum_m \frac{v_m(\sum_m \varphi_t^m(z))}{v_m(z)} \rho\left(\sum_m \varphi^m(z), \sum_m \varphi_t^m(z)\right)$$
$$\leq \lim_{|z|\to 1^-} max \sum_m \left\{\frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} \rho\left(\sum_m \varphi^m(z), \sum_m \phi^m(z)\right), \frac{v_m(\sum_m \phi^m(z))}{v_m(z)} \rho\left(\sum_m \varphi^m(z), \sum_m \phi^m(z)\right)\right\} = 0$$

Using Proposition 2.4 once again, we conclude that $C_{\sum_m \varphi^m} - C_{\sum_m \varphi_t^m}$ is compact on X_{v_m} for every $t \in [0,1]$. Thus, $C_{\sum_m \varphi^m u} \in [C_{\sum_m \varphi^m}]$ for all $t \in [0,1]$ and, to finish the proof, it remains to show that the map

$$[0,1] \to \mathcal{C}(X_{v_m}), t \mapsto C_{\sum_m \varphi_t^m},$$

is continuous on [0,1]. That is, $\|C_{\sum_m \varphi_x^m} - C_{\sum_m \varphi^m t}\| \to 0$ as $s \to t$ for all $t \in [0,1]$.

Fix a number $t \in [0,1]$. For every $r \in (0,1), s \in [0,1]$, and $\sum_m f_m \in X_{v_m}$, by (2.2) we get

$$\sum_m \left\|C_{\sum_m \varphi^m} \sum_m f_m - C_{\sum_m \varphi^m t} \sum_m f_m\right\|_{v_m}$$
$$= \sup_{z\in D} \sum_m \frac{|\sum_m f_m(\sum_m \varphi_s^m(z)) - \sum_m f_m(\sum_m \varphi_t^m(z))|}{v_m(z)}$$
$$\leq C \sum_m \| \sum_m f_m \|_{v_m} \sup_{z\in\mathbb{D}} \rho\left(\sum_m \varphi_s^m(z), \sum_m \varphi_t^m(z)\right) \frac{\text{m } \{v_m(\sum_m \varphi_s^m(z)), v_m(\sum_m \varphi_t^m(z))\}}{v_m(z)}$$
$$\leq C \sum_m \| \sum_m f_m \|_l \sup_{z\in\mathbb{D}} \rho\left(\sum_m \varphi_s^m(z), \sum_m \varphi_t^m(z)\right) \frac{\text{m } \{v_m(\sum_m \varphi^m(z)), v_m(\sum_m \phi^m(z))\}}{v_m(z)}.$$

Consequently, for every $r \in (0,1)$ and $s \in [0,1]$,

$$\|C_{\sum_m \sum_m \varphi_e^m} - C_{\sum_m \varphi^m t}\| \leq C \text{ m } \{\mathcal{I}(r,s), \mathcal{J}(r,s)\},$$

where

$$\mathcal{I}(r,s): \quad = \sup_{|z|\leq r} \sum_m \rho\left(\sum_m \varphi_s^m(z), \sum_m \varphi_t^m(z)\right) \frac{\text{m } \{v_m(\sum_m \varphi^m(z)), v_m(\sum_m \phi^m(z))\}}{v_m(z)}$$
$$\leq |s-t| \sum_m \frac{v_m(M_r)}{v_m(0)} \sup_{|z|\leq r} \frac{|\sum_m \varphi^m(z) - \sum_m \phi^m(z)|}{(1 - |\sum_m \varphi_s^m(z)||\sum_m \varphi_t^m(z)|)}$$
$$\leq \sum_m \frac{2v_m(M_r)}{v_m(0)(1-M_r^2)} |s-t| \text{ with } M_r := max \sum_m \{M(\sum_m \varphi^m, r), M(\sum_m \phi^m, r)\}$$

and, by Lemma 2.6,

$$\mathcal{J}(r,s) \quad := \sup_{|z|>r} \sum_m \rho\left(\sum_m \varphi_s^m(z), \sum_m \varphi_t^m(z)\right) \frac{\text{m } \{v_m(\sum_m \varphi^m(z)), v_m(\sum_m \phi^m(z))\}}{v_m(z)}$$
$$\leq \sup_{|z|>r} \sum_m \rho(\sum_m \varphi^m(z), \sum_m \phi^m(z)) \frac{\text{m } \{v_m(\sum_m \varphi^m(z)), v_m(\sum_m \phi^m(z))\}}{v_m(z)} =: J(r).$$

Therefore, for every $r \in (0,1)$ and $s \in [0,1]$,

$$\|C_{\sum_m \varphi_*^m} - C_{\sum_m \varphi_t^m}\| \leq C\max \sum_m \left\{\frac{2v_m(M_r)}{v_m(0)(1-M_r^2)} |s-t|, J(r)\right\}.$$

By letting $s \to t$, this implies that

$$\limsup_{s\to t} \|C_{\sum_m \varphi_n^m} - C_{\sum_m \varphi_t^m}\| \leq CJ(r) \text{ for all } r \in (0,1)$$

Next, letting $r \to 1^-$ and using that $J(r) \to 0$ as $r \to 1^-$ by (3.1), we then get

$$\lim_{s\to t}\left\|C_{\sum_m \varphi_2^m}-C_{\sum_m \varphi_t^m}\right\|=0,$$

which completes the proof.

Corollary 3.2. The set $\mathcal{C}_0(X_{v_m})$ of all compact composition operators on X_{v_m} is path connected in $\mathcal{C}(X_{v_m})$.

Remark 3.3 [24]. Corollary 3.2 follows also from [6, Theorem 3.2]. Indeed, putting $\sum_m \psi^{m(1)}\equiv 1$ and $\sum_m \psi^{m(2)}\equiv 1$ in the proof of this theorem we get that two compact composition operators $C_{\sum_m \phi^{m}(1)}$ and $C_{\sum_m \phi^{m}(2)}$ on X_{v_m} belong to the same path component of the space $\mathcal{C}(X_{v_m})$.

Now we use the main idea from [19, Example 1] to show that in general $[C_{\sum_m \varphi^m}]$ might not be a whole path component of $\mathcal{C}(X_{v_m})$.

Example 3.4 [24]. Let $\sum_m \varphi_0^m(z)=1+a(z-1)$ with $0<a<1$. Then the class $[C_{\sum_m \varphi^{m}0}]$ is not a path component in $\mathcal{C}(X_{v_m})$.

Proof. Let $\delta=\frac{a(1-a)}{4}$. For each $t\in[-\delta,\delta]$, we put

$$\sum\nolimits_m \varphi_t^m(z)=\sum\nolimits_m \varphi_0^m(z)+t(z-1)^2$$

and show that, for every $0<|t|\le\delta, C_{\sum_m \varphi_*^m}\sim C_{\sum_m \varphi_t^m}$ in $\mathcal{C}(X_{v_m})$ but $C_{\sum_m \varphi^m u}\notin[C_{\sum_m \varphi_*^m \sum_m \varphi_0^m}]$. Evidently, this gives the result. Since $1-|\sum_m \varphi_0^m(z)|^2\ge a(1-a)|z-1|^2$ for all $z\in\mathbb{D}$

$$\begin{aligned}\left|\sum\nolimits_m \varphi_t^m(z)\right| &\le\left|\sum\nolimits_m \varphi_0^m(z)\right|+|t||z-1|^2\le\sqrt{1-a(1-a)|z-1|^2}+|t||z-1|^2\\ &\le 1-\frac{a(1-a)}{2}|z-1|^2+|t||z-1|^2<1\end{aligned}$$

for all $z\in\mathbb{D}$ and $t\in[-\delta,\delta]$. Hence, $\sum_m \varphi_t^m\in\mathcal{S}(\mathbb{D})$, and, by Proposition 2.2(a), $C_{\sum_m \varphi_t^m}\in\mathcal{C}(X_{v_m})$ for all $t\in[-\delta,\delta]$

Next, similarly to [19, Example 1], consider a sequence $(z_n)\subset\mathbb{D}$ such that $z_n\to 1$ along the arc $|1-z|^2=1-|z|^2$. For each $n\in\mathbb{N}$, we have

$$1-\left|\sum\nolimits_m \varphi_0^m(z_n)\right|^2=a(2-a)|z_n-1|^2=a(2-a)(1-|z_n|^2)$$

Hence,

$$\left|\sum\nolimits_m \varphi_0^m(z_n)\right|=\sqrt{(1-a)^2+a(2-a)|z_n|^2}\ge|z_n|$$

and

$$\begin{aligned}\sum\nolimits_m \rho\left(\sum\nolimits_m \varphi_0^m(z_n),\sum\nolimits_m \varphi_t^m(z_n)\right) &\ge\frac{|t\parallel z_n-1|^2}{1-\sum_m|\sum_m \varphi_0^m(z_n)|^2+\sum_m|\sum_m \varphi_0^m(z_n)||t||z_n-1|^2}\\ &\ge\frac{|t|}{a(2-a)+|t|}\end{aligned}$$

Thus, for all $n\ge 1$ and $t\in[-\delta,\delta]$,

$$\sum\nolimits_m \frac{v_m(\sum_m \varphi_0^m(z_n))}{v_m(z_n)}\rho\left(\sum\nolimits_m \varphi_0^m(z_n),\sum\nolimits_m \varphi_t^m(z_n)\right)\ge\frac{|t|}{a(2-a)+|t|}$$

From this and Proposition 2.4 it follows that $C_{\rho u}-C_{\varphi_0}$ is not compact on X_{v_m} and, consequently, $C_{\sum_m \varphi_t^m}\notin[C_{\sum_m \varphi_0^m}]$ for all $0<|t|\le\delta$.

To complete the proof, it is enough to check that the path $C_{\sum_m \varphi_t^m}, t\in[-\delta,\delta]$, is continuous in $\mathcal{C}(X_{v_m})$. For every $t,s\in[-\delta,\delta]$ and $\sum_m f_m\in X_{v_m}$

using (2.2), we have

$$\begin{aligned}&\sum\nolimits_m\left\|C_{\sum_m \varphi_2^m}\sum\nolimits_m f_m-C_{\sum_m \varphi^m t}\sum\nolimits_m f_m\right\|_{v_m}\\ &=\sup_{z\in D}\sum\nolimits_m\frac{|\sum_m f_m(\sum_m \varphi_s^m(z))-\sum_m f_m(\sum_m \varphi_t^m(z))|}{v_m(z)}\\ &\le C\sum\nolimits_m\parallel\sum\nolimits_m f_m\parallel_{v_m}\sup_{z\in D}\rho\left(\sum\nolimits_m \varphi_s^m(z),\sum\nolimits_m \varphi_t^m(z)\right)\frac{\mathrm{m}\ \{v_m(\sum_m \varphi_s^m(z)),v_m(\sum_m \varphi_t^m(z))\}}{v_m(z)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\|C_{\sum_m \varphi_t^m}-C_{\sum_m \varphi_t^m}\right\|\\ &\le C\sup\nolimits_{z\in D}\sum\nolimits_m \rho\left(\sum\nolimits_m \varphi_s^m(z),\sum\nolimits_m \varphi_t^m(z)\right)\frac{\max\{v_m(\sum_m \varphi_s^m(z)),v_m(\sum_m \varphi_t^m(z))\}}{v_m(z)}\qquad(3.2)\end{aligned}$$

Moreover, for every $t,s\in[-\delta,\delta]$ and $z\in\mathbb{D}$,

$$\sum_m \rho\left(\sum_m \varphi_s^m(z), \sum_m \varphi_t^m(z)\right)$$
$$= \frac{|t-s \parallel z-1|^2}{\sum_m \mid 1-\sum_m \bar{} \varphi_s^m(z)} \varphi_t^m(z) \mid$$
$$\leq \frac{|t-s||z-1|^2}{1-\sum_m |\sum_m \varphi_0^m(z)|^2-(|t|+|s|)\sum_m |\sum_m \varphi_0^m(z)||z-1|^2-|ts||z-1|^4} \quad (3.3)$$
$$\leq \frac{|t-s||z-1|^2}{a(1-a)|z-1|^2-(|t|+|s|)|z-1|^2-|ts||z-1|^4}$$
$$\leq \frac{|t-s|}{a(1-a)-2\delta-4\delta^2} \leq \frac{4|t-s|}{a(1-a)}.$$

Next, for each $s \in [-\delta, \delta]$, we put

$$a_s = \sum_m \varphi_s^m(0) = 1-a+s, \beta_s(z) = \frac{z-a_s}{1-a_s z}, \text{ and } \sum_m \phi_s^m = \beta_s \circ \sum_m \varphi_s^m.$$

Then $\sum_m \phi_s^m(0) = \beta_s \circ \sum_m \varphi_s^m(0) = 0$. Hence, by the Schwarz lemma, $|\sum_m \phi_s^m(z)| \leq |z|$ for every $z \in \mathbb{D}$. From this it follows that, for every $r \in (0,1)$,

$$\sup_{|z|\leq r} \sum_m \left|\sum_m \varphi_s^m(z)\right| = \sup_{|z|\leq r} \sum_m \left|\left(\beta_s^{-1} \circ \sum_m \phi_s^m\right)(z)\right| \leq \sup_{|z|\leq r} |\beta_s^{-1}(z)| = \frac{r+|a_s|}{1+r|a_s|}$$
$$\leq \frac{r+r_0}{1+rr_0}$$

where $r_0 = 1-a+\delta \in (0,1)$. This, (3.2), and (3.3) imply that

$$\left\|C_{\sum_m \varphi_e^m} - C_{\sum_m \varphi_t^m}\right\| \leq \frac{4C}{a(1-a)}|t-s| \sup_{r\in[0,1)} \sum_m \frac{v_m\left(\frac{r+r_0}{1+rr_0}\right)}{v_m(r)}$$

and it remains to check that the last supremum is finite. By [1, Lemma 2.6], there is some constant $M > 0$, dependent only on v_m, such that

$$(\log v_m)'(r) = \frac{v_m'(r)}{v_m(r)} \leq \frac{M}{1-r} \; for\ all\ r \in \quad (0,1)$$

Then, using the arguments in the proof of [1, Theorem 2.8, (i) $\Longrightarrow$ (vii)], we get

$$\log v_m\left(\frac{r+r_0}{1+rr_0}\right) - \log v_m(r) \quad \leq (\log v_m)'\left(\frac{r+r_0}{1+rr_0}\right)\frac{r+r_0}{1+rr_0}\log\frac{r+r_0}{(1+rr_0)r}$$
$$\leq \frac{M(r+r_0)}{(1-r)(1-r_0)}\log\left(1+\frac{r_0(1-r^2)}{(1+rr_0)r}\right)$$
$$\leq \frac{M(r+r_0)r_0(1+r)}{(1-r_0)(1+rr_0)r} \leq \frac{8M}{1-r_0}$$

for every $r \in \left[\frac{1}{2}, 1\right)$. Thus, there is some number $M_0 > 1$, dependent only on v_m and r_0, such that

$$v_m\left(\frac{r+r_0}{1+rr_0}\right) \leq M_0 v_m(r) \text{ for all } r \in [0,1) .$$

Consequently,

$$\sup_{r\in[0,1)} \frac{v_m\left(\frac{r+r_0}{1+r_0}\right)}{v_m(r)} \leq M_0$$

which completes the proof.

Remark 3.5. Note that to characterize components in the space of composition operators on Hardy space H^2, [22] conjectured that the set of all composition operators that differ from the given one by a compact operator forms a component. Later, [7], and [20] independently showed that this conjecture is false. [19] also gave a negative answer to this conjecture for the setting of space H^∞. In fact, in Theorem 3.1 we proved that the sets of such a type are path connected in the space $\mathcal{C}(X_{v_m})$. Example 3.4 shows that, in general, they are not components of $\mathcal{C}(X_{v_m})$. Therefore, the conjecture is also not true for all spaces X_{v_m} given by weights v_m from the class $\mathcal{V}$.

We end with a result concerning isolated points in the spaces $\mathcal{C}(X_{v_m})$. The result in [6, Theorem 5.7] can be reformulated as follows: If the set

$$E(v_m, \sum_m \varphi^m) = \{\omega \in \partial\mathbb{D} \mid \exists (z_n) \subset \mathbb{D}: \lim_{n\to\infty} z_n = \omega \text{ and } \lim_{n\to\infty} \sum_m \frac{v_m(\sum_m \varphi^m(z_n))}{v_m(z_n)} > 0\Big\}$$

has Lebesgue measure strictly positive, then $C_{\sum_m \varphi^m}$ is isolated in $\mathcal{C}(X_{v_m})$. This is an analog of [19, Corollary 8]. On the other hand, in [19,, Corollary9 it was established that if

$$\int_0^{2\pi} \sum_m \log\left(1 - \left|\sum_m \varphi^m(e^{i\theta})\right|\right) d\theta > -\infty, \tag{3.4}$$

then $C_{\sum_m \varphi^m}$ is not isolated in $\mathcal{C}(H^\infty)$. Equivalently, the condition

$$\int_0^{2\pi} \sum_m \log\left(1 - \left|\sum_m \varphi^m(e^{i\theta})\right|\right) d\theta = -\infty \tag{3.5}$$

is necessary for the operator $C_{\sum_m \varphi^m}$ to be isolated in $\mathcal{C}(H^\infty)$. In the following proposition we extend this result to all weighted spaces X_{v_m} with $v_m \in \mathcal{V}$. Note that [12, Theorem 4.1] proved that (3.5) gives the complete description of isolated operators $C_{\sum_m \varphi^m}$ in $\mathcal{C}(H^\infty)$ (see [24]).

Proposition 3.6. If $\sum_m \varphi^m \in \mathcal{S}(\mathrm{D})$ satisfies (3.4), then the operator $C_{\sum_m \varphi^m}$ is not isolated in $\mathcal{C}(X_{v_m})$.

Proof. Following [19, Corollary 9], consider the next bounded outer function in :

$$\sum_m \phi^m(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \sum_m \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left(1 - \left|\sum_m \varphi^m(e^{i\theta})\right|\right) d\theta\right), z \in \mathbb{D}$$

As is known, $|\sum_m \phi^m| \le 1 - |\sum_m \varphi^m|$ in $\mathbb{D}$ and $|\sum_m \phi^m| = 1 - |\sum_m \varphi^m|$ almost everywhere on $\partial\mathbb{D}$. This implies, in particular, that the functions $\sum_m \varphi_t^m(z) = \sum_m \varphi^m(z) + t\sum_m \phi^m(z)$ are in $\mathcal{S}(\mathbf{D})$ for every $|t| < 1$. Hence, by Proposition 2.2(a), all operators $C_{\sum_m \varphi_t^m}, |t| < 1$, belong to $\mathcal{C}(X_{v_m})$ We will show that the path $C_{\sum_m \varphi^m u}, |t| \le \frac{1}{4}$, is continuous in $\mathcal{C}(X_{v_m})$ and, consequently, $C_{\sum_m \varphi^m}$ is not isolated.

By the proof of (i) $\Longrightarrow$ (vii) in [1, Theorem 2.8], there exists a constant $M > 0$ such that

$$\sum_m v_m\left(\frac{1+r}{2}\right) \le M \sum_m v_m(r) \text{ for all } r \in [0,1)$$

From this it follows that, for each $|t| \le \frac{1}{2}$ and all $z \in \mathbb{D}$,

$$\begin{aligned}\sum_m v_m\left(\sum_m \varphi_t^m(z)\right) &= \sum_m v_m(|\sum_m \varphi^m(z) + t\sum_m \phi^m(z)|) \le \sum_m v_m(|\sum_m \varphi^m(z)| + |t||\sum_m \phi^m(z)|) \\ &\le \sum_m v_m(|\sum_m \varphi^m(z)| + |t|(1 - |\sum_m \varphi^m(z)|)) \le \sum_m v_m\left(\frac{1 + |\sum_m \varphi^m(z)|}{2}\right) \le M\sum_m v_m(\sum_m \varphi^m(z))\end{aligned}$$

Using this and (2.2), we get that, for each $\sum_m f_m \in X_{v_m}$ and $s, t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\begin{aligned}\sum_m \left\|C_{\sum_m \varphi_t^m}\sum_m f_m - C_{\sum_m \varphi_*^m}\sum_m f_m\right\|_{v_m} &= \sup_{z\in D}\sum_m \frac{|\sum_m f_m(\sum_m \varphi_t^m(z)) - \sum_m f_m(\sum_m \varphi_s^m(z))|}{v_m(z)} \\ \le C\sum_m \| \sum_m f_m \|_{v_m} \sup_{z\in D}\rho\left(\sum_m \varphi_t^m(z), \sum_m \varphi_s^m(z)\right)&\frac{\max\{v_m(\sum_m \varphi_t^m(z)), v_m(\sum_m \varphi_s^m(z))\}}{v_m(z)}\end{aligned}$$

$$\begin{aligned}&\le CM\sum_m \| \sum_m f_m \|_{v_m} \sup_{z\in D}\rho\left(\sum_m \varphi_t^m(z), \sum_m \varphi_s^m(z)\right)\frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} \\ &\le M_0\sum_m \| \sum_m f_m \|_{v_m} \sup_{z\in D}\rho\left(\sum_m \varphi_t^m(z), \sum_m \varphi_s^m(z)\right)\end{aligned}$$

where $M_0 = CM\sup_{z\in D}\sum_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)} < \infty$ by Proposition 2.2(a). Thus, for every s, $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\|C_{\sum_m \varphi_t^m} - C_{\sum_m \varphi_2^m}\| \le M_0 \sup_{\Sigma\in D}\sum_m \rho\left(\sum_m \varphi_t^m(z), \sum_m \varphi_s^m(z)\right).$$

Next, for each $s, t \in \left[-\frac{1}{4}, \frac{1}{4}\right]$ and $z \in \mathbb{D}$

$$\sum_m \rho\left(\varphi_t^m(z), \sum_m \varphi_s^m(z)\right) = \left|\sum_m \frac{\sum_m \varphi_t^m(z) - \sum_m \varphi_s^m(z)}{1 - \bar{\varphi}^m{}_t(z)}\sum_m \varphi_s^m(z)\right|$$

$$\leq |t-s|\sum_m \frac{|\sum_m \phi^m(z)|}{1-|\sum_m \varphi^m(z)|^2-(|t|+|s|)|\sum_m \varphi^m(z)||\sum_m \phi^m(z)|-|ts||\sum_m \phi^m(z)|^2}$$
$$=|t-s\sum_m |\frac{1}{\frac{1-|\sum_m \varphi^m(z)|^2}{|\sum_m \varphi^m(z)|}-(|t|+|s|)|\sum_m \varphi^m(z)|-\sum_m |ts||\sum_m \phi^m(z)|}$$
$$\leq |t-s|\sum_m \frac{1}{\frac{1-|\sum_m \varphi^m(z)|}{|\sum_m \phi^m(z)|}-(|t|+|s|)-|ts|}$$
$$\leq |t-s|\frac{1}{1-(|t|+|s|)-|ts|}\leq \frac{16}{7}|t-s|.$$

Consequently,

$$\|C_{\sum_m \varphi_t^m}-C_{\sum_m \varphi_*^m}\|\leq \frac{16}{7}M_0|t-s| \text{ for all } s,t\in\left[-\frac{1}{4},\frac{1}{4}\right],$$

which implies that $C_{\sum_m \varphi_e^m}, t\in\left[-\frac{1}{4},\frac{1}{4}\right]$, is a continuous path in $\mathcal{C}(X_{v_m})$.

We study the topological structure of the space $\mathcal{C}_{w_m}(X_{v_m})$ of all nonzero bounded weighted composition operators on X_{v_m} under the operator norm topology. We write $W_{\sum_m \psi^m,\sum_m \varphi^m}\sim W_{\chi,\sum_m \phi^m}$ in $\mathcal{C}_{w_m}(X_{v_m})$ if the operators $W_{\sum_m \psi^m,\sum_m \varphi^m}$ and $W_{\chi,\sum_m \phi^m}$ are in the same path component of $\mathcal{C}_{w_m}(X_{v_m})$.

We know that the space $C^0_{w_m}(X_{v_m})$ of all bounded weighted composition operators is always path connected. Then Theorem 3.2 in [6] should be revised for the setting of nonzero weighted composition operators. To prove that two compact operators $W_{\sum_m \psi^{m}(1),\sum_m \phi^{m}(1)}$ and $W_{\sum_m \psi^{m}(2),\sum_m \phi^{m}(z)}$ in $C^0_{w_m}(H^0_{v_m}(\mathbf{D}))$ are path connected, [6] showed that

$$W_{\sum_m \psi^{m^{(1)}},\sum_m \phi^{m^{(1)}}}\sim W_{\sum_m \psi^{m^{(1)}}(0),}\sum_m \phi^{m^{(1)}}(0)\sim W_{\sum_m \psi^{m^{(2)}}(0),\sum_m \phi^{m^{(1)}}(0)}$$
$$\sim W_{\sum_m \psi^{m^{(2)}},\sum_m \phi^{m^{(2)}}}\ in\ C^0_{w_m}(H^0_{v_m}(\mathbb{D})),$$

which cannot be applied to the space $\mathcal{C}_{w_m}(H^0_{v_m}(\mathbf{D}))$ when $\sum_m \psi^{m^{(1)}}(0)=0$ or $\sum_m \psi^{m^{(2)}}(0)=0$. Hence, we develop some new ideas to establish this result for the spaces $\mathcal{C}_{w_m}(X_{v_m})$. So, we prove a bit more by showing that the set of all nonzero compact weighted composition operators on X_{v_m} is not a path component of $\mathcal{C}_{w_m}(X_{v_m})$ for all $v_m\in\mathcal{V}$.

We need the following, which is proved similarly to [23, Lemma 4.8].

Lemma 4.1. Every operator $W_{\sum_m \psi^m,\sum_m \varphi^m}\in\mathcal{C}_{w_m}(X_{v_m})$ is path connected with the operator $C_{\sum_m \varphi^m}$ in $\mathcal{C}_{w_m}(X_{v_m})$.

Theorem 4.2 (see [24]). The set $\mathcal{C}_{w_m,0}(X_{v_m})$ of all nonzero compact weighted composition operators on X_{v_m} is path connected in the space $\mathcal{C}_{w_m}(X_{v_m})$; but it is not a path component in this space.

Proof. (a) To prove that the set $\mathcal{C}_{w_m,0}(X_{v_m})$ is path connected in the space $\mathcal{C}_{w_m}(X_{v_m})$, it suffices to show that every operator $W_{\sum_m \phi^m,\sum_m \varphi^m}$ in $\mathcal{C}_{w_m,0}(X_{v_m})$ and the operator C_0 belong to the same path component of $\mathcal{C}_{w_m}(X_{v_m})$ via a path in $\mathcal{C}_{w_m,0}(X_{v_m})$.

If $\sum_m \psi^m(z)\equiv$ const, then the assertion follows from Lemma 4.1 and Corollary 3.2. Now suppose that $\sum_m \psi^m\in X_{v_m}$ is non-constant. We put

$$\sum_m \psi_t^m(z)=1-t+t\sum_m \psi^m(z) \text{ and } \sum_m \varphi_t^m(z)=t\sum_m \varphi^m(z), z\in\mathbb{D}, t\in[0,1]$$

Then, for every $t\in[0,1),\sum_m \psi_t^m$ is a nonzero function in X_{v_m} and $\overline{\sum_m \varphi_t^m(\mathbb{D})}\subset t\overline{\sum_m \varphi^m(D)}\subset\mathbb{D}$. From this and Proposition 2.5(b) it follows that all operators $W_{\sum_m \psi_u^m,\sum_m \varphi_t^m}, t\in[0,1)$, are compact on X_{v_m}. Hence, $W_{\sum_m \psi^m_{t,\sum_m \varphi_t^m}}\in\mathcal{C}_{w_m,0}(X_{v_m})$ for all $t\in[0,1]$; moreover, $W_{\sum_m \psi_b^m,\sum_m \psi_0^m}=C_0$ and $W_{\sum_m \psi_1^m,\sum_m \varphi_1^m}=W_{\sum_m \psi^m,\sum_m \varphi^m}$. We claim that the map

$$[0,1]\to\mathcal{C}_{w_m}(X_{v_m}), t\mapsto W_{\sum_m \psi_t^m,\sum_m \varphi_t^m}$$

is continuous on [0,1]. Then $W_{\sum_m \psi^m,\sum_m \varphi^m}\sim C_0$ in $\mathcal{C}_{w_m}(X_{v_m})$ via a path $W_{\sum_m \psi_t^m,\sum_m \varphi^m t}$ in $\mathcal{C}_{w_m,0}(X_{v_m})$

It remains to prove the claim. Obviously, $W_{\sum_m \psi_t^m,\sum_m \varphi_t^m}=(1-t)C_{t,\rho}+W_{t\dot{y},t\sum_m \varphi^m}$, and hence,

$\|W_{\sum_m \psi_s^m,\sum_m \varphi_x^m}-W_{\sum_m \psi_k^m,\sum_m \varphi_t^m}\|\leq\|(1-s)C_{s\sum_m \varphi^m}-(1-t)C_{t\sum_m \varphi^m}\|+\|W_{s\sum_m \phi^m,s\sum_m \varphi^m}-W_{t\sum_m \psi^m,t\sum_m \varphi^m}\|$,

for every $t,s\in[0,1]$. Consequently, to prove the claim, it is enough to show that for every $t\in[0,1]$

(i) $\lim_{s\to t}\sum_m \|(1-s)C_{s\sum_m \varphi^m}-(1-t)C_{t\sum_m \varphi^m}\|=0$ and (ii) $\lim_{s\to t}\sum_m \left\|W_{s\sum_m \psi^m,s\sum_m \varphi^m}-W_{t\dot{\psi},t_{\sum_m \varphi^m}}\right\|=0$.

In our further demonstration we will use the next obvious inequality for functions $\in H(\mathbb{D})$:

$$\left|\sum_m f_m(sz)-\sum_m f_m(tz)\right|\le |t-s||z|\max_{\tau\in[s,t]}\sum_m |f_m'(\tau z)|, z\in\mathbb{D}, t,s\in[0,1] \quad (4.1)$$

where we briefly write $[s,t]$ for the interval between s and t. First, we prove (i). If $t=1$, then by Proposition 2.2(a),

$$\|(1-s)C_{s\sum_m \varphi^m}\|\le(1-s)\sup_{x\in D}\sum_m \frac{v_m(s\sum_m \varphi^m(z))}{v_m(z)}\le(1-s)\sup_{z\in D}\sum_m \frac{v_m(\sum_m \varphi^m(z))}{v_m(z)}\to 0, s\to 1$$

Let now $t\in[0,1)$ and $t_0\in(t,1)$. For every $s\in[0,t_0)$ and $\sum_m f_m\in X_{v_m}$, using (2.1) and (4.1), we get

$$\begin{aligned}
&\sum_m \left\|(1-s)C_{s\sum_m \varphi^m}\sum_m f_m-(1-t)C_{t\sum_m \varphi^m}\sum_m f_m\right\|_{v_m}\\
&=\sup_{z\in D}\sum_m \frac{|(1-s)\sum_m f_m(s\sum_m \varphi^m(z))-(1-t)\sum_m f_m(t\sum_m \varphi^m(z))|}{v_m(z)}\\
&\le(1-s)\sup_{x\in D}\sum_m \frac{|\sum_m f_m(s\sum_m \varphi^m(z))-\sum_m f_m(t\sum_m \varphi^m(z))|}{v_m(z)}+|s-t|\sup_{z\in\mathbb{D}}\sum_m \frac{|\sum_m f_m(t\sum_m \varphi^m(z))|}{v_m(z)}\\
&\le|s-t|\sup_{z\in D}\sum_m \frac{|\sum_m \varphi^m(z)|}{v_m(z)}\max_{\tau\in[s,t]}\left|f_m'(\tau\sum_m \varphi^m(z))\right|+|s-t|\sup_{z\in\mathbb{D}}\sum_m \frac{v_m(t\sum_m \varphi^m(z))}{v_m(z)}\|\sum_m f_m\|_{v_m}\le C|s-t|\sum_m\\
&\|\sum_m f_m\|_{v_m}\sup_{z\in D}\frac{|\sum_m \varphi^m(z)|}{v_m(z)}\max_{\tau\in[s,t]}\frac{v_m(\tau\sum_m \varphi^m(z))}{1-|\tau\sum_m \varphi^m(z)|}+|s-t|\sup_{z\in\mathbb{D}}\sum_m \frac{v_m(t\sum_m \varphi^m(z))}{v_m(z)}\|\sum_m f_m\|_{v_m}\le\sum_m \frac{Cv_m(t_0)}{(1-t_0)v_m(0)}|s-t|\\
&\|\sum_m f_m\|_{v_m}+\sum_m \frac{v_m(t_0)}{v_m(0)}|s-t|\,\|\sum_m f_m\|_{v_m}=\left(\frac{C}{1-t_0}+1\right)\sum_m \frac{v_m(t_0)}{v_m(0)}|s-t|\\
&\|\sum_m f_m\|_{v_m}.
\end{aligned}$$

Therefore,

$$\|(1-s)C_{s\sum_m \varphi^m}-(1-t)C_{t\sum_m \varphi^m}\|\le\left(\frac{C}{1-t_0}+1\right)\sum_m \frac{v_m(t_0)}{v_m(0)}|s-t|\to 0 \text{ as } s\to t$$

which completes the proof of (i). Next, we prove (ii). Fix a number $t\in[0,1]$. For every $s\in[0,1]$ and $\sum_m f_m\in X_{v_m}$, we have

$$\begin{aligned}
&\sum_m \left\|W_{s\sum_m \psi^m,s\sum_m \varphi^m}\sum_m f_m-W_{t\sum_m \psi^m,t\sum_m \varphi^m}\sum_m f_m\right\|_{v_m}\\
&=\sup_{z\in D}\sum_m \frac{|s\sum_m \psi^m(z)\sum_m f_m(s\sum_m \varphi^m(z))-t\sum_m \psi^m(z)\sum_m f_m(t\sum_m \varphi^m(z))|}{v_m(z)}\\
&\le|s|\sup_{z\in D}\sum_m \frac{|\sum_m \psi^m(z)(\sum_m f_m(s\sum_m \varphi^m(z))-\sum_m f_m(t\sum_m \varphi^m(z)))|}{v_m(z)}+|s-t|\sup_{z\in\mathbb{D}}\sum_m \frac{|\sum_m \psi^m(z)\sum_m f_m(t\sum_m \varphi^m(z))|}{v_m(z)}.
\end{aligned}$$

To continue, we need several auxiliary estimates.

Estimate 1: We have

$$\begin{aligned}
\sup_{z\in D}\sum_m \frac{|\sum_m \psi^m(z)\sum_m f_m(t\sum_m \varphi^m(z))|}{v_m(z)}&\le\sum_m \|\sum_m f_m\|_{v_m}\sup_{z\in D}\frac{|\sum_m \psi^m(z)|v_m(t\sum_m \varphi^m(z))}{v_m(z)}\\
&\le\sum_m \|\sum_m f_m\|_{v_m}\sup_{z\in\mathbb{D}}\frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}=M\sum_m \|\sum_m f_m\|_{v_m}.
\end{aligned}$$

where $M:=\sup_{z\in D}\sum_m \frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}$ is finite by Proposition 2.5(a).

Estimate 2: Obviously, for every $r\in(0,1)$ and $s\in[0,1]$,

$$\sup_{2\in D}\sum_m \frac{|\sum_m \psi^m(z)(\sum_m f_m(s\sum_m \varphi^m(z))-\sum_m f_m(t\sum_m \varphi^m(z)))|}{v_m(z)}$$
$$=\max\sum_m \{\mathcal{I}(r,s,\sum_m f_m),\mathcal{J}(r,s,\sum_m f_m)\}$$

where, by using (2.1) and (4.1),

$$\begin{aligned}\mathcal{I}(r,s,\sum_m f_m) \quad &:= \sup_{|\sum_m f_m(z)|\le r}\sum_m \frac{|\sum_m \psi^m(z)(\sum_m f_m(s\sum_m \varphi^m(z))-\sum_m f_m(t\sum_m \varphi^m(z)))|}{v_m(z)}\\ &= |s-t|\sup_{|\sum_m f_m(z)|\le r}\sum_m \frac{|\sum_m \psi^m(z)\sum_m \varphi^m(z)|}{v_m(z)}\max_{\tau\in[s,t]}\left|f_m'(\tau\sum_m \varphi^m(z))\right|\\ &\le C|s-t|\sum_m \|\sum_m f_m\|_{v_m}\sup_{|\sum_m f_m(z)|\le r}\frac{|\sum_m \psi^m(z)|}{v_m(z)}\max_{\tau\in[s,t]}\frac{v_m(\tau\sum_m \varphi^m(z))}{1-|\tau\sum_m \varphi^m(z)|}\\ &\le \sum_m \frac{Cv_m(r)}{1-r}\|\sum_m \psi^m\|_{v_m}\|\sum_m f_m\|_{v_m}|s-t|,\end{aligned}$$

and

$$\begin{aligned}&\mathcal{J}(r,s,\sum_m f_m)\\ &:= \sup_{|o(z)|>r}\sum_m \frac{|\sum_m \psi^m(z)(\sum_m f_m(s\sum_m \varphi^m(z))-\sum_m f_m(t\sum_m \varphi^m(z)))|}{v_m(z)}\\ &\le \sup_{|p(z)|>r}\sum_m \frac{|\sum_m \psi^m(z)|(|\sum_m f_m(s\sum_m \varphi^m(z))|+|\sum_m f_m(t\sum_m \varphi^m(z))|)}{v_m(z)}\\ &\le \sum_m \|\sum_m f_m\|_{v_m}\sup_{|\sum_m f_m(z)|>r}\frac{|\sum_m \psi^m(z)|(v_m(s\sum_m \varphi^m(z))+v_m(t\sum_m \varphi^m(z)))}{v_m(z)}\\ &\le 2\sum_m \|\sum_m f_m\|_{v_m}\sup_{|\sum_m f_m(z)|>r}\frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}.\end{aligned}$$

Using the above estimates, we obtain

$$\|W_{s\sum_m \psi^m,s\sum_m \varphi^m}-W_{t\sum_m \psi^m,t\sum_m \varphi^m}\|$$
$$\le max\sum_m \left\{\frac{Cv_m(r)}{1-r}\|\sum_m \psi^m\|_{v_m}|s-t|,2\sup_{|\sum_m f_m(z)|>r}\frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}\right\}+M|s-t|$$

for every $r\in(0,1)$ and $s\in[0,1]$. By letting $s\to t$, and then $r\to 1^-$ in the last inequality, we get

$$\limsup_{s\to t}\left\|W_{s\sum_m \phi^m,s\sum_m \varphi^m}-W_{t\sum_m \phi^m,t_{\sum_m \varphi^m}}\right\|\le 2\lim_{r\to 1^-}\sup_{|\sum_m \varphi^m(z)|>r}\sum_m \frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}$$

Moreover, applying Proposition 2.5(b) to the compact operator $W_{v_m,\sum_m \varphi^m}$ on X_{v_m}, we obtain

$$\lim_{|\ell(z)|\to 1^-}\sum_m \frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}=0 \text{ if } X_{v_m}=H_{v_m}(\mathbb{D})$$

or

$$\lim_{|z|\to 1^-}\sum_m \frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}=0 \text{ if } X_{v_m}=H_{v_m}^0(\mathbb{D})$$

which both imply, by Remark 2.3, that

$$\lim_{r\to 1^-}\sup_{|\sum_m \varphi^m(z)|>r}\sum_m \frac{|\sum_m \psi^m(z)|v_m(\sum_m \varphi^m(z))}{v_m(z)}=0$$

Consequently, $\lim_{s\to t}\sum_m \left\|W_{s\sum_m \psi^m,\sum_m \varphi^m}-W_{t\sum_m \psi^m,t_p}\right\|=0$. This establishes the result claimed.

(b) Now we consider the operators $W_{to,(f_m)_0}$ and $C_{\sum_m \varphi_0^m}$, where $\sum_m \psi_0^m(z)=1-z$ and $\sum_m \varphi_0^m(z)=1+a(z-1)$ with $0<a<1$. Obviously, $W_{\text{èo.fo}}$ and $C_{\sum_m \varphi_0^m}$ belong to $\mathcal{C}_{w_m}(X_{v_m})$. However, it is easy to check that $W_{\sum_m \psi_b^m,\sum_m \psi_0^m}$ is compact, while $C_{\sum_m \psi_0^m}$ is not compact on X_{v_m}. Indeed, for all $r\in(0,1)$

$$\sum_m \frac{v_m(\sum_m \varphi_0^m(r))}{v_m(r)}=\sum_m \frac{v_m(1+a(r-1))}{v_m(r)}\ge 1$$

Hence, by Proposition 2.2(b), $C_{\sum_m \varphi_0^m}$ is not compact on X_{v_m}. Next, for any sequence $(z_n)_n$ in $\mathbb{D}$ with $|z_n|\to 1$ as $n\to\infty$, without loss of generality we suppose that $z_n\to\eta\in\partial D$. If $\eta\ne 1$, then $1+a(\eta-1)\in\mathbb{D}$, hence,

$$\sum_m \frac{|\sum_m \psi_0^m(z_n)|v_m(\sum_m \varphi_0^m(z_n))}{v_m(z_n)}\le 2\sum_m \frac{v_m(\sum_m \varphi_0^m(z_n))}{v_m(z_n)}\to 0 \text{ as } n\to\infty$$

If $\eta=1$, then $\sum_m \psi_0^m(z_n)\to 0$ as $n\to\infty$, hence,

$$\sum_m \frac{|\sum_m \psi_0^m(z_n)|v_m(\sum_m \varphi_0^m(z_n))}{v_m(z_n)}\le\sum_m \left|\sum_m \psi_0^m(z_n)\right|\sup_{z\in\mathbb{D}}\frac{v_m(\sum_m \varphi_0^m(z))}{v_m(z)}\to 0 \text{ as } n\to\infty$$

since the last supremum is finite by Proposition 2.2(a). Consequently,

$$\lim_{|z|\to 1^-} \sum_m \frac{|\sum_m \psi_0^m(z)| v_m(\sum_m \varphi_0^m(z))}{v_m(z)} = 0$$

which implies, by Remark 2.3 and Proposition 2.5(b), that $W_{\sum_m \phi_b^m, \sum_m \psi_0^m}$ is compact on X_{v_m}.

It remains to note that, by Lemma 4.1, $W_{\sum_m \psi_0^m, \sum_m \varphi_0^m} \sim C_{\sum_m \varphi_0^m}$ in $\mathcal{C}_{w_m}(X_{v_m})$. From this it follows that the set $\mathcal{C}_{w_m,0}(X_{v_m})$ is not a path component of $\mathcal{C}_{w_m}(X_{v_m})$.

We restate some validity results for weighted composition operator (see [24]).

Proposition 4.3. Let $\sum_m \varphi^m$ and $\sum_m \phi^m$ be two functions in $\mathcal{S}(\mathrm{D})$. If the difference $C_{\sum_m \varphi^m} - C_{\sum_m \phi^m}$ is compact on X_{v_m}, then all the operators $W_{\sum_m \psi^m, \sum_m \varphi^m}$ and $W_{\chi, \sum_m \phi^m}$ from the space $\mathcal{C}_{w_m}(X_{v_m})$ belong to the same path component of this space.

Proof. By Lemma 4.1, $W_{\sum_m \psi^m, \sum_m \varphi^m} \sim C_{\sum_m \varphi^m}$ and $W_{\chi, \sum_m \phi^m} \sim C_{\sum_m \phi^m}$ in $\mathcal{C}_{w_m}(X_{v_m})$. On the other hand, by Theorem 3.1, $C_{\sum_m \phi^m} \sim C_{\sum_m \varphi^m}$ in $\mathcal{C}(X_{v_m})$ and, hence, in $\mathcal{C}_{w_m}(X_{v_m})$. Consequently, $W_{\chi, \sum_m \phi^m} \sim W_{\sum_m \psi^m, \sum_m \varphi^m}$ in $\mathcal{C}_{w_m}(X_{v_m})$.

Remark 4.4. In [6, Theorem 4.2] a similar result to Proposition 4.3 was stated in the setting of the space $C^0_{w_m}(H^0_{v_m}(\mathbb{D}))$ under some additional restrictions on functions $\sum_m \varphi^m, \sum_m \phi^m, \sum_m \psi^m$, and χ that are strictly stronger than the ones in Proposition 4.3. In particular, in this theorem the authors required that $\lim_{|z|\to 1^-} \rho(\sum_m \varphi^m(z), \sum_m \phi^m(z)) = 0$, which implies, by Proposition 2.4, that the difference $C_{\sum_m \varphi^m} - C_{\sum_m \phi^m}$ is a compact operator on X_{v_m}.

For $\sum_m \varphi^m \in \mathcal{S}(\mathbb{D})$, denote by $\mathcal{W}([C_{\sum_m \varphi^m}])$ the set of all weighted composition operators $W_{\sum_m \psi^m, \sum_m \phi^m} \in \mathcal{C}_{w_m}(X_{v_m})$ with $C_{\sum_m \phi^m} \in [C_{\sum_m \varphi^m}]$. The following result follows immediately from Proposition 4.3.

Corollary 4.5. Each set $\mathcal{W}([C_{\sum_m \varphi^m}]), \sum_m \varphi^m \in \mathcal{S}(\mathbb{D})$, is path connected in $\mathcal{C}_{w_m}(X_{v_m})$

Now we show that the sets $\mathcal{W}([C_{\sum_m \varphi^m}])$ may be path components of the space $\mathcal{C}_{w_m}(X_{v_m})$ and may be not. To see this, we consider the next examples (see [24]).

Example 4.6. For $\sum_m \varphi_0^m(z) = 1 + a(z-1)$ with $0 < a < 1$, the set $\mathcal{W}([C_{\sum_m \varphi_0^m}])$ is not a path component of $\mathcal{C}_{w_m}(X_{v_m})$. More precisely, $\mathcal{W}([C_{\sum_m \varphi_0^m}])$ is a proper subset of the path component of $\mathcal{C}_{w_m}(X_{v_m})$ containing $\mathcal{C}_{w_m,0}(X_{v_m})$.

Proof. By part (b) in the proof of Theorem 4.2, the operator $W_{\sum_m \phi_0^m \text{.fo}}$ with $\sum_m \psi_0^m(z) = 1 - z$ and $\sum_m \varphi_0^m(z) = 1 + a(z-1)$ is compact, while $C_{\sum_m \varphi_0^m}$ is not compact on X_{v_m}. Then, by Theorem 4.2, $W_{\text{\psi}_0, (f_m)_0} \sim C_0$ in $\mathcal{C}_{w_m}(X_{v_m})$. But $C_{\sum_m \varphi_0^m} - C_0$ is not compact on X_{v_m}, which implies that the operator C_0 does not belong to $\mathcal{W}\left([C_{\sum_m \varphi_0^m}]\right)$ and completes the proof.

Remark 4.7. The arguments in Example 4.6 work as well for those sets $\mathcal{W}([C_{\sum_m \varphi^m}])$ that generated by $\sum_m \varphi^m \in \mathcal{S}(\mathbb{D})$ with the finite set $E(v_m, \sum_m \varphi^m)$. Thus, all these sets being path connected in the space $\mathcal{C}_{w_m}(X_{v_m})$ are proper subsets of the corresponding path components of $\mathcal{C}_{w_m}(X_{v_m})$ containing $\mathcal{C}_{w_m,0}(X_{v_m})$

Example 4.8. For $\sum_m \varphi_1^m(z) = z$, the set $\mathcal{W}([C_{\sum_m \varphi_1^m}])$ is a path component of $\mathcal{C}_{w_m}(X_{v_m})$

Proof. Obviously, $E(v_m, \sum_m \varphi_1^m) = \partial D$. Hence, by [6, Theorem 5.7], $C_{\sum_m \varphi_1^m}$ is isolated in $\mathcal{C}(X_{v_m})$, which implies that $[C_{\sum_m \varphi_1^m}] = \{C_{\sum_m \varphi_1^m}\}$. From this and Proposition 2.5(a) it follows that

$$\mathcal{W}([C_{\sum_m \varphi_1^m}]) = \left\{W_{\sum_m \psi^m, \sum_m \varphi_1^m} : 0 < \| \sum_m \psi^m \|_\infty < \infty\right\}$$

We will prove that $\mathcal{W}([C_{\sum_m \varphi_1^m}])$ is open and, simultaneously, closed in $\mathcal{C}_{w_m}(X_{v_m})$, from which the assertion follows.

Let $(W_{\sum_m \psi_n^m, \sum_m \varphi_1^m})$ be a sequence in $\mathcal{W}([C_{\sum_m \varphi_1^m}])$ converging to some operator $W_{\chi, \sum_m \phi^m}$ in $\mathcal{C}_{w_m}(X_{v_m})$. Then $W_{\sum_m \psi_n^m, \sum_m \varphi_1^m}(\sum_m f_m) \to W_{\chi, \sum_m \phi^m}(\sum_m f_m)$ in X_{v_m} for all $\sum_m f_m \in X_{v_m}$ Taking here $\sum_m f_m(z) \equiv 1$ and $\sum_m f_m(z) \equiv z$, we obtain that $\sum_m \psi_n^m \to \chi$ and $\sum_m \psi_n^m \sum_m \varphi_1^m \to \chi \sum_m \phi^m$ in X_{v_m} as $n \to \infty$. Therefore,

$$\sum_m \chi\left(\sum_m \varphi_1^m - \sum_m \phi^m\right) = \sum_m \left(\chi - \sum_m \psi_n^m\right) \sum_m \varphi_1^m + \sum_m \left(\sum_m \psi_n^m \sum_m \varphi_1^m - \chi \sum_m \phi^m\right) \to 0 \text{in}\, X$$

Since $\chi \not\equiv 0$, this implies that $\sum_m \phi^m = \sum_m \varphi_1^m$. Thus, the set $\mathcal{W}([C_{\sum_m \varphi_1^m}])$ is closed in $\mathcal{C}_{w_m}(X_{v_m})$. The fact that it is open in $\mathcal{C}_{w_m}(X_{v_m})$ follows immediately from the following auxiliary lemma, in which we will use the next notation: $F(\sum_m \psi^m, \varepsilon) := \{\omega \in \partial\mathrm{D} : |\sum_m \psi^m(\omega)| \geq \varepsilon\}$ and $\| \sum_m \psi^m \|_e := \inf\{\varepsilon > 0 : |F(\sum_m \psi^m, \varepsilon)| = 0\}$

Lemma 4.9 (see [24]). Let $W_{\sum_m \phi^m, \sum_m \sum_m \varphi_1^m}$ be an operator in $\mathcal{W}([C_{\sum_m \varphi^m 1}])$. Then, for every operator $W_{\chi,\sum_m \phi^m}$ in $\mathcal{C}_{w_m}(X_{v_m})$ with $\sum_m \phi^m \neq \sum_m \varphi_1^m$,

$$\|W_{\sum_m \psi^m, \sum_m \varphi_1^m} - W_{\chi,\sum_m \phi^m}\| \geq \sum_m \| \sum_m \psi^m \|_e$$

Proof. Since $\sum_m \psi^m$ is a nonzero function, $\| \sum_m \psi^m \|_e > 0$. Take an arbitrary number $r \in (0, \| \sum_m \psi^m \|_c)$. Then $|F(\sum_m \psi^m, r)| > 0$.

Since $\sum_m \phi^m \neq \sum_m \varphi_1^m$, $|\{\omega \in \partial\mathbb{D}: \sum_m \phi^m(\omega) = \omega\}| = 0$. So there exist a point $\omega \in F(\sum_m \psi^m, r)$ and a sequence $(z_n) \subset \mathbb{D}$ such that $z_n \to \omega$, $|\sum_m \psi^m(z_n)| \to |\sum_m \psi^m(\omega)| \geq r$, and $\sum_m \phi^m(z_n) \to \eta \neq \omega$. Then $\sum_m \rho(z_n, \sum_m \phi^m(z_n)) \to 1$ as $n \to \infty$.

Next, by [3, Subsection 1.2(iv), Theorem 1.13 and comments after it], for each $n \in \mathbb{N}$, there is a function $(f_m)_n$ in the unit ball of X_{v_m} such that $(f_m)_n(z_n) = \tilde{v_m}(z_n)$ (recall that by $\tilde{v_m}$ it is denoted the weight associated with v_m). We put

$$h_n(z) = \sum_m (f_m)_n(z) \frac{z - \sum_m \phi^m(z_n)}{1 - z\sum_m \bar{\phi}^m(z_n)}, z \in \mathbb{D}.$$

Then $h_n \in X_{v_m}$ with $\|h_n\|_e \leq 1$ for all n. Taking into account that $\sum_m W_{\sum_m \psi^m, \sum_m \varphi_1^m} h_n(z_n) = \sum_m \psi^m(z_n)\rho(z_n, \sum_m \phi^m(z_n))\widetilde{v_m}(z_n)$ and $\sum_m W_{\chi,\sum_m \phi^m} h_n(z_n) = 0$, we get

$$\sum_m \|W_{\sum_m \phi^m, \sum_m \varphi_1^m} - W_{\chi,\sum_m \phi^m}\| \geq \sum_m \|W_{\sum_m \psi^m, \sum_m \varphi_1^m} h_n - W_{\chi,\sum_m \phi^m} h_n\|_j$$

$$\geq \sum_m \frac{|W_{\sum_m \phi^m, \sum_m \varphi_1^m} h_n(z_n) - W_{\chi,\sum_m \phi^m} h_n(z_n)|}{\tilde{v_m}(z_n)} = \sum_m \left|\sum_m \psi^m(z_n)\right| \rho\left(z_n, \sum_m \phi^m(z_n)\right)$$

for all $n \in \mathbb{N}$. Thus,

$$\sum_m \|W_{\sum_m \phi^m, \sum_m \varphi_1^m} - W_{\chi,\sum_m \phi^m}\| \geq \limsup_{n\to\infty} \sum_m \left|\sum_m \psi^m(z_n)\right| \rho\left(z_n, \sum_m \phi^m(z_n)\right) \geq r$$

and, consequently, $\sum_m \|W_{v_m, \sum_m \varphi_1^m} - W_{\chi,\sum_m \phi^m}\| \geq \sum_m \| \sum_m \psi^m \|_e$.

Some of the arguments used in the proof of Example 4.8 work as well for any isolated operator $C_{\sum_m \varphi^m}$ in $C(X_{v_m})$. More precisely, by the same reasons as in this example, one can easily check that the corresponding sets $\mathcal{W}([C_{\sum_m \varphi^m}]) = \mathcal{W}(\{C_{\sum_m \varphi^m}\})$ are all closed in the space $\mathcal{C}_{w_m}(X_{v_m})$. Moreover, by Corollary 4.5 they are path connected in this space. They may be also open in $\mathcal{C}_{w_m}(X_\varepsilon)$.

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