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Review Paper

Analytical Solution of a Steady and Transient Single Phase Flow Equation in a Petroleum Reservoir

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Abstract
Industrial revolution had steadily mounted pressure on the demand for oil and its subproducts, which had necesitated the study and understanding ofoilfields and oil exploration processes. In order to increase production, information obtained from fluid filtration process as well as pressure drop profile using appropriate mathematical model is essential.This article investigatedthe presure distribution when a single phase fluid flow through a horizontal homogeneous porous medium. The scenario was represented by a partial dfferental equation which we solved analytically, using seperation of variables techniques. We first obtained the steady state solution and there after, the transient solution. Our result, from the general solution obtained revealed that as time becomes very large, the transient solution reduces to the steady state solution.

Keywords*: Single phase flow, steady state, pressure profile, differential equation*

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I. INTRODUCTION

Fluid flow and heat transfer through porous channels with constatn and variable cross sections have important applications in many fields of engineering such as underground water flow, filtration and purification processes, petroleum exploration studies and geological studies just to mention a few. Fluid flow scenarios through porous media are modelled using partial differential equations.

Partial differential equation (pde) is a fundamental equation not only for fluid flow processes, but also used in modelling physical processess ranging from those in Chemistry, physics, biology, finance, and engineeringKim, Y., Gostick, J.T. (2019) . In fluid mechanics, Partial differential equations are vital modelling tools in oil wells and oil flow processes. It is used in the simulation of fluid flow through porous materials.

Darcy's law is very reliabe in describing a linear relationship between volumetric flow rate and pressure gradient and it has been the fundamental principle in flow and transport processes in porous media (Muskat, 1946)

Drilona et.al (2024) analyed oil reservoir dynamics and determine the pressure drop in the near-wellbore subject to constant oil production. Their findings underscore the relevance of depletion time, production rate, and reservoir radius in calculating pressure drops. Due to the complexities of the porous media structure, modelling the phenomena of fluid flow through porous structure is very challenging (Zuonaki and Adokiye, 2024; Komal *et.al*, 2023; Vincent *et.al* 2022; Pan and Miller 2003; Nagi, 2009).This article investigatedthe presure distribution when a single phase fluid flows through a horizontal homogeneous porous medium. The scenario was represented by a partial dfferental equation which we solved analytically, using seperation of variables techniques.

II. MATHEMATICAL FORMULATION

Consider a simple horizonal channel containing homogeneous porous material shown in figure 1, having an initial pressure of P_o with the pressure at $x = 0$,given as P_L
while the pressure at x= L is P_g

Figure 1: Horizontal Porous Channel

Also, consider a single phase liquid flowing through the channel from left to right, assuming constant viscosity, constant permeability, constant porosity and compressibility. The partial differential equation which best describes the given scenario is modelled as

$$\frac{\partial^2 P}{\partial x^2} = \left(\frac{\phi\mu c}{k}\right)\frac{\partial P}{\partial t} \qquad (1)$$

where $P = P(x,t)$ is the pressure, ϕ is the porosity, μ is the viscosity c is the compressibility and k, the permeability.

Equation (1) obviously is a time dependent or transient flow. Now suppose the flow conditions no longer depend on time, it is classified to as steady flow. The equation (1) is reduced to

$$\frac{\partial^2 P}{\partial x^2} = 0 \qquad (2)$$

It is our aim to solve equations (1) and equation (2) and determine the pressure distributions associated with both of them. We solve equation (2) first since it is simple.

$$\frac{\partial^2 P}{\partial x^2} = 0$$

by twice integration, we have

$$P(x) = Cx + D$$

using the boundary condition $P(x=0) = P_L$

implies $P(x=0) = D = P_L$

and for $P(x=L) = P_g$, we have

$$P(x=L) = CL + D = P_g$$

$$CL + P_L = P_g$$

$$\therefore \ \mathrm{C} = \frac{P_g - P_L}{L}$$

Thus $P(x) = P_L + \left(P_g - P_L\right)\dfrac{x}{L}$ (3)

Equation (3) is the steady state solution of equation (1).

Now, we provide a detailed solution for the transient or time dependent flow given in equation (1) by the method of seperation of variables.

$$\frac{\partial^2 P}{\partial x^2} = \left(\frac{\phi\mu c}{k}\right)\frac{\partial P}{\partial t}$$

$$\frac{\partial^2 P}{\partial x^2} = \alpha\frac{\partial P}{\partial t} \quad \left[\alpha = \frac{\phi\mu c}{k}\right], \quad \left(0 < x < L, t > 0\right) \qquad (4)$$

subject to the following boundary/initial conditions

$$\left.\begin{aligned} P(0,t) &= P_L \quad (t>0) \\ P(L,t) &= P_g \quad (t>0) \\ P(x,0) &= P_0 \quad (t=0) \end{aligned}\right\} \qquad (5)$$

Let $P(x,t) = X(x)T(t)$ (6)

be the solution to equation (4), satisfying equation (5)

$$X''T = \alpha X\dot{T}$$

$$\frac{X''}{X} = \alpha\frac{\dot{T}}{T} = k$$

Using the boundary conditions (5), we have:

$$X'' - kX = 0 \quad (7)$$

$$\dot{T} - \frac{k}{\alpha}T = 0 \text{ or } \dot{T} - k\alpha' T = 0 \quad [\alpha' = \alpha^{-1}] \quad (8)$$

The only realistic solution to equation (7) is when

$$k = -\lambda_n^2 = -\left(\frac{n\pi}{L}\right)^2$$

$$\Rightarrow X = X_n = \sin(\lambda_n x) = \sin\frac{n\pi}{L}x, \quad n \in N \quad (8)$$

Also for T; we have

$$\dot{T} - \alpha' kT = 0 \text{ or } \dot{T} + \left(\sqrt{\alpha'}\frac{n\pi}{L}\right)^2 T = 0$$

$$\Rightarrow \dot{T} + \lambda_n^* T = 0 \text{ where } \lambda_n^* = \left(\sqrt{\alpha'}\frac{n\pi}{L}\right)^2$$

$$\dot{T} = -\lambda_n^* T \Rightarrow T = T_n = c_n e^{-\lambda_n^{*2} t}$$

Thus the normal mode of the one dimensional single phase solution (1) becomes

$$P_n(x,t) = X_n(x)T_n(t) = c_n e^{-\lambda_n^{*2} t}\sin\lambda_n x$$

Now, by the principle of superposition;

$$P(x,t) = \sum_{n=1}^{\infty} P_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^{*2} t}\sin\lambda_n x \quad (9)$$

using $P(x,0) = P_0$ in (9)

$$P_0 = \sum_{n=1}^{\infty} c_n \sin\lambda_n x$$

and using Fourier series, we have

$$c_n = \frac{2}{L}\int_0^L P_0 \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{2}{L}P_0\frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]\Bigg|_0^L$$

$$= -\frac{2}{n\pi}P_0 - \frac{2}{n\pi}P_0\left[(-1)^n - 1\right]$$

$$= \left.\begin{matrix} \frac{4}{n\pi}P_0, & \text{for } n = \text{odd} \\ 0. & \text{for } n = \text{even} \end{matrix}\right\}$$

Thus the transient solution (9) is given as

$$P(x,t)=\sum_{n=1}^{\infty}c_n\exp\left(-\frac{n^2\pi^2}{L^2}\frac{k}{\phi\mu c}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$=\sum_{n=1}^{\infty}\frac{4P_0}{n\pi}\exp\left(-\frac{n^2\pi^2}{L^2}\frac{k}{\phi\mu c}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$=\frac{4P_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{n^2\pi^2}{L^2}\frac{k}{\phi\mu c}t\right)\sin\left(\frac{n\pi}{L}x\right) \qquad (10)$$

Therefore the analytical solution of the time-dependent pressure profile through a horizontal homogeneous porous medium is given by

$$P(x,t)=P(x)=P_L+\left(P_g-P_L\right)\frac{x}{L}+\left[\frac{4P_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{n^2\pi^2}{L^2}\frac{k}{\phi\mu c}t\right)\sin\left(\frac{n\pi}{L}x\right)\right] \qquad (11)$$

Observed that at time becomes very large, the term in the closed bracket approaches zero and equation (11) becomes equation (3); ie the steady state solution

$$P(x,t)=P(x)=P_L+\left(P_g-P_L\right)\frac{x}{L}$$

To this end, we will the adapt fundamental rule of mass conservation and Darcy equation for each phase as well as their constitutive relations. We begin with the modeling of single phase mass conservation equation in a petroleum reservoir.

III. Discussion

We considered a single phase fluid, flowing through a horizontal porous structure which is homogeneous. We assumed that the fluid is incompressible, the porosity and the permeability of the channel are constant. The scenario is captured using partial differental equation which is a vital tool for modelling such processes. We solved the modelled equation analytically using seperation of variables technique and determined the steady state solution as well as the transient solution. Our results, revealed that as time becomes very large, the transient solution reduces to the steady state solution.

IV. Conclusion

Analytical solution of both steady and transient single-phase flow equations in a petroleum reservoir provides significant insights into reservoir behavior, which is crucial for effective reservoir management and optimization of production strategies. The steady-state flow analysis offers a simplified view of reservoir conditions, assuming constant fluid properties and flow rates, thereby helping to predict long-term production trends under stable conditions. On the other hand, the transient flow analysis addresses the more complex and realistic variations in pressure and flow over time, capturing the effects of reservoir depletion and fluid dynamics in the early stages of production. Additionally, the ability to predict pressure transient behaviors allows for improved well testing, reservoir characterization, and more accurate forecasting of recovery. Despite their assumptions and simplifications, these analytical models serve as powerful tools in the absence of extensive field data, guiding decision-making and helping to optimize resource extraction in petroleum reservoirs.

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