Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 4 (April 2025) pp: 97-110
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Piecewise fractional interpolation with application to the Coupled System of Nonlinear Fractional Ordinary Differential Equations

Yinghong Zhang[1] Qingzhu Wang[2]
[1,2]School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, China.

Abstract: *In this paper, we consider the numerical method for the fractional differential equations. It is based on the piecewise fractional interpolants (PFIs), which is a new family of interpolants. We extend the technique to solve the Coupled system of fractional ordinary differential equations (FODEs) and present a predictor-corrector methods for the numerical solution of the coupled system of FODEs. By using the present method, we are able to solve efficiently the case that the solution of FODEs has lower smoothness. The stability and convergence of the present method is rigorously established. Some numerical examples are provided to confirm the theoretical claims.*

Key words: *Nonlinear coupled fractional differential equations; piecewise fractional interpolants; predictor-corrector method; error analysis; stability analysis.*

***R**eceived 20 Apr., 2025; **R**evised 28 Apr., 2025; **A**ccepted 02 May., 2025*

I. Introduction:

In recent years, fractional calculus has developed rapidly in various scientific fields, and plays an important role in signal processing, anomalous diffusion, physics and engineering (see[1][2][3][4][5]). Fractional differential equations can describe various complex physical and mechanical behaviors, and many phenomena in life can also be described by fractional differential equations, such as the problem of blood alcohol concentration, the problem of video tapes, and the problem of world population growth (see[6] and references therein). So solving its numerical solution has practical significance. The exact solutions of many fractional differential equations cannot be obtained exactly, so the research on numerical algorithms of fractional differential equations has attracted much attention. At present, some progress has been made in the research of numerical algorithms for fractional differential equations, and they are gradually applied in different fields such as mechanics, viscoelasticity [7], biology [8], and simulated fluid flow [4] and so on.

There are some existing methods for solving the coupled system of nonlinear fractional ordinary differential equations. For example, the methods in [9] constructed a Legendre spectral collocation method for coupled systems of nonlinear fractional differential equations. The linear incompatibility fractional differential equations with caputo derivatives two-dimensional coupled systems and corresponding inhomogeneous systems have been considered in [10]. It proposed a high-order scheme for numerical solutions of the coupled system of FODEs (cf. [11]). In [12], it proved the existence and uniqueness of a class of implicit nonlinear coupling systems of fractional differential equations under non-local conditions. In [13], it constructed legend-jacobi spectral configuration method to solve two point boundary value problems nonlinear systems with the fractional derivative orders at most two. It analyzed the stability and asymptotic stability of systems of nonlinear non-autonomous equations with generalized proportional caputo fractional derivatives in [14]. With the terminal problem of nonlinear systems of fractional differential equations, it was mentioned in [15], and so on.

It is well known that the predictor-corrector method plays an important role in solving differential equations, such as, a new finite-difference predictor-corrector method was proposed in [16] to solve nonlinear fractional differential equations (FDEs) and further extended to the system of FDEs. In [17], it constructed second-order linear interpolation and third-order quadratic interpolation predictor-corrector methods to solve fractional-order nonlinear differential equations. A numerical method for predictor and corrector of fractional differential equations based on Newton interpolation was proposed in [18]. For a general right hand side function, especially the function is nonlinear, it is usually difficult to obtain the analytical solution for systems of fractional differential equations. In addition, due to the singularity in some fractional differential equations, the polynomial cannot capture the singular term, especially when the solution has a low smoothness, which prompts us to propose a new numerical solution to solve this problem.

The outline of the paper is as follows. In Section 2, we describe in detail the predictor-corrector algorithm based on PFIs. Then in Section 3, we give the stability and convergence analysis. In Section 4, we verify the feasibility of the method through some numerical examples. In Section 5 gives some concluding remarks. The final research questions of this paper are as follows

$$\begin{cases} {}_0^C D_t^\alpha u(t) = f(t,u(t),v(t)) & \alpha > 0 \\ {}_0^C D_t^\alpha v(t) = g(t,u(t),v(t)) & \alpha > 0 \\ u^{(k)}(0) = u_0^{(k)}, \quad v^{(k)}(0) = v_0^{(k)}, \quad k = \lceil \alpha \rceil - 1 \end{cases} \tag{1.1}$$

II. A new predictor-corrector Scheme

2.1 Preliminary knowledge

Start with some simple knowledge points.

The caputo fractional derivative is defined by

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad 0 \le n-1 \le \alpha \le n, \tag{2.1}$$

where $\Gamma(.)$ is the Gamma function.

Incomplete beta function $I_t(a,b)$ is defined by

$$I_t(a,b) = \frac{1}{B(a,b)} \int_0^t s^{a-1}(1-s)^{b-1} ds, a,b > 0, 0 < t < 1, \tag{2.2}$$

here $B(a,b)$ is the beta function.

It is well-known that the (1.1) is equivalent to

$$\begin{cases} u(t) = g_1(t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau,u(\tau),v(\tau)) d\tau \cdot \\ v(t) = g_2(t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} g(\tau,u(\tau),v(\tau)) d\tau \cdot \\ g_1(t) = \displaystyle\sum_{k=0}^{n-1} u_0^{(k)} \frac{t^k}{k!}, \ g_2(t) = \displaystyle\sum_{k=0}^{n-1} v_0^{(k)} \frac{t^k}{k!}, \quad 0 < n-1 \le \alpha \le n \end{cases} \tag{2.3}$$

We investigate a new approach to problem (1.1). The predictor-corrector scheme is still used, but the difference from the classic predictor-corrector scheme is that we no longer use the original Lagrangian interpolation polynomial, but use a new fractional interpolation family, with the regularity behavior that is exactly adapted to that of the right-hand side of (1.1) near the initial point t=0. To construct the fractional interpolation function family, it is first necessary to know the singular exponent of the caputo fractional derivative, assuming that the ${}_0^C D_t^\alpha u, {}_0^C D_t^\alpha v$ behaves as

$$\begin{aligned} {}_0^C D_t^\alpha u(t) &:= f(t) = \sum_{j=1}^{\infty} d_j t^{\beta_j}, \quad 0 \le \beta_1 \le \beta_2 ..., \quad \beta_j \in \mathbb{R}, \quad j = 1,2,.. \\ {}_0^C D_t^\alpha v(t) &:= g(t) = \sum_{j=1}^{\infty} h_j t^{\gamma_j}, \quad 0 \le \gamma_1 \le \gamma_2 ..., \quad \gamma_j \in \mathbb{R}, \quad j = 1,2,... \end{aligned} \tag{2.4}$$

where d_j, h_j are some constant. The parameter β_j, γ_j are called the singular exponent of the function f, g. According to ([19], Theorem 2.5), the equation system (1.1) has a singular exponent on the right side of the equation. In [20], a pre-algorithm method was proposed, which can accurately find $\beta_1, \beta_2, \beta_3, \beta_4$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. This paper is based on this pre-algorithm method to determine the singularity index of the equation and then carry out related work.

Piecewise fractional interpolation is the core idea of this paper, and the following piecewise fractional interpolation function is constructed.

For $t \in [t_j, t_{j+1}],\ j = 1, 2, ..., n\text{-}1$, there is the following fractional interpolation function

$$\begin{aligned}\tilde{f}_j(t,u(t),v(t)) &= (\frac{t}{t_j})^{\beta 1}(\frac{t^{\beta_2-\beta_1}-t_{j+1}^{\beta_2-\beta_1}}{t_j^{\beta_2-\beta_1}-t_{j+1}^{\beta_2-\beta_1}})f(t_j,u_j,v_j)\\ &+(\frac{t}{t_{j+1}})^{\beta 1}(\frac{t^{\beta_2-\beta_1}-t_j^{\beta_2-\beta_1}}{t_{j+1}^{\beta_2-\beta_1}-t_j^{\beta_2-\beta_1}})f(t_{j+1},u_{j+1},v_{j+1})\\ \tilde{g}_j(t,u(t),v(t)) &= (\frac{t}{t_j})^{\gamma 1}(\frac{t^{\gamma_2-\gamma_1}-t_{j+1}^{\gamma_2-\gamma_1}}{t_j^{\gamma_2-\gamma_1}-t_{j+1}^{\gamma_2-\gamma_1}})g(t_j,u_j,v_j)\\ &+(\frac{t}{t_{j+1}})^{\gamma 1}(\frac{t^{\gamma_2-\gamma_1}-t_j^{\gamma_2-\gamma_1}}{t_{j+1}^{\gamma_2-\gamma_1}-t_j^{\gamma_2-\gamma_1}})g(t_{j+1},u_{j+1},v_{j+1})\end{aligned} . \tag{2.5}$$

For $t \in [t_0, t_1]$, there is the following fractional interpolation function

$$\begin{aligned}\tilde{f}_0(t,u(t),v(t)) &= t^{\beta 1}(\frac{t^{\beta_2-\beta_1}-t_1^{\beta_2-\beta_1}}{t_0^{\beta_2-\beta_1}-t_1^{\beta_2-\beta_1}})\lim_{x\to 0^+}\frac{f(t,u(t),v(t))}{t^{\beta 1}}\\ &+(\frac{t}{t_1})^{\beta 1}(\frac{t^{\beta_2-\beta_1}-t_0^{\beta_2-\beta_1}}{t_1^{\beta_2-\beta_1}-t_0^{\beta_2-\beta_1}})f(t_1,u_1,v_1)\\ \tilde{g}_0(t,u(t),v(t)) &= t^{\gamma 1}(\frac{t^{\gamma_2-\gamma_1}-t_1^{\gamma_2-\gamma_1}}{t_0^{\gamma_2-\gamma_1}-t_1^{\gamma_2-\gamma_1}})\lim_{x\to 0^+}\frac{g(t,u(t),v(t))}{t^{\lambda 1}}\\ &+(\frac{t}{t_1})^{\gamma 1}(\frac{t^{\gamma_2-\gamma_1}-t_0^{\gamma_2-\gamma_1}}{t_1^{\gamma_2-\gamma_1}-t_0^{\gamma_2-\gamma_1}})g(t_1,u_1,v_1)\end{aligned} . \tag{2.6}$$

For $t \in [0,T]$, $f(t),\ g(t)$ can be approximated by the following fractional interpolation function $p_n^1(t),\ p_n^2(t)$ respectively

$$\begin{aligned}p_n^1(t) &= \begin{cases} c_{10}^1 t^{\beta_1} + c_{20}^1 t^{\beta_2}, & t \in [t_0, t_1] \\ \vdots & \\ c_{1n-1}^1 t^{\beta_1} + c_{2n-1}^1 t^{\beta_2}, & t \in [t_{n-1}, t_n] \end{cases}\\ p_n^2(t) &= \begin{cases} c_{10}^2 t^{\gamma_1} + c_{20}^2 t^{\gamma_2}, & t \in [t_0, t_1] \\ \vdots & \\ c_{1n-1}^2 t^{\gamma_1} + c_{2n-1}^2 t^{\gamma_2}, & t \in [t_{n-1}, t_n] \end{cases}\end{aligned} , \tag{2.7}$$

where the coefficient $\left\{c_{1j}^i, c_{2j}^i\right\}_{j=1}^{n-1}, i = 1, 2$ is obtained by solving the following equations

$$\begin{aligned}&\begin{bmatrix} t_j^{\beta_1} & t_j^{\beta_2} \\ t_{j+1}^{\beta_1} & t_{j+1}^{\beta_2} \end{bmatrix}\begin{bmatrix} c_{1j}^1 \\ c_{2j}^1 \end{bmatrix} = \begin{bmatrix} f(t_j, u(t_j), v(t_j)) \\ f(t_{j+1}, u(t_{j+1}), v(t_{j+1})) \end{bmatrix},\ j = 1, ..., n-1\\ &\begin{bmatrix} t_j^{\gamma_1} & t_j^{\gamma_2} \\ t_{j+1}^{\gamma_1} & t_{j+1}^{\gamma_2} \end{bmatrix}\begin{bmatrix} c_{1j}^2 \\ c_{2j}^2 \end{bmatrix} = \begin{bmatrix} g(t_j, u(t_j), v(t_j)) \\ g(t_{j+1}, u(t_{j+1}), v(t_{j+1})) \end{bmatrix},\ j = 1, ..., n-1\end{aligned} , \tag{2.8}$$

and for c_{10}^i, c_{20}^i, it is necessary to solve the following equations separately

$$\begin{bmatrix} 1 & t_0^{(\beta_2-\beta_1)} \\ t_1^{\beta_1} & t_1^{\beta_2} \end{bmatrix}\begin{bmatrix} c_{10}^1 \\ c_{20}^1 \end{bmatrix} = \begin{bmatrix} \lim\limits_{x\to 0^+} \dfrac{f(t,u(t),v(t))}{t^{\beta_1}} \\ f_i(t_1,u(t_1),v(t_1)) \end{bmatrix}$$
$$\begin{bmatrix} 1 & t_0^{(\gamma_2-\gamma_1)} \\ t_1^{\gamma_1} & t_1^{\gamma_2} \end{bmatrix}\begin{bmatrix} c_{10}^2 \\ c_{20}^2 \end{bmatrix} = \begin{bmatrix} \lim\limits_{x\to 0^+} \dfrac{g(t,u(t),v(t))}{t^{\gamma_1}} \\ g(t_1,u(t_1),v(t_1)) \end{bmatrix}. \quad (2.9)$$

It is evident from (2.4) and (2.7) that the regularity behavior of $p_n^1(t), p_n^2(t)$ is exactly the same as that of the function $f(t), g(t)$.

2.2 Description of the predictor-corrector scheme

We will use the piecewise fractional interpolation function defined above to construct a predictor-corrector algorithm for solving the numerical solution of (1.1). Let $u_j, v_j, j = 0,1,...,k$ be the approximate solution of $u(t_j), v(t_j)$, and assume that u_k, v_k has been solved, now we discuss the solution process of u_{k+1}, v_{k+1}.

Firstly, the grid is divided uniformly $0 = t_0 < t_1 < ... < t_n = T$, and it satisfies

$$\tau_j = t_{1+j} - t_j = (j+1)\mu\ , \quad \mu = \frac{2T}{n(n+1)}\ , \quad j = 0,1,...,n-1. \quad (2.10)$$

Let us consider the following definite integral

$$I_{k+1} = \int_0^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} f(\tau,u(\tau),v(\tau))d\tau,\ \ k = 0,...,n-1$$
$$L_{k+1} = \int_0^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-1} g(\tau,u(\tau),v(\tau))d\tau,\ \ k = 0,...,n-1, \quad (2.11)$$

an approximation to this definite integral

$$I_{k+1} \approx \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-\tau)^{\alpha-1} \tilde{f}_j(\tau,u(\tau),v(\tau))d\tau$$
$$L_{k+1} \approx \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-\tau)^{\alpha-1} \tilde{g}_j(\tau,u(\tau),v(\tau))d\tau, \quad (2.12)$$

functions $\tilde{f}_j(\tau,u(\tau),v(\tau))$ and $\tilde{g}_j(\tau,u(\tau),v(\tau))$ on interval $[t_j, t_{j+1}]$ are approximations of functions f and g respectively, so choosing different $\tilde{f}_j(\tau,u(\tau),v(\tau))$ and $\tilde{g}_j(\tau,u(\tau),v(\tau))$ will lead to different approximation schemes.

Let $\tilde{p}_n^i(t), i = 1,2$ be the piecewise fraction function as follows

$$\tilde{p}_n^1(t) = \begin{cases} \tilde{c}_{10}^1 t^{\beta_1} + \tilde{c}_{20}^1 t^{\beta_2}, & t \in [t_0, t_1] \\ \vdots & \\ \tilde{c}_{1n-1}^1 t^{\beta_1} + \tilde{c}_{2n-1}^1 t^{\beta_2}, & t \in [t_{n-1}, t_n] \end{cases}$$
$$\tilde{p}_n^2(t) = \begin{cases} \tilde{c}_{10}^2 t^{\gamma_1} + \tilde{c}_{20}^2 t^{\gamma_2}, & t \in [t_0, t_1] \\ \vdots & \\ \tilde{c}_{1n-1}^2 t^{\gamma_1} + \tilde{c}_{2n-1}^2 t^{\gamma_2}, & t \in [t_{n-1}, t_n] \end{cases}, \quad (2.13)$$

where the coefficient $\left\{\tilde{c}_{1j}^i, \tilde{c}_{2j}^i\right\}_{j=1}^{k}, i = 1,2$ is obtained by solving the following equations

$$\begin{bmatrix} t_j^{\beta_1} & t_j^{\beta_2} \\ t_{j+1}^{\beta_1} & t_{j+1}^{\beta_2} \end{bmatrix}\begin{bmatrix} \tilde{c}_{1j}^1 \\ \tilde{c}_{2j}^1 \end{bmatrix} = \begin{bmatrix} f(t_j,u_j,v_j) \\ f(t_{j+1},u_{j+1},v_{j+1}) \end{bmatrix}, \ j=1,...,k$$
$$\begin{bmatrix} t_j^{\gamma_1} & t_j^{\gamma_2} \\ t_{j+1}^{\gamma_1} & t_{j+1}^{\gamma_2} \end{bmatrix}\begin{bmatrix} \tilde{c}_{1j}^2 \\ \tilde{c}_{2j}^2 \end{bmatrix} = \begin{bmatrix} g(t_j,u_j,v_j) \\ g(t_{j+1},u_{j+1},v_{j+1}) \end{bmatrix}, \ j=1,...,k \quad , \qquad (2.14)$$

and for $\tilde{c}_{10}^i, \tilde{c}_{20}^i, i=1,2$, need to solve the following equations to get

$$\begin{bmatrix} 1 & 0 \\ t_1^{\beta_1} & t_1^{\beta_2} \end{bmatrix}\begin{bmatrix} \tilde{c}_{10}^1 \\ \tilde{c}_{20}^1 \end{bmatrix} = \begin{bmatrix} \lim\limits_{t\to 0^+} \dfrac{f(t,u(t),v(t))}{t^{\beta_1}} \\ f(t_1,u_1,v_1) \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ t_1^{\gamma_1} & t_1^{\gamma_2} \end{bmatrix}\begin{bmatrix} \tilde{c}_{10}^2 \\ \tilde{c}_{20}^2 \end{bmatrix} = \begin{bmatrix} \lim\limits_{t\to 0^+} \dfrac{g(t,u(t),v(t))}{t^{\gamma_1}} \\ g(t_1,u_1,v_1) \end{bmatrix} . \qquad (2.15)$$

Let $\tilde{f}_j(t,u(t),v(t)) = \tilde{c}_{1j}^1 t^{\beta_1} + \tilde{c}_{2j}^1 t^{\beta_2}$, $\tilde{g}_j(t,u(t),v(t)) = \tilde{c}_{1j}^2 t^{\gamma_1} + \tilde{c}_{2j}^2 t^{\gamma_2}$, $j=0,1,...,k$ and brought into (2.12), the following implicit scheme can be obtained

$$u_{k+1} = g_1(t_{k+1}) + \sum_{j=0}^{k} (\tilde{c}_{1j}^1 H_{1j}^{1k} + \tilde{c}_{2j}^1 H_{2j}^{1k}), k=0,1,...,n-1$$
$$v_{k+1} = g_2(t_{k+1}) + \sum_{j=0}^{k} (\tilde{c}_{1j}^2 H_{1j}^{2k} + \tilde{c}_{2j}^2 H_{2j}^{2k}), k=0,1,...,n-1 \quad , \qquad (2.16)$$

The coefficient $H_{lj}^{ik}, l=1,2, \ i=1,2$ is obtained by the following formula

$$\begin{aligned} H_{lj}^{1k} &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}} (t_{k+1} - s)^{\alpha-1} s^{\beta_l} ds \\ &= \frac{t_{k+1}^{\alpha+\beta_l}}{\Gamma(\alpha)} B(1+\beta_l,\alpha)(I_{\frac{t_{j+1}}{t_{k+1}}}(1+\beta_l,\alpha) - I_{\frac{t_j}{t_{k+1}}}(1+\beta_l,\alpha)) \\ H_{lj}^{2k} &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}} (t_{k+1} - s)^{\alpha-1} s^{\gamma_l} ds \\ &= \frac{t_{k+1}^{\alpha+\gamma_l}}{\Gamma(\alpha)} B(1+\gamma_l,\alpha)(I_{\frac{t_{j+1}}{t_{k+1}}}(1+\gamma_l,\alpha) - I_{\frac{t_j}{t_{k+1}}}(1+\gamma_l,\alpha)) \end{aligned} . \qquad (2.17)$$

Now, to arrive at an explicit scheme, we let the interpolation function used in the interval $[t_k, t_{k+1}]$ be the same as that used in the previous subinterval $[t_{k-1}, t_k]$, and an explicit scheme can be obtained as follows

$$u_{k+1} = g_1(t_{k+1}) + \sum_{j=0}^{k-1} (\tilde{c}_{1j}^1 H_{1j}^{1k} + \tilde{c}_{2j}^1 H_{2j}^{1k}) + \tilde{c}_{1k-1}^1 H_{1k}^{1k} + \tilde{c}_{2k-1}^1 H_{2k}^{1k}, k=0,1,...,n-1$$
$$v_{k+1} = g_2(t_{k+1}) + \sum_{j=0}^{k-1} (\tilde{c}_{1j}^2 H_{1j}^{2k} + \tilde{c}_{2j}^2 H_{2j}^{2k}) + \tilde{c}_{1k-1}^2 H_{1k}^{2k} + \tilde{c}_{2k-1}^2 H_{2k}^{2k}, k=0,1,...,n-1 \qquad (2.18)$$

Since (2.16) and (2.18) we can get a prediction-correction scheme to calculate $u_{k+1}, v_{k+1}, k=1,...,n-1$.

(1) Prediction stage

$$u_{k+1}^p = g_1(t_{k+1}) + \sum_{j=0}^{k-1} (\tilde{c}_{1j}^1 H_{1j}^{1k} + \tilde{c}_{2j}^1 H_{2j}^{1k}) + \tilde{c}_{1k-1}^1 H_{1k}^{1k} + \tilde{c}_{2k-1}^1 H_{2k}^{1k}$$
$$v_{k+1}^p = g_2(t_{k+1}) + \sum_{j=0}^{k-1} (\tilde{c}_{1j}^2 H_{1j}^{2k} + \tilde{c}_{2j}^2 H_{2j}^{2k}) + \tilde{c}_{1k-1}^2 H_{1k}^{2k} + \tilde{c}_{2k-1}^2 H_{2k}^{2k}$$

where the coefficient $\tilde{c}_{1k}^{ip}, \tilde{c}_{2k}^{ip}, i=1,2$ is obtained by solving the following system of equations

$$\begin{bmatrix} t_k^{\beta_1} & t_k^{\beta_2} \\ t_{k+1}^{\beta_1} & t_{k+1}^{\beta_2} \end{bmatrix} \begin{bmatrix} \tilde{c}_{1k}^{1p} \\ \tilde{c}_{2k}^{1p} \end{bmatrix} = \begin{bmatrix} f(t_k, u_k, v_k) \\ f(t_{k+1}, u_{k+1}^p, v_{k+1}^p) \end{bmatrix}$$
$$\begin{bmatrix} t_k^{\gamma_1} & t_k^{\gamma_2} \\ t_{k+1}^{\gamma_1} & t_{k+1}^{\gamma_2} \end{bmatrix} \begin{bmatrix} \tilde{c}_{1k}^{2p} \\ \tilde{c}_{2k}^{2p} \end{bmatrix} = \begin{bmatrix} g(t_k, u_k, v_k) \\ g(t_{k+1}, u_{k+1}^p, v_{k+1}^p) \end{bmatrix}.$$

(2) Correction stage

$$u_{k+1} = g_1(t_{k+1}) + \sum_{j=0}^{k-1} (\tilde{c}_{1j}^1 H_{1j}^{1k} + \tilde{c}_{2j}^1 H_{2j}^{1k}) + \tilde{c}_{1k}^{1p} H_{1k}^{1k} + \tilde{c}_{2k}^{1p} H_{2k}^{1k}$$
$$v_{k+1} = g_2(t_{k+1}) + \sum_{j=0}^{k-1} (\tilde{c}_{1j}^2 H_{1j}^{2k} + \tilde{c}_{2j}^2 H_{2j}^{2k}) + \tilde{c}_{1k}^{2p} H_{1k}^{2k} + \tilde{c}_{2k}^{2p} H_{2k}^{2k}$$

where the coefficient $\tilde{c}_{1k}^i, \tilde{c}_{2k}^i$ is obtained by (2.14).

Remark 2.1 The startup item u_1, v_1 should be calculated through the following steps

(1) Let the initial predicted value $u_1^p = 0, v_1^p = 0$.

(2) Obtain $\tilde{c}_{10}^{ip}, \tilde{c}_{20}^{ip}, i=1,2$ by solving the following system of equations

$$\begin{bmatrix} 1 & 0 \\ t_1^{\beta_1} & t_1^{\beta_2} \end{bmatrix} \begin{bmatrix} \tilde{c}_{10}^{1p} \\ \tilde{c}_{20}^{1p} \end{bmatrix} = \begin{bmatrix} \lim_{t\to 0^+} \dfrac{f(t, u(t), v(t))}{t^{\beta_1}} \\ f(t_1, u_1^p, v_1^p) \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ t_1^{\gamma_1} & t_1^{\gamma_2} \end{bmatrix} \begin{bmatrix} \tilde{c}_{10}^{2p} \\ \tilde{c}_{20}^{2p} \end{bmatrix} = \begin{bmatrix} \lim_{t\to 0^+} \dfrac{g(t, u(t), v(t))}{t^{\gamma_1}} \\ g(t_1, u_1^p, v_1^p) \end{bmatrix}$$

(3) Calculate the initial correction value

$$u_1 = g_1(t_1) + \tilde{c}_{10}^{1p} H_{10}^{10} + \tilde{c}_{20}^{1p} H_{20}^{10}$$
$$v_1 = g_2(t_1) + \tilde{c}_{10}^{2p} H_{10}^{20} + \tilde{c}_{20}^{2p} H_{20}^{20}$$

(4) Obtain $\tilde{c}_{10}^i, \tilde{c}_{20}^i, i=1,2$ by solving the following system of equations

$$\begin{bmatrix} 1 & 0 \\ t_1^{\beta_1} & t_1^{\beta_2} \end{bmatrix} \begin{bmatrix} \tilde{c}_{10}^{1} \\ \tilde{c}_{20}^{1} \end{bmatrix} = \begin{bmatrix} \lim_{t\to 0^+} \dfrac{f(t, u(t), v(t))}{t^{\beta_1}} \\ f(t_1, u_1, v_1) \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ t_1^{\gamma_1} & t_1^{\gamma_2} \end{bmatrix} \begin{bmatrix} \tilde{c}_{10}^{2} \\ \tilde{c}_{20}^{2} \end{bmatrix} = \begin{bmatrix} \lim_{t\to 0^+} \dfrac{g(t, u(t), v(t))}{t^{\gamma_1}} \\ g(t_1, u_1, v_1) \end{bmatrix}$$

We refer to the algorithms (1)-(2) above as predictor-corrector schemes. Next, the stability and error of the algorithm will be analyzed.

III. Error estimation and stability analysis

3.1 Auxiliary results

We will mention some lemmas, which are essential for the analysis of stability and error. Let $\Lambda_n = \{\beta_1, \beta_2, ..., \beta_n\}$ and $\mathrm{P}_n = \{\gamma_1, \gamma_2, ..., \gamma_n\}$, where $n \in \square$, are two different non-negative real numbers, and satisfy the following relationship

$$\begin{aligned} &0 \le \beta_1 < \beta_2 < ... < \beta_n, \quad 0 < \beta_{i+1} - \beta_i < 1, \quad i = 1, ..., n\text{-}1 \\ &0 \le \gamma_1 \le \gamma_2 < ... < \gamma_n, \quad 0 < \gamma_{i+1} - \gamma_i < 1, \quad i = 1, ..., n\text{-}1 \end{aligned}. \tag{3.1}$$

We define the space Λ_n (or P_n) of the continuous function on $C([a,b], \Lambda_n)$ (or $C([a,b], \mathrm{P}_n)$) as follows

$$C([a,b],\Lambda_n)=\left\{ f\in C[a,b]\,|\, t^{-\beta_1}f\in C[a,b], D_{\beta_n-\beta_{n-1}}...D_{\beta_2-\beta_1}(t^{-\beta_1}f)\in C[a,b]\right\}.$$

The operator $D_\alpha f, \alpha>0$ is defined in [1] as

$$D_\alpha f = t^{1-\alpha}f'. \tag{3.2}$$

Lemma 3.1 (Discrete Gronwall Inequality, [21]) Let $a_i, 0\le i\le N$, be a sequence of non-negative real numbers satisfying

$$a_i \le b_i + M\Delta t^{\gamma}\sum_{j=0}^{i-1}(i-j)^{\gamma-1}a_j, 1\le i\le N,$$

where $0<\gamma\le 1, M>0$ is bounded independently of Δt, and $b_i, 0\le i\le N$, is a monotonic increasing sequence of non-negative real numbers. Then

$$a_i \le b_i E_\gamma(M\Gamma(\gamma)(i\Delta t)^{\gamma}), 0\le i\le N,$$

where

$$E_\gamma(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+1)}.$$

is the Mittag-Leffler function of order γ.

In particular, when $\gamma=1$, (3.4) becomes $a_i\le b_i\exp(Mi\Delta t), 0\le i\le N.$

3.2 Error Analysis

In the error analysis, we assume that equation system (1.1) satisfies the following Lipschitz condition with respect to the second and third variables: that is, there exists a constant L that satisfies the following conditions

$$\begin{aligned} &|f(t,u_1,v_1)-f(t,u_2,v_2)|\le L(|u_1-v_1|+|u_2-v_2|)\quad \forall u_1,v_1,u_2,v_2\in R\\ &|g(t,u_1,v_1)-g(t,u_2,v_2)|\le L(|u_1-v_1|+|u_2-v_2|)\quad \forall u_1,v_1,u_2,v_2\in R \end{aligned}\tag{3.6}$$

Let $\Lambda_3=\{\beta_1,\beta_2,\beta_3\}$ and $\mathrm{P}_3=\{\gamma_1,\gamma_2,\gamma_3\}$ be real numbers satisfying (3.1), and assuming that the right-hand function of equation group (1.1) satisfies $f(t,u(t),v(t))\in([0,T],\Lambda_3)$ and $g(t,u(t),v(t))\in([0,T],\mathrm{P}_3)$, and combined with (3.6), the predictor-corrector scheme proposed in this paper has the following error estimates

$$|u(t_{k+1})-u_{k+1}|\le Cn^{-\min\{2,2(\alpha+\beta_3)\}},\qquad k=1,...,n\text{-}1, \tag{3.7}$$

$$|v(t_{k+1})-v_{k+1}|\le Cn^{-\min\{2,2(\alpha+\gamma_3)\}},\qquad k=1,...,n\text{-}1, \tag{3.8}$$

where C is a constant independent of k, n and grid size.

Rremark 3.3 The proof of (3.2) is easily obtained from ([20], Theorem 2). For a better understanding, the following will briefly introduce.

Let: $e^1_{k+1}=|u(t_{k+1})-u_{k+1}|,\ e^2_{k+1}=|v(t_{k+1})-v_{k+1}|, k=0,1,...,n-1$, we have: From([20],

$$\begin{aligned} e^1_{k+1} &=\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}f(s,u(s),v(s))ds\\ &-\sum_{j=0}^{k-1}(\tilde c^1_{1j}H^{1k}_{1j}+\tilde c^1_{2j}H^{1k}_{2j})-(\tilde c^{1p}_{1k}H^{1k}_{1k}+\tilde c^{1p}_{2k}H^{1k}_{2k})\Big|\\ &\le\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}f(s,u(s),v(s))ds\\ &-\sum_{j=0}^{k}(c^1_{1j}H^{1k}_{1j}+c^1_{2j}H^{1k}_{2j})\Big|+\Big|\sum_{j=0}^{k-1}(c^1_{1j}H^{1k}_{1j}+c^1_{2j}H^{1k}_{2j})-\sum_{j=0}^{k-1}(\tilde c^1_{1j}H^{1k}_{1j}+\tilde c^1_{2j}H^{1k}_{2j})\Big|\\ &+|c^1_{1k}H^{1k}_{1k}+c^1_{2k}H^{1k}_{2k}-\tilde c^{1p}_{1k}H^{1k}_{1k}-\tilde c^{1p}_{2k}H^{1k}_{2k}|:=I_1+I_2+I_3 \end{aligned}\tag{3.9}$$

lemma 1) we know: $I_1 \le C_1 n^{-\min\{2,2(\alpha+\beta_3)\}}$.

Using (2.8), (2.9), (2.14), (2.15) and the Lipschitz conditions of f, g with the Lipschitz constant $L>0$, the following results hold

$$
\begin{aligned}
&I_2+I_3 \\
&\le |(c_{10}^1-\tilde{c}_{10}^1)H_{10}^{1k}+(c_{20}^1-\tilde{c}_{20}^1)H_{20}^{1k}|+\sum_{j=0}^{k-1}|(c_{1j}^1-\tilde{c}_{1j}^1)H_{1j}^{1k} \\
&+(c_{2j}^1-\tilde{c}_{2j}^1H_{2j}^{1k})|+|(c_{1k}^1-\tilde{c}_{1k}^{1p})H_{1k}^{1k}+(c_{2k}^1-\tilde{c}_{2k}^{1p})H_{2k}^{1k}| \\
&\le L\Bigg(|\frac{H_{20}^{1k}-t_0^{\beta_2-\beta_1}H_{10}^{1k}}{t_1^{\beta_1}(t_1^{\beta_2-\beta_1}-t_0^{\beta_2-\beta_1})}|(|e_1^1|+|e_1^2|)+\sum_{j=1}^{k-1}|\frac{t_{j+1}^{\beta_2-\beta_1}H_{1j}^{1k}-H_{2j}^{1k}}{t_j^{\beta_1}(t_{j+1}^{\beta_2-\beta_1}-t_j^{\beta_2-\beta_1})}|(|e_j^1|+|e_j^2|) \\
&+\sum_{j=1}^{k-1}|\frac{H_{2j}^{1k}-t_j^{\beta_2-\beta_1}H_{1j}^{1k}}{t_{j+1}^{\beta_1}(t_{j+1}^{\beta_2-\beta_1}-t_j^{\beta_2-\beta_1})}|(|e_{j+1}^1|+|e_{j+1}^2|)+|\frac{t_{k+1}^{\beta_2-\beta_1}H_{1k}^{1k}-H_{2k}^{1k}}{t_k^{\beta_1}(t_{k+1}^{\beta_2-\beta_1}-t_k^{\beta_2-\beta_1})}|(|e_k^1|+|e_k^2|) \\
&+|\frac{H_{2k}^{1k}-t_k^{\beta_2-\beta_1}H_{1k}^{1k}}{t_{k+1}^{\beta_1}(t_{k+1}^{\beta_2-\beta_1}-t_k^{\beta_2-\beta_1})}|(|u(t_{k+1})-u_{k+1}^p|+|v(t_{k+1})-v_{k+1}^p|)\Bigg) \\
&\le L\left(\sum_{j=1}^{k}M_{j,k+1}^*(|e_j^1|+|e_j^2|)+C\tau_{\max}^{\alpha}(|u(t_{k+1})-u_{k+1}^p|+|v(t_{k+1})-v_{k+1}^p|)\right)
\end{aligned} \qquad (3.10)
$$

The coefficient $M_{j,k+1}^*$ \$ is defined in [20].

Assume $\beta_3<\gamma_3$ (the assumption here does not affect the numerical results), from ([20], Theorem 2), we have

$$|e_j^1|+|e_j^2|\le Cn^{-\min\{2,2(\alpha+\beta_3)\}},\ \ j=1,\ldots,k, \qquad (3.11)$$

$$|u(t_{k+1})-u_{k+1}^p|+|v(t_{k+1})-v_{k+1}^p|\le Cn^{-\min\{2,2(\alpha+\beta_3)\}}. \qquad (3.12)$$

From (3.10), 3.11), (3.12) and the analysis of the coefficient $M_{j,k+1}^*$ in [20], the following error estimates are obtained

$$|u(t_{k+1})-u_{k+1}|\le Cn^{-\min\{2,2(\alpha+\beta_3)\}}. \qquad (3.13)$$

In a similar way, we can derive

$$|v(t_{k+1})-v_{k+1}|\le Cn^{-\min\{2,2(\alpha+\gamma_3)\}}. \qquad (3.14)$$

3.3 Stability Analysis

For the convenience of stability analysis, we note that

$$
N_{j,k+1}^*=\begin{cases}
\dfrac{\lim\limits_{t\to0^+}\dfrac{f(t,u(t),v(t))}{t^{\beta_1}}(t_1^{\beta_2}H_{10}^{1k}-t_1^{\beta_1}H_{20}^{1k})}{t_0^{\beta_1}(t_1^{\beta_2-\beta_1}-t_0^{\beta_2-\beta_1})} & j=0 \\
|\dfrac{t_{j+1}^{\beta_2-\beta_1}H_{1j}^{1k}-H_{2j}^{1k}}{t_j^{\beta_1}(t_{j+1}^{\beta_2-\beta_1}-t_j^{\beta_2-\beta_1})}|+|\dfrac{H_{2j-1}^{1k}-t_{j-1}^{\beta_2-\beta_1}H_{1j-1}^{1k}}{t_j^{\beta_1}(t_j^{\beta_2-\beta_1}-t_{j-1}^{\beta_2-\beta_1})}| & j=1,\ldots,k, \\
\dfrac{H_{2k}^{1k}-t_k^{\beta_2-\beta_1}H_{1k}^{1k}}{t_{k+1}^{\beta_1}(t_{k+1}^{\beta_2-\beta_1}-t_k^{\beta_2-\beta_1})} & j=k
\end{cases} \qquad (3.15)
$$

$$\tilde{N}_{j,k+1}=\begin{cases}\dfrac{\lim\limits_{t\to0^+}\dfrac{f(t,u(t),v(t))}{t^{\beta_1}}(t_1^{\beta_2}H_{10}^{1k}-t_1^{\beta_1}H_{20}^{1k})}{t_0^{\beta_1}(t_1^{\beta_2-\beta_1}-t_0^{\beta_2-\beta_1})} & j=0\\ \dfrac{t_{j+1}^{\beta_2-\beta_1}H_{1j}^{1k}-H_{2j}^{1k}}{t_j^{\beta_1}(t_{j+1}^{\beta_2-\beta_1}-t_j^{\beta_2-\beta_1})}+\dfrac{H_{2j-1}^{1k}-t_{j-1}^{\beta_2-\beta_1}H_{1j-1}^{1k}}{t_j^{\beta_1}(t_j^{\beta_2-\beta_1}-t_{j-1}^{\beta_2-\beta_1})}, & 1\le j\le k-2\\ |\dfrac{t_k^{\beta_2-\beta_1}H_{1k-1}^{1k}-H_{2k-1}^{1k}}{t_{k-1}^{\beta_1}(t_k^{\beta_2-\beta_1}-t_{k-1}^{\beta_2-\beta_1})}|+|\dfrac{t_k^{\beta_2-\beta_1}H_{1k}^{1k}-H_{2k}^{1k}}{t_{k-1}^{\beta_1}(t_k^{\beta_2-\beta_1}-t_{k-1}^{\beta_2-\beta_1})}| & \\ +|\dfrac{H_{2k-2}^{1k}-t_{k-2}^{\beta_2-\beta_1}H_{1k-2}^{1k}}{t_{k-1}^{\beta_1}(t_{k-1}^{\beta_2-\beta_1}-t_{k-2}^{\beta_2-\beta_1})}|, & j=k-1\\ \dfrac{H_{2k-1}^{1k}-t_{k-1}^{\beta_2-\beta_1}H_{1k-1}^{1k}}{t_k^{\beta_1}(t_k^{\beta_2-\beta_1}-t_{k-1}^{\beta_2-\beta_1})}+\dfrac{H_{2k}^{1k}-t_{k-1}^{\beta_2-\beta_1}H_{1k}^{1k}}{t_k^{\beta_1}(t_k^{\beta_2-\beta_1}-t_{k-1}^{\beta_2-\beta_1})}, & j=k\end{cases}. \quad (3.16)$$

In same way, the definitions of $Q^*_{j,k+1}$ and $\tilde{Q}_{j,k+1}$ similar to $N^*_{j,k+1}$ and $\tilde{N}_{j,k+1}$, i.e. replace β, f in $N^*_{j,k+1}$ and $\tilde{N}_{j,k+1}$ with γ, g respectively, we won't go into detail here, then the algorithm format can be rewritten as follows

(1) Prediction stage

$$u^p_{k+1}=g_1(t)+\sum_{j=0}^{k}\tilde{N}_{j,k+1}f(t_j,u_j,v_j)\,, \quad (3.17)$$

$$v^p_{k+1}=g_2(t)+\sum_{j=0}^{k}\tilde{Q}_{j,k+1}g(t_j,u_j,v_j)\,. \quad (3.18)$$

(2) Correction stage

$$u_{k+1}=g_1(t)+\sum_{j=0}^{k}N^*_{j,k+1}f(t_j,u_j,v_j)+N^*_{k+1,k+1}f(t_j,u^p_{k+1},v^p_{k+1})\,, \quad (3.19)$$

$$v_{k+1}=g_2(t)+\sum_{j=0}^{k}Q^*_{j,k+1}g(t_j,u_j,v_j)+Q^*_{k+1,k+1}g(t_j,u^p_{k+1},v^p_{k+1})\,, \quad (3.20)$$

where $k=1,...,n-1$.

Similar to the analysis of coefficient $M^*_{j,k+1}$ in [20], we can get that

$$N^*_{j,k+1}\le C(\tau_j(t_{k+1}-t_{j+1})^{\alpha-1}+\tau_{j-1}(t_{k+1}-t_j)^{\alpha-1})\le C(\tau_j(t_{k+1}-t_j)^{\alpha-1})\,, \quad (3.21)$$

$$\tilde{N}_{j,k+1}\le C(\tau_j(t_{k+1}-t_{j+1})^{\alpha-1}+\tau_{j-1}(t_{k+1}-t_j)^{\alpha-1})\le C(\tau_j(t_{k+1}-t_j)^{\alpha-1})\,, \quad (3.22)$$

where $j=1,\cdots,k,$, for $j=k+1,$ we have

$$N^*{}_{k+1,k+1}\le C(t_{k+1}-t_k)^{\alpha}\le C\tau^{\alpha}_{\max}\,. \quad (3.23)$$

Similarly, $Q^*_{j,k+1}$ and $\tilde{Q}_{j,k+1}$ satisfy the following relationship

$$Q^*_{j,k+1}\le C(\tau_j(t_{k+1}-t_j)^{\alpha-1})\,, \quad (3.24)$$

$$\tilde{Q}_{j,k+1}\le C(\tau_j(t_{k+1}-t_j)^{\alpha-1})\,, \quad (3.25)$$

where $j=1,\cdots,k,$, for $j=k+1,$, we have

$$Q^*{}_{k+1,k+1}\le C(t_{k+1}-t_k)^{\alpha}\le C\tau^{\alpha}_{\max}\,. \quad (3.26)$$

For the convenience of discussion, we transform the right-hand side of (1.1) into an equivalent model, as follows:

$$f(t,u(t),v(t)) := \lambda u(t)$$
$$g(t,u(t),v(t)) := \mu u(t), \qquad (3.27)$$

where λ and μ are real numbers.

Theorem3.4 Let $0<\alpha<1$, for f and g given by (3.27), the algorithm (3.19), (3.20) is stable with respect to the initial value, that is, it satisfies

$$|u_j| \le Cu_0 , \quad |v_j| \le Cu_1 , \quad j=1,...,n\text{-}1, \qquad (3.28)$$

where C only depends on λ (or μ), α and T.

Proof: First prove that $|u_j| \le Cu_0, \quad j=1,...,n\text{-}1,$

we can easily know that $g_1(t)$ satisfies

$$|g_1(t)| = |\sum_{k=0}^{n-1} u_0^{(k)} \frac{t^k}{k!}| \le u_0 \exp(t) \le u_0 \exp(T). \qquad (3.29)$$

Substituting (3.29) into (3.17), noting the definition of f in (3.27) and the estimation of the coefficients in (3.22), we derive

$$|u_{k+1}^p| = |g_1(t) + \lambda \sum_{j=0}^{k} \tilde{N}_{j,k+1} u_j| \le Cu_0 \exp(T)$$
$$+ C|\lambda| \sum_{j=0}^{k} \tau_j (t_{k+1}-t_j)^{\alpha-1} |u_j| \qquad (3.30)$$

From ([22], Lemma 3.1, 3.2) we know that $\sum_{j=0}^{k} (\tau_j (t_{k+1}-t_j)^{\alpha-1})$ is bounded when $0<\alpha<1$. Then we have

$$(\tau_j (t_{k+1}-t_j)^{\alpha-1}) \le C(k+2+j)^{\alpha-1}(k+1-j)^{\alpha-1}(j+1) \le C(k+j-1)^{\alpha-1}. \qquad (3.31)$$

Substituting (3.31) into (3.30) yields

$$|u_{k+1}^p| \le Cu_0 \exp(T) + C|\lambda| \sum_{j=0}^{k} (k+1-j)^{\alpha-1} |u_j|, \qquad (3.32)$$

applying the discrete Gronwall Lemma 3.1 to (3.33) gives

$$|u_{k+1}^p| \le Cu_0. \qquad (3.33)$$

Substituting (3.33) into (3.19) and paying attention to (3.21), (3.23) and (3.32) we have

$$|u_{k+1}| \le |g_1(t)| + |\lambda| \sum_{j=0}^{k} (N_{j,k+1}^*) |u_j| + |\lambda| (N_{k+1,k+1}^*) |u_{k+1}^p|$$
$$\le u_0(\exp(T)+\lambda C) + |\lambda| \sum_{j=0}^{k} (N_{j,k+1}^*) |u_j| \qquad . \qquad (3.34)$$
$$\le u_0(\exp(T)+\lambda C) + C|\lambda| \sum_{j=0}^{k} (k+1-j)^{\alpha-1} |u_j|$$

Applying the discrete Gronwall Lemma 3.1 to (3.34) gives

$$|u_{k+1}| \le Cu_0. \qquad (3.35)$$

In a similar way, we can derive

$$|v_{k+1}| \le Cu_0. \qquad (3.36)$$

IV. Numerical results

In this section, we will verify the accuracy of the predictor-corrector scheme proposed in this paper through numerical examples, and define the error and convergence order by

$$e_n^1 = \max_{0\le k\le n-1} |u(t_{k+1}) - u_{k+1}| \quad , \quad R_n^1 = \ln\left(\frac{e_n^1}{e_n^1}\right) / \ln\left(1+\frac{1}{n}\right)$$

$$e_n^2 = \max_{0\le k\le n-1} |v(t_{k+1}) - v_{k+1}| \quad , \quad R_n^2 = \ln\left(\frac{e_n^2}{e_n^2}\right) / \ln\left(1+\frac{1}{n}\right)$$

In order to calculate the singularity index of the right-hand function of (1.1) more conveniently, we design its exact solution as

$$\begin{cases} u(t) = t^{\sigma_1} + t^{\sigma_2} + t^{\sigma_3} + t^{\sigma_4}, & 0 \le \sigma_1 < \sigma_2 < \sigma_3 < \sigma_4 \\ v(t) = t^{\rho_1} + t^{\rho_2} + t^{\rho_3} + t^{\rho_4}, & 0 \le \rho_1 < \rho_2 < \rho_3 < \rho_4 \end{cases},$$

where $e_n^i, R_n^i, i=1,2$ is the experimental error and the experimental convergence order, in addition, we denote $R_i, i=1,2$ as the theoretical convergence order.

Example 1: Consider (1.1) with $0<\alpha<1$, we set the right-hand function of (1.1) as (4.1).

$$\begin{cases} f(t,u(t),v(t)) = \sum_{i=1}^{4} \frac{\Gamma(1+\sigma_i)}{\Gamma(1+\sigma_i-\alpha)} t^{\sigma_i-\alpha} + t\sin(\sum_{i=1}^{4} t^{\sigma_i}) \\ -t\sin(u(t)) - tu(t) - t\sum_{i=1}^{4} t^{\sigma_i} - tv(t) + t\sum_{i=1}^{4} t^{\rho_i}, \\ g(t,u(t),v(t)) = \sum_{i=1}^{4} \frac{\Gamma(1+\sigma_i)}{\Gamma(1+\rho_i-\alpha)} t^{\rho_i-\alpha} + t\sin(\sum_{i=1}^{4} t^{\rho_i}) \\ -t\sin(v(t)) - tu(t) + t\sum_{i=1}^{4} t^{\sigma_i} - tv(t) + t\sum_{i=1}^{4} t^{\rho_i}. \end{cases}$$

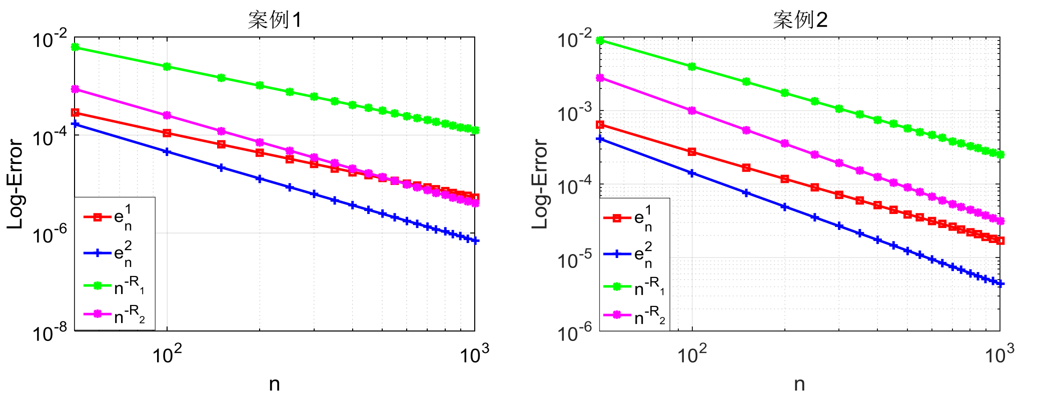

Figure 1: Comparison of experimental error and theoretical error of Case I (left) and Case II (right) for Example 1.

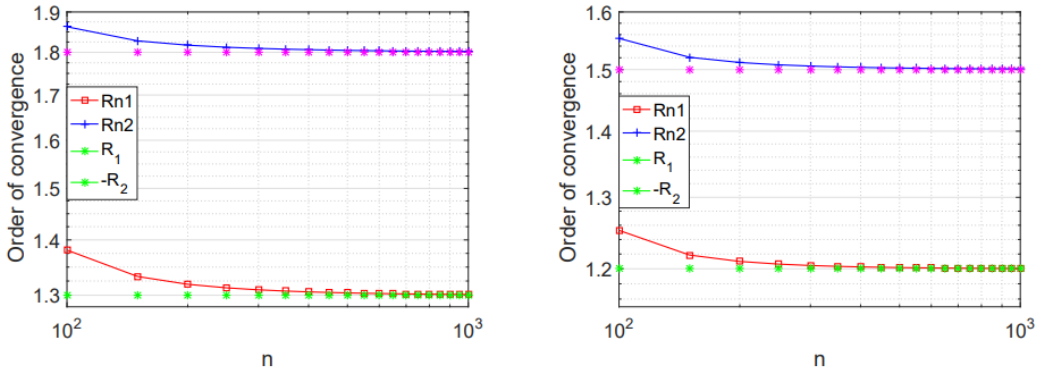

Figure 2: The comparison of the experimental convergence order and the theoretical convergence order of case I (left) and case II (right) under different n for example 1.

We design two cases for different singularity exponents

Case(I), $\begin{cases} \rho_1 = 0.5 , & \rho_2 = 0.6 , & \rho_3 = 0.65 , & \rho_4 = 1.35 \\ \sigma_1 = 0.4 , & \sigma_2 = 0.7 , & \sigma_3 = 0.9 , & \sigma_4 = 1.45 \end{cases}$

Case(II), $\begin{cases} \rho_1 = 0.4 , & \rho_2 = 0.5 , & \rho_3 = 0.6 , & \rho_4 = 1.3 \\ \sigma_1 = 0.5 , & \sigma_2 = 0.55 , & \sigma_3 = 0.75 , & \sigma_4 = 1.35 \end{cases}$

In Fig.1, the experimental error $Log(e_n^i), i = 1,2$ and the theoretical expected error $Log(e_n^i), i = 1,2$ of Case(I) (left) and Case(II) (right) under different n are compared, It is easy to see that the experimental error e_n^1, e_n^2 of case(I) and case(II) decays to $O(n^{-R_1})$ and $O(n^{-R_2})$, respectively, which meets the theoretical requirements. In addition, we show in Fig.2 that the experimental convergence order $R_n^i, i = 1,2$ of Case(I) and Case(II) gradually approaches the theoretical convergence order $R_i, i = 1,2$ with the change of n. We observe that the experimental convergence order agrees with our theoretical results.

Table 1 The comparison of the experimental convergence order and the theoretical convergence order of Example 1 .

ρ_3	0.8	0.9	1	1.05	1.1	1.15	1.2	1.25
σ_3	1.25	1.2	1.15	1.1	1.05	1	0.9	0.8
R_n^1	1.6050	1.8114	2.0242	2.0168	2.0131	2.0098	2.0074	2.0060
R_1	1.6	1.8	2	2	2	2	2	2
R_n^2	2.0120	2.0146	2.0183	2.0220	2.0251	2.0172	1.8077	1.6033
R_2	2	2	2	2	2	2	1.8	1.6

In Table 1, we analyzed the experimental convergence order and theoretical convergence order of the equation group (1.1) under different ρ_3 and σ_3 conditions in Example 1 when $n = 500, 600$, and it can be seen from the table that the respective experimental convergence order of the equations at this time the order of convergence matches the theoretical order of convergence.

Example 2: Consider (1.1) with $0 < \alpha < 1$, we set the right-hand function of (1.1) as 4.2.

$$\begin{cases} f(t,u(t),v(t)) = \sum_{i=1}^{4} \frac{\Gamma(1+\sigma_i)}{\Gamma(1+\sigma_i - \alpha)} t^{\sigma_i - \alpha} + t\sin(\sum_{i=1}^{4} t^{\sigma_i}) - t\sin(u(t)) \\ \qquad -tu^2(t) + t(\sum_{i=1}^{4} t^{\sigma_i})^2 + tv(t) - t\sum_{i=1}^{4} t^{\rho_i} \\ g(t,u(t),v(t)) = \sum_{i=1}^{4} \frac{\Gamma(1+\rho_i)}{\Gamma(1+\rho_i - \alpha)} t^{\rho_i - \alpha} + t\sin(\sum_{i=1}^{4} t^{\rho_i}) - t\sin(v(t)) \\ \qquad -tv^2(t) + t(\sum_{i=1}^{4} t^{\rho_i})^2 + tu(t) - t\sum_{i=1}^{4} t^{\sigma_i} \end{cases}$$

where $\begin{cases} \rho_1 = 0.62, & \rho_2 = 0.65 , & \rho_3 = 0.8 , & \rho_4 = 1.5 \\ \sigma_1 = 0.65 , & \sigma_2 = 0.7 , & \sigma_3 = 0.9 , & \sigma_4 = 1.5 \end{cases}$ and $\alpha = 0.6$.

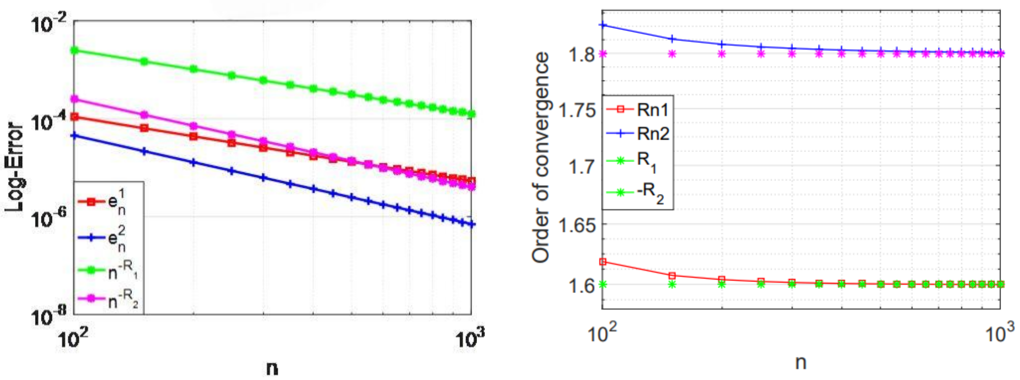

Figure 3: Comparison of error(left) and convergence order (right) for Example 2.

In Fig.3 (left), the experimental error $Log(e_n^i), i=1,2$ and the theoretical expected error $Log(n^{-R_i}), i=1,2$ of example 2 under different n are compared, It is easy to see that the experimental error e_n^1, e_n^2 of example 2 decays to $O(n^{-R_1})$ and $O(n^{-R_2})$ respectively, which meets the theoretical requirements. In addition, we show in Fig.3 (right) that the experimental convergence order $R_n^i, i=1,2$ $ of example 2 gradually approaches the theoretical convergence order $R_i, i=1,2$ with the change of n We observe that the experimental convergence order agrees with our theoretical results.

Table 2 The comparison of the experimental convergence order and the theoretical convergence order of Example 2.

ρ_3	0.75	0.85	0.95	1.05	1.1	1.15	1.2	1.25
σ_3	1.25	1.2	1.15	1.1	1.05	0.95	0.85	0.75
R_n^1	1.5028	1.7054	1.9115	2.0015	2.0015	2.0019	2.0025	2.0032
R_1	1.5	1.7	1.9	2	2	2	2	2
R_n^2	2.0066	2.0056	2.0047	2.0040	2.0037	1.9054	1.7027	1.5031
R_2	2	2	2	2	2	1.9	1.7	1.5

In Table 2, we analyzed the experimental convergence order and theoretical convergence order of the equation group (1..1) under different ρ_3 and σ_3 conditions in Example 2 when $n=500,600$, and it can be seen from the table that the respective experimental convergence order of the equations at this time the order of convergence matches the theoretical order of convergence.

V. Concluding remarks

In order to better handle the low regularity of the solutions of coupled system of fractional differential equations, we use a new interpolation family, namely fractional interpolation, which is applied to construct a new predictor-corrector scheme to solve coupled system of fractional differential equations. The analysis of error and stability of the proposed scheme are carried out, and some numerical examples are given to verify the theoretical results. In the future, we will use this scheme to solve coupled system of fractional partial differential equations.

Acknowledgments

The authors would like to thank the referees and editor for their valuable comments and suggestions which helped to improve the results of this paper.

References

[1]. Bouchaud, J. P. and Georges, A. Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. Phys. Rep, 1990, 195: 127-293.
[2]. Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. Theory and Applications of Fractional Differential Equations. Elsevier, 2006.
[3]. Miller, K. S. and Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Willey and Sons, 1993.

[4]. Podlubny, I. Fractional Differential Equations. Academic Press, New York, 1999.
[5]. Samko, S. G., Kilbas, A. A. and Marichev, O. I. Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Switzerland, 1993.
[6]. Almeida, R., Bastos, N. and Monteiro, M. Modeling some real phenomena by fractional differential equations. Math. Methods Appl. Sci, 2016, 39: 4846-4855.
[7]. Koeller, R. C. Applcation of fractional calculus to the theory of viscoelasticity. J.Appl. Mech, 1984,51: 229-307.
[8]. Benson,D. A., Wheatcraft, S. W. and Meerschaert, M. M. Application of a fractional advection-dispersion equation. Nonlinear Anal, 2006, 36: 1403-1412.
[9]. Zhou, X. and Dai, Y. A spectral collocation method for the coupled system of nonlinear fractional differential equations. Aims Math, 2022, 7: 5670-5689.
[10]. Huseynov, I.T., Ahmadova, A. and Fernandez, A. Explicit analytical solutions of incommensurate fractional differential equation systems. Appl. Math. Comput, 2021, 390: 125590.
[11]. Zhou, X. and Xu, C. Numerical solution of the coupled system of nonlinear fractional ordinary differential equations. Advances in Applied Mathematics and Mechanics, 2017, 9: 574-595.
[12]. Nabil, T. Ulam stabilities of nonlinear coupled system of fractional differential equations including generalized caputo fractional derivative. Aims Mathematics, 2021, 6: 5088-5105.
[13]. Zaky , M. A. and Ameen, I. G. A priori error estimates of a jacobi spectral method for nonlinear systems of fractional boundary value problems and related volterra-fredholm integral equations with smooth solutions. Numer. Algorithms, 2019: 1-27.
[14]. Agarwal, R., Hristova, S. and O'Regan, D. Stability of generalized proportional caputo fractional differential equations by lyapunov functions. Fractal and Fractional, 2022, 6(1): 34.
[15]. Shiri, B., Wu, G. and Baleanu, D. Terminal value problems for the nonlinear systems of fractional differential equations. Appl. Numer. Math, 2021, 170: 162-178.
[16]. Jhinga, A. and Daftardar-Gejji, V. A new finite-difference predictor-corrector method for fractional differential equations. Appl. Math. Comput, 2018, 336: 418-432.
[17]. Nguyen, T.B. and Jang, B. A high-order predictor-corrector method for solving nonlinear differential equations of fractional order. Fract. Calc. Appl. Anal, 2017, 20: 447-476.
[18]. Douaifia, R., Bendoukha, S. and Abdelmalek, S. A newton interpolation based predictor--orrector numerical method for fractional differential equations with an activator--nhibitor case study. Mathematics and Computers in Simulation, 2021, 187: 391-413.
[19]. Kai, D. The analysis of fractional differential equations. Lecture notes in mathematics, 2010.
[20]. Yarmohammadi , M. and Javadi, S. Piecewise fractional interpolation with application to fractional differential equation. Journal of Scientific Computing, 2021, 86: 1-22.
[21]. Dixon, J. and McKee, S. Weakly singular discrete gronwall inequalities. Z. Angew. Math. Mech, 1986, 66: 535-544.
[22]. Li, C. and Yi, Q. Finite difference methods with non-uniform meshes for nonlinear fractional differential equations. J. Comput. Phys, 2016, 316: 614-631.