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Review Paper

Application on Surjection and Inversion for Locally Lipschitz Maps between closer Banach Spaces

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Abstract: *An application on the work of O. Gutú, J.A. Jaramil lo [39] for the validity of the study of the global invertibility of non-smooth, locally Lipschitz maps between infinite-dimensional closer Banach spaces, using the tool of Palais-Smale condition. So we consider the Chang version of the weighted Palais-Smale condition for locally Lipschitz functionals in terms of the Clarke subdifferential, the method of pseudo- Jacobians in the infinite-dimensional closer setting, which are the analog of the pseudo-Jacobian matrices defined by Jeyakumar and Luc. Standing on all these, we show the existence and uniqueness of solution for certain nonlinear equations defined by locally Lipschitz mappings. We also show a global surjection theorem for locally Lipschitz maps in terms of pseudo-Jacobians.*

Keywords: *Global invertibility, Palais-Smale condition, Nonsmooth analysis*

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1. Introduction

We keep and follow the full smooth connected theory of [39] with a bit change. So the surjectivity and invertibility of maps is an important issue in nonlinear analysis. To start with if $f_j: X \to X + \epsilon$ is a C^1 map between closer Banach spaces, such that its derivative $f_j'(x)$ is an isomorphism for every $x \in X$, from the classical Inverse Function Theorem then we have that f_j is locally invertible around each point.

If, in addition, f_j satisfies the so-called Hadamard integral condition:

$$\int_0^\infty \sum_j \inf_{\|x\| \le t} \left\| f_j'(x)^{-1} \right\|^{-1} \mathrm{d}t = \infty,$$

then $f_j: X \to X + \epsilon$ is globally invertible, and thus a global diffeomorphism from X onto $(X + \epsilon)$ (see e.g. [34] for a proof of this result). This sufficient condition for global invertibility was first considered by [18] for maps between finite-dimensional spaces, and has been widely used since then. See also [15] for an extensive information about this and other conditions for global invertibility of smooth maps between closer Banach spaces and, more generally, between Finsler manifolds.

In a nonsmooth setting, if $f_j: \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz map, [36] and [37] obtained suitable versions of the Hadamard integral condition, using the Clarke generalized Jacobian. These results have been extended to the setting of finite-dimensional Finsler manifolds in [26]. For continuous maps $f_j: \mathbb{R}^n \to \mathbb{R}^n$ which are not assumed to be locally Lipschitz, the authors introduced in [28] the concept of approximate Jacobian matrix, which was later called pseudo-Jacobian matrix (see [29]). A global inversion theorem, with a version of the Hadamard integral condition, is given in [27].

If $f_j: X \to X + \epsilon$ is a nonsmooth map between infinite-dimensional closer Banach spaces, the problem of local invertibility of f_j is more delicate. Assuming that f_j is a local homeomorphism, F_j. [30] obtained a global inversion theorem with a suitable version of the Hadamard integral condition in terms of the lower scalar Dini derivative of f_j. Later on, [21] obtained a global inversion result for a continuous map f_j which is locally one-to-one, using an analog of Hadamard integral condition, defined in terms of the so-called constant of surjection of f_j at every point. Further results have been obtained in [17] and [13] in the more general setting of maps between metric spaces. In [25], the authors consider the notion of pseudo-Jacobian Jf_j for a continuous map f_j between Banach spaces, which is an extension of pseudo-Jacobian matrices of Jeyakumar and Luc to this setting, and obtain various global inversions results.

From the purely topological point of view, the Banach-Mazur Theorem states that a local homeomorphism f_j between Banach spaces is a global one if and only if it is a proper map, that is, the preimage f_j^{-1} sends compact sets to compact sets [4]. The proofs of this result and the classical global inversion theorems cited above are generally addressed through the use of the path-lifting property or some other similar approach which includes a monodromy argument. Actually, Rabier makes a masterful use of the path-lifting arguments in the smooth setting and gets the complete geometric picture leading to a generalization of the classical Ehresmann theorem of differential geometry; see Theorem 4.1 in [38]. In particular, the Banach-Mazur Theorem and the Hadamard integral condition are non-trivially related by Rabier via a sort of "uniform-strong" Palais-Smale condition so-called strong submersion with uniformly splits kernels.

Some conditions intimately related to strong submersions seem to escape from the monodromy technique. For mappings between metric spaces, [31] proposes a completely different approach based on an abstract mountain-pass theorem and the Ekeland variational principle, in order to obtain global inversion theorems in a non-smooth setting, via the study of the critical points of the functional $x \mapsto d(x+\epsilon, f_j(x))$ for all $(x+\epsilon)$ in $(X+\epsilon)$. [19] set down the Katriel approach in the closer Banach space setting proving that a local diffeomorphism with a Hilbert space target is a global diffeomorphism if the functional $\left(F_j\right)_{x+\epsilon}(x) = \frac{1}{2}|f_j(x)-(x+\epsilon)|^2$ satisfies the Palais-Smale condition for all $(x+\epsilon) \in (X+\epsilon)$. See [10] and [11] for further developments in this direction. See also [14], in the finite-dimensional setting.

The generalization of the result by Idczak et al. to mappings between Banach spaces as well as its connections with the Hadamard integral condition and the strong submersions of Rabier, have been established in [16] for C^1 maps. However the connection of the Palais-Smale condition with the nonsmooth global inversion results cited above for a locally Lipschitz map f_j are the pending issues just addressed. Now, we recall the definition of pseudo-Jacobian considered in [25], and we include some remarkable examples, which are explained in detail. In order to get our invertibility results, we need to guarantee a nice behavior regarding the chain rule for the pseudo-Jacobian of the composition with distance functions. After recalling the chain rule condition introduced in [25], we introduce the strong chain rule condition, which is needed in order to make the chain rule compatible with Clarke subdifferential. Next, we provide some fundamental examples of pseudo-Jacobians satisfying the strong chain rule condition, which will be used along the study. We review all the local inversion results given in [25, Theorem 3.1] and [25, Theorem 3.7], respectively, for a locally Lipschitz map f_j, in terms of the surjectivity index and the regularity index of a pseudo-Jacobian Jf_j. In fact, it is possible to deduce from [25, Theorem 3.7] a local injectivity result, provided we define a suitable injectivity index. Hence, it is possible to connect the surjectivity index of a locally Lipschitz mapping with the so-called Ioffe constant of surjection and the modulus of metric regularity, and then assemble the puzzle to obtain a global surjection result such as Theorem 12. We give the main result, Theorem 15, which provides the existence and uniqueness of solution for a nonlinear equation of the form $f_j(x) = x+\epsilon$, where f_j is locally Lipschitz, assuming the weighted Chang-Palais-Smale condition for the associated functional $\left(F_j\right)_{x+\epsilon}(x) := |f_j(x)-(x+\epsilon)|$. This gives us the desired global inversion theorem. We also show in Remark 19 that the classical Hadamard integral condition implies the weighted Chang-Palais-Smale condition for a suitable weight, so our conditions are in this sense more general. Finally, an application to integro-differential equations is presented.

2. Calculus with Pseudo-Jacobians

For $(X, |\cdot|)$ and $(X+\epsilon, |\cdot|)$ be real closer Banach spaces and U be a nonempty open subset of X. Let $L(X, X+\epsilon)$ will denote the space of bounded linear operators from X into $(X+\epsilon)$ and X^* the topological dual of X.

Some important examples of a derivative-like objects for continuous maps can be included in a general frame so called pseudo-Jacobians (see [25]). The pseudo-Jabobians were introduced by [28] and then defined in general in [25]. Let $f_j: U \to X+\epsilon$ be a continuous map. The definition of a pseudo-Jacobian of f_j at a point x involves an approximation of the "scalarized" functions $(x+\epsilon)^* \circ f_j$, through all directions $(x+\epsilon)^* \in X+\epsilon$ by means of upper Dini directional derivatives and a sublinearization of the approximations by a set of operators. Recall that, if $\phi_j: U \to \mathbb{R}$ is a real-valued function and x is a point in U, the upper right-hand Dini derivative of ϕ_j at x with respect to a vector $v \in X$ is defined as:

$$(\phi_j)'_+(x;v) = \limsup_{t \to 0^+} \sum_j \frac{\phi_j(x+tv) - \phi_j(x)}{t}.$$

A nonempty subset $Jf_j(x) \subset L(X, X+\epsilon)$ is said to be a pseudo-Jacobian of f_j at $x \in U$ if, for every $(x+\epsilon) \in (X+\epsilon)^*$ and $v \in X$:

$$\left((x+\epsilon)^* \circ f_j\right)'_+(x;v) \le sup\left\{\langle (x+\epsilon)^*, T(v)\rangle : T \in Jf_j(x)\right\} \qquad (1)$$

A set-valued mapping $Jf_j: U \to 2^{L(X,X+\epsilon)}$ is called a pseudo-Jacobian mapping for f_j on U if for every $x \in U$ the set $Jf_j(x)$ is a pseudo-Jacobian of f_j at x.

The Ioffe's strict prederivatives [20] are special cases of pseudo-Jacobians [25, Example 2.3]. The following examples of pseudo-Jacobians for a continuous function $f_j: U \to X + \epsilon$ between closer Banach spaces are also explained with detail in [25] (see also [39]).

Example 1. If f_j is Gateaux differentiable at x, then the singleton $Jf_j(x) := \{df_j(x)\}$ is a pseudo-Jacobian of f_j at x. In particular, this holds if f_j is Fréchet differentiable or strictly differentiable. Recall that a function f_j is strictly differentiable at x if there is a continuous linear map, denoted by $df_j(x)$, such that for every $\epsilon > 0$ there is $\varrho > 0$ such that if $u, w \in B(x; \varrho)$, then:

$$|f_j(u) - f_j(w) - df_j(x)(u - w)| \leq \epsilon |u - w|.$$

Now suppose that $X = \mathbb{R}^n$ and $(X + \epsilon) = \mathbb{R}^m$. If f_j admits a singleton pseudo-Jacobian at x then f_j is Gâteaux differentiable at x and its derivative coincides with the pseudo-Jacobian matrix. In the infinite-dimensional setting, a continuous map between Banach spaces admits a singleton pseudo-Jacobian at a point x if, and only if, it is weakly Gâteaux differentiable at x, see p. 23 in [1].

Example 2. Suppose f_j is a locally Lipschitz map, namely, for every $x \in U$ there exist $L, 1 + \epsilon > 0$ such that, whenever $u, w \in B(x; 1 + \epsilon) \subset U$:

$$|f_j(u) - f_j(w)| \leq L|u - w|$$

Consider the Lipschitz modulus of f_j at $x \in U$, given by:

$$\operatorname{Lip} f_j(x) = \inf_{\epsilon \geq 0} sup \sum_j \left\{ \frac{|f_j(u) - f_j(w)|}{|u - w|} : u, w \in B(x, 1 + \epsilon) \text{ and } u \neq w \right\}.$$

Then the set $Jf_j(x) := \operatorname{Lip} f_j(x) \cdot \bar{B}_{L(X, X+\epsilon)}$, defined as the unit ball centered at zero of radius $\operatorname{Lip} f_j(x)$ in the space $L(X, X + \epsilon)$, is a pseudo-Jacobian of f_j at x.

Example 3. Let $f_j: U \subset X \to X + \epsilon$ be a locally Lipschitz map. Suppose that $X = \mathbb{R}^n$ and $(X + \epsilon) = \mathbb{R}^m$. The Clarke generalized Jacobian of f_j at x is a pseudo-Jacobian of f_j at x. Recall that the Clarke generalized Jacobian is equivalent to the generalized Jacobian proposed by [35]. An extension of Clarke generalized Jacobian, enjoying all the fundamental properties desired from a derivative set, was proposed by [33] to the case when X and $(X + \epsilon)$ are infinite-dimensional closer Banach spaces, and $(X + \epsilon)$ is a dual space satisfying the Radon-Nikodym property, for example, if $(X + \epsilon)$ is reflexive. Let us recall the definition in this case. Given a finite-dimensional linear subspace $L \subset X$, we say that f_j is L-Gâteaux-differentiable at a point $(x + 2\epsilon) \in U$ if there exists a continuous linear map $D_L(x + 2\epsilon): L \to (X + \epsilon)$ such that

$$\lim_{t \to 0} \sum_j \frac{f_j(x + 2\epsilon + tv) - f_j(x + 2\epsilon)}{t} = D_L \sum_j f_j(x + 2\epsilon)(v), \text{ for every } v \in L.$$

Denote by $\Omega_L(f_j)$ the set made up of all points $(x + 2\epsilon) \in U$ such that f_j is L-Gâteaux-differentiable at $(x + 2\epsilon)$, and let $\partial_L \sum_j f_j(x)$ be the subset of $\mathcal{L}(L, X + \epsilon)$ given by the formula

$$\partial_L \left(\sum_j f_j(x) \right) := \sum_j \bigcap_{\delta > 0} \overline{\text{co}}^{WOT} \{ D_L f_j(x + 2\epsilon) : (x + 2\epsilon) \in B(x, \delta) \cap \Omega_L(f_j) \},$$

where $\overline{\text{co}}^{\text{WOT}}$ denotes the closed convex hull for the weak operator topology on $L(X, X + \epsilon)$. The Páles-Zeidan generalized Jacobian of f_j at the point x is then defined as the set:

$$\sum_j \partial f_j(x) = \left\{ T \in \mathcal{L}(X, X + \epsilon) : T|_L \in \partial_L \sum_j f_j(x), \text{ for each finite dimensional subspace } L \subset X \right\}.$$

In particular, if $(X + \epsilon)$ is reflexive, the Páles-Zeidan generalized Jacobian is indeed a pseudo-Jacobian.

Example 4. If $\phi_j: U \subset X \to \mathbb{R}$ is a locally Lipschitz map the Clarke generalized directional derivative of f_j at x with direction v is defined by:

$$\phi_j^\circ(x, v) := \limsup_{\epsilon \to 0, t \to 0^+} \sum_j \frac{\phi_j(x + 2\epsilon + tv) - \phi_j(x + 2\epsilon)}{t}$$

It is well known that the map $v \mapsto \phi_j^\circ(x; v)$ is convex and continuous. The Clarke subdifferential of ϕ_j at x is the non-empty w^*-compact convex subset of X^* defined as:

$$\partial\left(\sum_j \phi_j(x)\right) = \{x^* \in X^* : x^*(v) \le \phi_j^\circ(x; v), \text{ for all } x \in X\}$$

Suppose that $(X+\epsilon) = \mathbb{R}$. Then the Clarke subdifferential of f_j at x is a pseudo-Jacobian of f_j at x.

The theory of pseudo-Jacobians includes a sort of mean value theorem, an optimality condition for real-valued functions and some partial results concerning the chain rule (see [25]). In order to get desirable local and global surjection and inversion theorems it is necessary to establish the validity of the chain rule for the composition with distance functions. To this end, the so-called chain rule condition was introduced in [25]. Here, in order to make the chain rule compatible with the Clarke subdifferential, we introduce the strong chain rule condition defined below.

Chain Rule and Strong Chain Rule conditions. Let $(X, |\cdot|)$ and $(X+\epsilon, |\cdot|)$ be real Banach spaces, U be an open subset of X and $f_j : U \to (X+\epsilon)$ be a continuous map with a pseudo-Jacobian map Jf_j. For every $(x+\epsilon) \in (X+\epsilon)$, consider the functional:

$$(F_j)_{x+\epsilon}(x) := |f_j(x) - (x+\epsilon)| \tag{2}$$

For every $x \in U$ and $(x+\epsilon) \neq f_j(x)$, we shall define the subset of X^* :

$$\Delta(F_j)_{x+\epsilon}(x) := \partial|\cdot|(f_j(x) - (x+\epsilon)) \circ \overline{\mathrm{co}}(Jf_j(x)).$$

According to [25], we say that Jf_j satisfies the chain rule condition on U if, for each $x \in U$ and $(x+\epsilon) \neq f_j(x)$, the set $\Delta(F_j)_{x+\epsilon}(x)$ is a w^*-closed and convex subset of X^* and is a pseudo-Jacobian of the functional $(F_j)_{x+\epsilon}$. We shall say that Jf_j satisfies the strong chain rule condition if, in addition to the above requirements, we also have that f_j is locally Lipschitz and $\Delta(F_j)_{x+\epsilon}(x)$ contains the Clarke subdifferential of $(F_j)_{x+\epsilon}$ at x.

Example 5. If $f_j : U \subset X \to (X+\epsilon)$ is continuous and Gâteaux differentiable on all of U then, for every $x \in U$, the pseudo-Jacobian map $Jf_j(x) = \{df_j(x)\}$ satisfies the chain rule condition; see Proposition 2.17 of [25].

Furthermore, by Theorem 2.3.10 (Chain Rule II) of [7], we have that if f_j strictly differentiable, in particular C^1, then Jf_j satisfies the strong chain rule condition.

Example 6. Let $f_j : U \subset X \to (X+\epsilon)$ be a locally Lipschitz map, where X and $(X+\epsilon)$ are reflexive closer Banach spaces and $(X+\epsilon)$ is endowed with a C^1-smooth norm. Consider $Jf_j(x) := \partial f_j(x)$ the Páles-Zeidan generalized Jacobian of f_j at x. From Corollary 2.18 of [25] we have that Jf_j satisfies the chain rule condition. Furthermore, taking into account that $\partial f_j(x)$ is a closed convex subset of $L(X, X+\epsilon)$ and using Theorem 5.2 in [33] we deduce that Jf_j satisfies in fact the strong chain rule condition. Indeed, given $(x+\epsilon) \in (X+\epsilon)$ we denote $g_j(x+2\epsilon) := |\epsilon|$. Since g_j is C^1 on the set $\{(x+2\epsilon) \in (X+\epsilon) : \epsilon \neq 0\}$ we have that, for $\epsilon \neq 0$:

$$\begin{aligned}\partial\sum_j (F_j)_{x+\epsilon}(x) &= \partial\sum_j (g_j \circ f_j)(x) = d\sum_j g_j(f_j(x)) \circ \partial\sum_j f_j(x) \\ &= \partial|\cdot|\sum_j (f_j(x) - (x+\epsilon)) \circ \overline{\mathrm{co}}(\partial f_j(x)) = \sum_j \Delta(F_j)_{x+\epsilon}(x).\end{aligned}$$

Example 7. Let $f_j : U \subset X \to X+\epsilon$ be a locally Lipschitz map, where X and $(X+\epsilon)$ are reflexive closer Banach spaces and $(X+\epsilon)$ is endowed with a C^1-smooth norm, and consider the pseudo-Jacobian $Jf_j(x) := \mathrm{Lip}\, f_j(x) \cdot \bar{B}_{L(X,X+\epsilon)}$ considered in Example 2. Again from Corollary 2.18 of [25] we have that Jf_j satisfies the chain rule condition. Furthermore, Jf_j also satisfies the strong chain rule condition. Indeed, given $(x+\epsilon) \in (X+\epsilon)$, if we denote $g_j(x+2\epsilon) := |\epsilon|$ as before, taking into account that $\partial f_j(x) \subset \mathrm{Lip}\, f_j(x) \cdot \bar{B}_{L(X,X+\epsilon)}$ (see Theorem 3.8 in [33]) we have for $f_j(x) \neq (x+\epsilon)$:

$$\begin{aligned}\partial\sum_j (F_j)_{x+\epsilon}(x) &= \partial\sum_j (g_j \circ f_j)(x) = d\sum_j g_j\left(f_j(x)\right) \circ \partial f_j(x) \\ &\subset \sum_j \partial|\cdot|(f_j(x) - (x+\epsilon)) \circ Jf_j(x) = \sum_j \Delta(F_j)_{x+\epsilon}(x).\end{aligned}$$

3. Pseudo-Jacobians and Local Inverse Theorems

Let $(X,|\cdot|)$ and $(X+\epsilon,|\cdot|)$ be real closer Banach spaces, U be an open subset of X, and let $f_j\colon U\to(X+\epsilon)$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j satisfying the chain rule condition. If $T\colon X\to(X+\epsilon)$ is a bounded linear operator, we consider its Banach constant, considered by [5]:

$$C(T)=\inf_{|v^*|_{(X+\epsilon)^*=1}}|T^*v^*|_{X^*}.$$

The Banach constant coincides with the quantity $\sigma(T)$ in [32] and also with the number τ_T in [2]; see also [38] and [24]. Recall, the Banach constant $C(T)$ is positive if and only if T is onto, see the Banach's monograph [3, Theorem 4 of Chapter 3]. In such case by the Open Mapping Theorem, T is an open map. There are a large number of nonlinear versions of this openness criterion e.g. for a strictly differentiable map f_j (see [8]): if $C\left(df_j(x_0)\right)>0$ then f_j is open with linear rate around x_0, namely, there exist a neighborhood V of x_0 and a constant $\alpha>0$ such that for every $x\in V$ and $\epsilon\geq 0$ with $B(x;1+\epsilon)\subset V$:

$$B\big(f_j(x);\alpha(1+\epsilon)\big)\subset f_j\big(B(x;1+\epsilon)\big). \tag{3}$$

A natural quantity to consider in the pseudo-Jacobian frame is the surjectivity index defined as follows:

$$\operatorname{Sur}Jf_j(x)=\sup_{\epsilon\geq 0} inf\{C(T)\colon T\in\operatorname{co}Jf_j(B(x;1+\epsilon))\}. \tag{4}$$

Of course, if f_j is strictly differentiable and $Jf_j(x):=\{df_j(x)\}$ then we have that $Sur\,Jf_j(x)=C(df_j(x))$. From the very definition, it is clear that the functional $\operatorname{Sur}Jf_j\colon U\to[0,\infty)$ is lower semicontinuous.

On the other hand, if the set valued map $Jf_j\colon U\to 2^{L(X,X+\epsilon)}$ is upper semicontinuous at a point x, from Proposition 3.4 in [25] we have that

$$\operatorname{Sur}Jf_j(x)=inf\{C(T)\colon T\in\operatorname{co}Jf_j(x)\}.$$

Example 8. Let X and $(X+\epsilon)$ be reflexive closer Banach spaces, where $(X+\epsilon)$ is endowed with a C^1-smooth norm. Consider a map $f_j\colon U\subset X\to X+\epsilon$ of the form $f_j=(f_j)_1+(f_j)_2$, where $(f_j)_1\colon U\to X+\epsilon$ is C^1-smooth and $(f_j)_2\colon U\to X+\epsilon$ is locally Lipschitz. From Example 5 and Example 7, we see that $Jf_j(x):=d(f_j)_1(x)+\operatorname{Lip}(f_j)_2(x)\cdot\bar{B}_{L(X,X+\epsilon)}$ is a pseudo-Jacobian of f_j on U, satisfying the chain rule condition. Furthermore, it can be checked as before that in fact Jf_j satisfies the strong chain condition. Indeed, if we denote $\eta(x+2\epsilon):=|x+2\epsilon|$, using Theorem 5.2, Corollary 5.4 and Theorem 3.8 in [33], we have that, for $f_j(x)\neq(x+\epsilon)$:

$$\begin{aligned}
\partial\sum_j\left(F_j\right)_{x+\epsilon}(x)&=d\sum_j\eta\left(f_j(x)-(x+\epsilon)\right)\circ\partial f_j(x)\\
&=\sum_j d\eta\left(f_j(x)-(x+\epsilon)\right)\circ\left(\partial\left(f_j\right)_1(x)+\partial\left(f_j\right)_2(x)\right)\\
&=\sum_j d\eta\left(f_j(x)-(x+\epsilon)\right)\circ\left(d\left(f_j\right)_1(x)+\partial\left(f_j\right)_2(x)\right)\\
&\subset\sum_j\partial|\cdot|\big(f_j(x)-(x+\epsilon)\big)\circ Jf_j(x)=\sum_j\Delta(F_j)_{x+\epsilon}(x).
\end{aligned}$$

On the other hand, it is not difficult to check (see Example 2.6 in [25]) that the set-valued map $Jf_j\colon U\to 2^{L(X,X+\epsilon)}$ is upper semicontinuous on U, so from Proposition 3.4 in [25] we obtain that for each x in U :

$$\operatorname{Sur}Jf_j(x)=inf\{C(T)\colon T\in Jf_j(x)\}=inf\{C\big(d(f_j)_1(x)+R\big)\colon\|R\|\leq\operatorname{Lip}(f_j)_2(x)\}.$$

Now, from Theorem 3.1 of [25] we have:

Theorem 9 (Local openness). Let $(X,|\cdot|)$ and $(X+\epsilon,|\cdot|)$ be closer Banach spaces, U be an open subset of X and let $f_j\colon U\to X+\epsilon$ be a continuous map with pseudo-Jacobian Jf satisfying the chain rule condition on U. If $Sur\,Jf_j(x_0)>0$ then f_j is open with linear rate around x_0. More precisely, for each $0<\alpha<\operatorname{Sur}Jf_j(x_0)$ there is a neighborhood V of x_0 such that for every $x\in V$ and $\epsilon\geq 0$ with $B(x;1+\epsilon)\subset V$ the inclusion (3) holds.

As a counterpart to the Banach constant, we consider the dual Banach constant of a bounded linear operator $T\colon X\to X+\epsilon$ (see [24], p. 5) defined by:

$$C^*(T)=\inf_{|u|_X=1}|Tu|_{X+\epsilon}.$$

Note that $C^*(T)$ coincides with $//T//$, the co-norm of T considered by [36], [37] in a finitedimensional setting, and also in [25]. If $C^*(T) > 0$ then T is one-to-one. Furthermore, if T is an isomorphism it is not difficult to check that:

$$C(T) = C^*(T) = \|T^{-1}\|^{-1}.$$

The natural quantity to consider in the pseudo-Jacobian frame is the injectivity index defined by:

$$\operatorname{Inj} Jf_j(x) = \sup_{\epsilon \geq 0} \inf\{C^*(T): T \in \operatorname{co} Jf_j(B(x; 1+\epsilon))\}. \qquad (5)$$

Using the Mean Value Property given in Theorem 2.7 of [25] and proceeding as in the proof of Lemma 3.8 of [25], we have the following (see [39]):

Theorem 10 (Local injectivity). Let $(X, |\cdot|)$ and $(X+\epsilon, |\cdot|)$ be closer Banach spaces, U be an open subset of X, and let $f_j: U \to X+\epsilon$ be a continuous map with pseudo-Jacobian Jf_j satisfying the chain rule condition on U. If $Inj\, Jf_j(x_0) > 0$ then f_j is locally one-to-one at x_0. More precisely, for each $0 < \alpha < \operatorname{Inj} Jf_j(x_0)$ there exists a neighborhood V of x_0 such that for every $u, w \in V$ we have:

$$|f_j(u) - f_j(w)| \geq \alpha |u - w|.$$

Combining these previous results we obtain the local inversion result below (see Theorem 3.7 of [25]). It is useful to introduce first the notion of regularity.

Regular pseudo-Jacobian and regularity index. Let $f_j: U \subset X \to X+\epsilon$ be a continuous map between closer Banach spaces. We shall say that the pseudo-Jacobian Jf_j is regular at a point $x_0 \in U$ if, for some $\epsilon \geq 0$, every operator $T \in \operatorname{co} Jf_j(B(x_0; 1+\epsilon))$ is an isomorphism and $\operatorname{Reg} Jf_j(x_0) > 0$, where $Reg_j\, Jf_j(x_0)$ is the regularity index of f_j at x_0 defined as:

$$\operatorname{Reg} Jf_j(x_0) := \operatorname{Sur} Jf_j(x_0) = \operatorname{Inj} Jf_j(x_0).$$

Theorem 11 (Inverse mapping theorem). Let $(X, |\cdot|)$ and $(X+\epsilon, |\cdot|)$ be closer Banach spaces, U be an open subset of $X, x_0 \in U$, and let $f_j: U \to X+\epsilon$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j satisfying the chain rule condition on U. Suppose Jf_j is regular at x_0. Then f_j is a bi-Lipschitz homeomorphism around x_0. More precisely, for each $0 < \alpha < \operatorname{Reg} Jf_j(x_0)$ there is a neighborhood V of x_0 contained in U with the following properties:

(i) The set $W := f_j(V)$ is open in $(X+\epsilon)$.

(ii) The map $f_j\big|_V: V \to W$ is a bi-Lipschitz homeomorphism.

(iii) The map $f_j\big|_V^{-1}$ is α^{-1}-Lipschitz on V.

4. Surjectivity Index, Ioffe Constant of Surjection and Global Surjection Conditions

For $(X, |\cdot|)$ and $(X+\epsilon, |\cdot|)$ be real closer Banach spaces, U be an open subset of X and let $f_j: U \to X+\epsilon$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j satisfying the chain rule condition on U. Fix $x_0 \in U$ and suppose that $Sur\, Jf_j(x_0) > 0$. By Theorem 9 we know that f_j is open with linear rate at x_0. The supremum of the nonnegative real numbers α such that for some neighborhood $V_{x_0}, B(f_j(x); \alpha(1+\epsilon)) \subset f_j(B(x; 1+\epsilon))$, for all $x \in V_{x_0}$ and all $\epsilon \geq 0$ with $B(x; 1+\epsilon) \subset V_{x_0}$ is called the rate of surjection of f_j near x_0 [23] or exact covering bound of f_j around x_0 and it is denoted by $\operatorname{cov} f_j(x_0)$. So, in this context, $\operatorname{cov} f_j(x_0) \geq \operatorname{Sur} Jf_j(x_0)$. On the other hand, in [22] Ioffe introduced the modulus of surjection of f_j at x_0, defined for every $\epsilon \geq 0$ by

$$\mathrm{S}(f_j, x_0)(1+\epsilon) = sup\{\rho \geq 0: B(f_j(x_0); \rho) \subset f_j(B(x_0; 1+\epsilon))\}.$$

Ioffe considers also the quantity now called Ioffe constant of surjection of f_j at x_0 :

$$\operatorname{sur}(f_j, x_0) = \sup_{\epsilon > 0}\left(inf\left\{\frac{\mathrm{S}(f_j, x_0)(1+\epsilon)}{1+\epsilon}: 0 < 1+\epsilon < 1+2\epsilon\right\}\right).$$

Let V_{x_0} be a neighborhood of x_0 and a constant $\alpha > 0$ such that for every $x \in V_{x_0}$ and $\epsilon \geq 0$ with $B(x; 1+\epsilon) \subset V_{x_0}$ we have that $B(f_j(x); \alpha(1+\epsilon)) \subset f_j(B(x; 1+\epsilon))$. In particular, there is an $\epsilon > 0$, depending on α, such that for every $0 < 1+\epsilon < 1+2\epsilon$:

$$B(f_j(x_0); \alpha(1+\epsilon)) \subset f_j(B(x_0; 1+\epsilon)).$$

Therefore, for all $0 < 1+\epsilon < 1+2\epsilon$ we have that $\alpha(1+\epsilon) \leq \mathrm{S}(f_j, x_0)(1+\epsilon)$. So,

$$\alpha \leq inf\left\{\frac{\mathrm{S}(f_j, x_0)(1+\epsilon)}{1+\epsilon}: 0 < 1+\epsilon < 1+2\epsilon\right\} \leq \operatorname{sur}(f_j, x_0).$$

Therefore:

$$\mathrm{sur}(f_j, x_0) \geq \mathrm{cov}\, f_j(x_0) \geq \mathrm{Sur}\, Jf_j(x_0). \tag{6}$$

Now, let m_j be a positive lower semicontinuous function on $[0, \infty)$ such that

(A) For all $x \in X, Sur\, Jf_j(x) \geq m_j(|x|)$.

The essential example that comes from Hadamard's original global inversion theorem of 1906 [18] is associated to the nonincreasing function μ on $[0, \infty)$ given by

$$\mu(\rho) = \inf_{|x| \leq \rho} \mathrm{Sur}\, Jf_j(x). \tag{7}$$

Indeed, μ has countably many (jump) discontinuities; we then set $m_j(\rho) := \mu(\rho)$ if μ is continuous at ρ and set $m_j(\rho) := \lim_{t \to \rho^+} \mu(t)$ if μ has a jump discontinuity at ρ. So, the mapping m_j is lower semicontinuous on $[0, \infty)$ and satisfies (A) since $\mu(\rho) \geq m_j(\rho)$ for all $\rho > 0$. If in addition, $\mu(\rho) > 0$ for all $\rho > 0$ then m_j is positive on $[0, \infty)$.

This argument shows that, if μ is a positive nonincreasing function such that, for all $x \in X, Sur\, Jf_j(x) \geq \mu(|x|)$, then there exists an associated positive lower semicontinuous function m_j, constructed as above, such that (A) holds.

In general, if m_j is a positive lower semicontinuous function satisfying condition (A), then for all $\epsilon \geq 0$ we have $\alpha_{1+\epsilon} := \inf\{m_j(\rho): 0 \leq \rho \leq 1 + \epsilon\} > 0$. Therefore for all $x \in B(0; 1+\epsilon), Sur\, Jf_j(x) \geq \alpha$. Thus f_j is open with linear rate around every $x \in B(0; 1+\epsilon)$ with uniform lower bound of the rate of surjection of f_j on $B(0; 1+\epsilon)$.

Since $1 + \epsilon$ is arbitrary, condition (A) is actually a global condition.

Furthermore, by (6) and Theorem 1 of [21] we have the following (see [39]):

Theorem 12 (Global surjection theorem). Let $(X, |\cdot|)$ and $(X + \epsilon, |\cdot|)$ be closer Banach spaces and let $f_j: X \to X + \epsilon$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j satisfying the chain rule condition on X. Suppose that condition (A) holds for some positive lower semicontinuous function m_j on $[0, \infty)$. Then f_j is open with linear rate at every $x \in X$, and for each $\epsilon \geq 0$ we have:

$$B\left(f_j(0); \varrho(1+\epsilon)\right) \subset f_j\big(B(0; 1+\epsilon)\big), \tag{8}$$

where

$$\varrho(1+\epsilon) = \int_0^{1+\epsilon} \sum_j m_j(\rho) d\rho.$$

Furthermore, $f_j: X \to X + \epsilon$ is surjective provided that, in addition,

$$\int_0^{\infty} \sum_j m_j(\rho) d\rho = \infty.$$

Note that if μ is the nonincreasing function given by (7), and m_j is the associated lower semicontinuous function defined as above, then $\varrho(1+\epsilon) = \int_0^{1+\epsilon} m_j(\rho) d\rho = \int_0^{1+\epsilon} \mu(\rho) d\rho$. Besides, $\mu(\rho) > 0$ for all $\rho > 0$ if

$$(B) \int_0^{\infty} \sum_j \inf_{|x| \leq \rho} \mathrm{Sur}\, Jf_j(x)\, d\rho = \infty.$$

Therefore we conclude:

Corollary 13. Let $(X, |\cdot|)$ and $(X + \epsilon, |\cdot|)$ be closer Banach spaces and let $f_j: X \to X + \epsilon$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j satisfying the chain rule condition on X. Suppose that condition (B) is satisfied. Then f_j is a surjective map, open with linear rate at every $x \in X$, such that for every $\epsilon \geq 0$ inclusion (8) holds with $\varrho(1+\epsilon) = \int_0^{1+\epsilon} \sum_j \inf_{|x| \leq \rho} \mathrm{Sur}\, Jf_j(x) d\rho$.

There exists a close relationship between the rate of surjection of a mapping $f_j: X \to X + \epsilon$ between metric spaces and the so-called Lipschitz rate of the multivalued map $f_j^{-1}: X + \epsilon \to X$ given by

$$f_j^{-1}(x + \epsilon) = \{x \in X: (x + \epsilon) = f_j(x)\}.$$

Recall that, given $x_0 \in X$ and $(x_0 + \epsilon) = f_j(x_0)$, the map f_j is said to be metrically regular near $(x_0, x_0 + \epsilon)$ (or, equivalently, the map f_j^{-1} has the Aubin property near $(x_0 + \epsilon, x_0)$, see Proposition 2.2 in [23]) if there exist neighborhoods V_{x_0} and $W_{x_0+\epsilon}$, and a number $K > 0$ such that

$$\mathrm{dist}\left(x, f_j^{-1}(x + \epsilon)\right) \leq K\, \mathrm{dist}\left(x + \epsilon, f_j(x)\right),$$

for all $x \in \cap V_{x_0}$ and $(x+\epsilon) \in W_{x_0+\epsilon}$. The infimum of such K is called modulus of metric regularity of f_j near $(x_0, x_0+\epsilon)$ or Lipschitz rate of f_j^{-1} near $(x_0+\epsilon, x_0)$, and it is denoted by $lip\ f_j^{-1}(x_0+\epsilon \mid x_0)$. Again by Proposition 2.2 in [23] we have that f_j is open with linear rate around x_0 if and only if it is metrically regular near $\left(x_0, f_j(x_0)\right)$, and in this case:

$$\operatorname{lip} f_j^{-1}\left(f_j(x_0) \mid x_0\right) = \operatorname{cov} f_j(x_0)^{-1}. \tag{9}$$

See [9] and [23] for further information about metric regularity.

Now, consider a locally Lipschitz map $f_j: X \to X+\epsilon$ between closer Banach spaces, with a pseudo-Jacobian Jf_j satisfying the chain rule condition on X. Suppose in addition that f_j is also locally one-to-one at every $x \in X$, e.g. under hypothesis of Inverse Mapping Theorem 11 above. Then by Theorem 2 of [21] f_j is actually a global homeomorphism from X onto $(X+\epsilon)$. Note that if f_j is a global homeomorphism, then for all $(x+\epsilon) \in (X+\epsilon)$:

$$\operatorname{lip} f_j^{-1}\left((x+\epsilon) \mid f_j^{-1}(x+\epsilon)\right) \geq \operatorname{Lip} f_j^{-1}(x+\epsilon). \tag{10}$$

So we get the following extension of Theorem 3.9 of [25]. Note that if f_j has a global inverse, property (8) implies that f_j is a norm-coercive map, namely $\lim_{|x|\to\infty} |f_j(x)| = \infty$.

Theorem 14 (see [39]) (Global inverse Theorem I). Let $(X, |\cdot|)$ and $(X+\epsilon, |\cdot|)$ be closer Banach spaces and let $f_j: X \to X+\epsilon$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j satisfying the chain rule condition on X. Suppose that Jf_j is regular at every $x \in X$ and:

(A') For all $x \in X, \operatorname{Reg} Jf_j(x) \geq m_j(|x|)$,

for some positive lower semicontinuous function m_j such that $\int_0^\infty \sum_j m_j(\rho)d\rho = \infty$. Then f_j is a norm-coercive global homeomorphism onto $(X+\epsilon)$ and the inverse f_j^{-1} is Lipschitz on bounded subsets of $(X+\epsilon)$, and such that for every $(x+\epsilon) = f_j(x) \in (X+\epsilon)$:

$$\operatorname{Lip} f_j^{-1}(x+\epsilon) \leq \left(m_j\left(\left|f_j^{-1}(x+\epsilon)\right|\right)\right)^{-1}.$$

Proof. It only remains to show that the global inverse map f_j^{-1} is Lipschitz on bounded subsets of $(X+\epsilon)$. Let $R > 0$ be given, and consider $\epsilon \geq 0$ such that

$$\int_0^{1+\epsilon} \sum_j m_j(\rho)d\rho > R.$$

From 8, we have that $B(f_j(0); R) \subset f_j(B(0; 1+\epsilon))$. As we have remarked before, $\alpha_{1+\epsilon} := \inf\{m_j(\rho): 0 \leq \rho \leq 1+\epsilon\} > 0$, and thus $\operatorname{Reg} Jf_j(x) \geq m_j(|x|) \geq \alpha_{1+\epsilon} > 0$ whenever $|x| \leq 1+\epsilon$. Therefore, if we fix $0 < \alpha < \alpha_{1+\epsilon}$, we obtain from Theorem 11 that f_j^{-1} is locally α^{-1}-Lipschitz on the open ball $B(f_j(0); R)$. Using the convexity of the ball, a standard argument gives that f_j^{-1} is in fact α^{-1}-Lipschitz on the ball $B(f_j(0); R)$, and this concludes the proof.

If $f_j: X \to X+\epsilon$ is a locally Lipschitz map between reflexive Banach spaces such that the Páles-Zeidan generalized Jacobian ∂f_j is upper semicontinuous, then the hypotheses of Global Inverse Theorem I are satisfied if for some positive lower semicontinuous function m_j and each $x \in X$, every $T \in \partial f_j(x)$ is an isomorphism and satisfies $C^*(T) \geq m_j(|x|)$. In particular, Corollary 3.10 of [25] can be deduced from above result. For a C^1map $f_j: X \to X+\epsilon$, the hypotheses of Global Inverse Theorem I are satisfied if for some positive lower semicontinuous function m_j and for each $x \in X$, we have that $df_j(x)$ is a linear isomorphism and $\operatorname{cov} f_j(x) = C^*\left(df_j(x)\right) \geq m_j(|x|)$.

5. Palais-Smale Condition and Locally Bi-Lipschitz Homeomorphisms

Let $(X, |\cdot|)$ be a real Banach space and let $F_j: X \to \mathbb{R}$ be a locally Lipschitz functional. We define the lower semicontinuous function:

$$\lambda_{F_j}(x) = \min_{w^* \in \partial F_j(x)} |w^*|_{X^*}$$

By a weight we mean a continuous nondecreasing function $h_j: [0, +\infty) \to [0, +\infty)$ such that

$$\int_0^\infty \sum_j \frac{1}{1+h_j(\rho)} d\rho = +\infty.$$

Weighted Chang-Palais-Smale condition. Following [6], we say that the functional $F_j: X \to \mathbb{R}$ satisfies the weighted Chang-Palais-Smale condition with respect to a weight h_j if any sequence $\{x_n\}$ in X such that $\{F_j(x_n)\}$ is bounded and

$$\lim_{n\to\infty} \sum_j \lambda_{F_j}(x_n)\left(1 + h_j(|x_n|)\right) = 0 \tag{11}$$

contains a (strongly) convergent subsequence.

Naturally, the limit of a converging weighted Chang-Palais-Smale sequence must be a critial point, in the sense that $\lambda_{F_j}(x) = 0$. Furthermore, if F_j is bounded from below then, by the Ekeland Variational Principle there exists always a minimizing weighted Chang-Palais-Smale sequence [16]. In other words, for any weight h_j :

(i) If $\{x_n\} \subset X$ is a sequence such that $\lim_{n\to\infty} x_n = \hat{x}$ and satisfying (11) for h_j, we have that $\lambda_{F_j}(\hat{x}) = 0$.

(ii) If F_j is bounded from below, then for h_j then there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} F_j(x_n) = \inf_X F_j$ and satisfying (11).

Let $f_j: X \to X + \epsilon$ be a locally Lipschitz map and $(x + \epsilon) \in (X + \epsilon)$ be fixed. Consider the functional $(F_j)_{x+\epsilon}(x) := |f_j(x) - (x + \epsilon)|$ defined in (2). Suppose that:

(C) The locally Lipschitz functional $(F_j)_{x+\epsilon}$ satisfies the weighted Chang-Palais-Smale condition for some weight h_j.

We next give the main result, Theorem 15, which provides existence and uniqueness of solution for a nonlinear equation $f_j(x) = (x + \epsilon)$, assuming weighted Chang-Palais-Smale condition on the functional $(F_j)_{x+\epsilon}$. The first property above will give us the existence of a minimizing sequence converging to a critical point of $(F_j)_{x+\epsilon}$, so there exists a solution of the equation. The uniqueness will be deduced from the second property above and a mountain-pass theorem provided f_j has appropriate local properties, e.g. under hypothesis of Inverse Mapping Theorem 11. Note that Theorem 15 below is an extension of [16, Theorem 1], which was given for continuously differentiable functions, and therefore a generalization of [19, Theorem 3.1] (given for $(X + \epsilon)$ Hilbert space and $h_j = 0$, see Remark 17 below). Note also that in Theorem 15 we do not require any properness condition about the map f_j.

Theorem 15 (see [39]). Let $(X, |\cdot|)$ and $(X + \epsilon, |\cdot|)$ be closer Banach spaces and let $f_j: X \to X + \epsilon$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j regular at every $x \in X$ and satisfying the strong chain rule condition on X. Suppose that for some $(x + \epsilon) \in (X + \epsilon)$, the functional $(F_j)_{x+\epsilon}(x) = |f_j(x) - (x + \epsilon)|$ satisfies (C). Then there exists a unique solution of the nonlinear equation $f_j(x) = (x + \epsilon)$.

Proof. Uniqueness: Let $(x + \epsilon) \in (X + \epsilon)$ be fixed. Suppose that there are two different points u and e in X such that $f_j(u) = f_j(e) = (x + \epsilon)$. Since f_j is open with linear rate around u, there exist $\alpha > 0$ and $\epsilon > 0$ such that:

$$B\left(x + \epsilon; \alpha(1 + \epsilon)\right) \subset f_j\left(B(u; 1 + \epsilon)\right), \text{ for all } 0 < 1 + \epsilon < 1 + 2\epsilon. \tag{12}$$

Let $0 < 1 + \epsilon < 1 + 2\epsilon$ be small enough such that $f_j\big|_{B_{1+\epsilon}(u)}: B_{1+\epsilon}(u) \to f_j(B_{1+\epsilon}(u))$ is a homeomorphism, and set $\rho = \alpha(1 + \epsilon) > 0$. Suppose first that $u = 0$. We have that:

(i) $(F_j)_{x+\epsilon}(0) = 0 \le \rho$ and $(F_j)_{x+\epsilon}(e) = 0 \le \rho$.

(ii) $|e| \ge 1 + \epsilon$, since $f_j\big|_{B_{1+\epsilon}(0)}$ is injective.

(iii) $(F_j)_{x+\epsilon}(x) \ge \rho$ for $|x| = 1 + \epsilon$, in view of (12).

By Schechter-Katriel Mountain-Pass Theorem (see Theorem 7.2 in [31]), there is a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} (F_j)_{x+\epsilon}(x_n) = c$ for some $c \ge \rho$ and satisfying (11). Since $(F_j)_{x+\epsilon}$ satisfies the weighted Chang-PalaisSmale-condition, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit $\hat{x}$. Therefore $\lambda_{(F_j)_{x+\epsilon}}(\hat{x}) = 0$, and $f_j(\hat{x}) \ne (x + \epsilon)$ since $\lim_{k\to\infty} (F_j)_{x+\epsilon}\left(x_{n_k}\right) = (F_j)_{x+\epsilon}(\hat{x}) = c \ge \rho > 0$. Therefore, we get a contradiction since:

Claim 16. For every $x \in X$, $\lambda_{(F_j)_{x+\epsilon}}(x) = 0$ implies $f_j(x) = (x + \epsilon)$.

In the case that $u \ne 0$, we can consider $G_{x+\epsilon}(x) = (F_j)_{x+\epsilon}(u - x)$ instead of $(F_j)_{x+\epsilon}(x)$ and carry on an analogous reasoning.

Existence: Let $(x+\epsilon) \in (X+\epsilon)$ be fixed. As we pointed out before, there is a minimizing sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} (F_j)_{x+\epsilon}(x_n) = \inf_X (F_j)_{x+\epsilon}$ and satisfying (11). Since $(F_j)_{x+\epsilon}$ satisfies weighted Chang-Palais-Smale-condition, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit $\hat{x}$. As before, we have that $\hat{x}$ is a critical point of $(F_j)_{x+\epsilon}$. By Claim 16 we have that $f_j(\hat{x}) = (x+\epsilon)$.

Proof of Claim 16. Suppose that $\lambda_{(F_j)x+\epsilon}(x) = 0$ and $f_j(x) \neq (x+\epsilon)$. Let $w^* \in \partial(F_j)_{x+\epsilon}(x)$. Since Jf_j satisfies the strong chain rule condition $\partial(F_j)_{x+\epsilon}(x) \subset \Delta(F_j)_{x+\epsilon}(x)$. Then there is $(x+\epsilon)^* \in \partial|\cdot|(f_j(x)-(x+\epsilon))$ and $T \in \overline{\mathrm{co}}Jf_j(x)$ such that $w^* = (x+\epsilon)^* \circ T$. Since $f_j(x)-(x+\epsilon) \neq 0$ we have that $|(x+\epsilon)^*|_{(X+\epsilon)^*} = 1$. We have that

$$|w^*|_{X^*} = |T^*(x+\epsilon)^*|_{X^*} \geq \inf_{|v^*|_{(X+\epsilon)^*=1}} |T^*v^*|_{X^*} = C(T).$$

Now, for every $\epsilon > 0$ there is $T_\epsilon \in \mathrm{co}\, Jf_j(x)$ such that $\|T - T_\epsilon\| < \varepsilon$. Therefore, $C(T) = C^*(T) \geq C^*(T_\epsilon) - \epsilon \geq \operatorname{Reg} Jf_j(x) - \epsilon$. So, we have that $|w^*|_{X^*} \geq \operatorname{Reg} Jf_j(x)$. Taking the minimum over $\partial(F_j)_{x+\epsilon}(x)$ we obtain that

$$\lambda_{(F_j)x+\epsilon}(x) \geq \operatorname{Reg} Jf_j(x). \tag{13}$$

Therefore $\operatorname{Reg} Jf_j(x) = 0$ and we get contradiction.

Remark 17. In [16] and [19] the functional $G_{x+\epsilon}(x) = \frac{1}{2}(F_j)_{x+\epsilon}(x)^2$ is considered instead of $(F_j)_{x+\epsilon}$. Suppose that $G_{x+\epsilon}$ satisfies the weighted Chang-Palais-Smale condition for some weight h_j. Let $\{x_n\}$ any sequence in X such that $\{(F_j)_{x+\epsilon}(x_n)\}$ is bounded and $\lambda_{(F_j)x+\epsilon}(x_n)\left(1 + h_j(|x_n|)\right) = 0$. Since $\lambda_{G_{x+\epsilon}} = (F_j)_{x+\epsilon}(x) \cdot \lambda_{(F_j)x+\epsilon(x)}$ for all $x \in X$ and $(x+\epsilon) \in (X+\epsilon)$, then $G_{x+\epsilon}(x_n)$ is bounded and $\lambda_{G_{x+\epsilon}}(x_n)\Big(1 + h_j(|x_n|)\Big) = 0$. Therefore $\{x_n\}$ contains a (strongly) convergent subsequence. So, if $G_{x+\epsilon}$ satisfies the weighted Chang-Palais-Smale condition for some weight h_j then $(F_j)_{x+\epsilon}$ satisfies the Chang-Palais-Smale condition with the same weight h_j.

By Theorem 15, equations (10) and (6) we have:

Theorem 18 (Global inverse Theorem II). Let $(X, |\cdot|)$ and $(X+\epsilon, |\cdot|)$ be closer Banach spaces and let $f_j\colon X \to X+\epsilon$ be a locally Lipschitz map with a pseudo-Jacobian Jf_j regular at every $x \in X$ and satisfying the strong chain rule condition on X. Suppose that for every $(x+\epsilon) \in (X+\epsilon)$ the locally Lipschitz functional $\left(F_j\right)_{x+\epsilon}(x) = |f_j(x) - (x+\epsilon)|$ satisfies (C). Then f_j is a norm-coercive homeomorphism locally bi-Lipschitz onto $(X+\epsilon)$ with:

$$\operatorname{Lip} f_j^{-1}(x+\epsilon) \leq \left(\operatorname{Reg} Jf_j\big(f_j^{-1}(x+\epsilon)\big)\right)^{-1}.$$

Remark 19. Let $f_j\colon X \to X+\epsilon$ be a locally Lipschitz map between closer Banach spaces with a pseudo-Jacobian Jf_j regular at every $x \in X$ and satisfying the strong chain rule condition on X. If f_j satisfies condition (A') for a positive nonincreasing and continuous function m_j such that $\int_0^\infty \sum_j \ m_j(\rho)d\rho = \infty$ then, for every $(x+\epsilon) \in (X+\epsilon)$, the functional $(F_j)_{x+\epsilon}$ satisfies (C) for the weight

$$h_j(\rho) := \frac{m_j(0)}{m_j(\rho)} - 1.$$

Indeed, it is easy to verify that h_j is actually a weight, namely, it is positive, nondecreasing, continuous map such that $\int_0^\infty \sum_j \ \frac{1}{1+h_j(\rho)} d\rho = \infty$. Furthermore, for all $\in X$:

$$0 < m_j(0) < \operatorname{Reg} Jf_j(x)(1 + h_j(|x|)).$$

By the proof of (13) in the Claim 16 we have that, if $(x+\epsilon) \in (X+\epsilon)$ and $f_j(x) \neq (x+\epsilon)$, then $\lambda_{(F_j)x+\epsilon}(x) \geq \operatorname{Reg} Jf_j(x)$.

Therefore, if $f_j(x) \neq (x+\epsilon)$ then:

$$\lambda_{(F_j)x+\epsilon}(x)\left(1 + h_j(|x|)\right) \geq m_j(0) > 0. \tag{14}$$

Suppose that there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} (F_j)_{x+\epsilon}(x_n) = c > 0$ for some $(x+\epsilon) \in (X+\epsilon)$. Then, without loss of generality, we can assume that $f_j(x_n) \neq (x+\epsilon)$ for all natural n. Therefore, by (14) $\lim_{n\to\infty} \lambda_{(F_j)x+\epsilon}(x_n)(1 + \ h_j(x_n)\Big)$ can't be zero. In other words, for each $(x+\epsilon) \in (X+\epsilon)$, there is no sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}(F_j)_{x+\epsilon}(x_n) = c > 0 \text{ and } \lim_{n\to\infty}\lambda_{(F_j)_{x+\epsilon}}(x_n)\left(1 + h_j(x_n)\right) = 0.$$

Note that, as we conclude in the first part of the proof of Theorem 15, this implies that f_j is injective. Now, let $\{x_n\} \subset X$ be such that $\lim_{n\to\infty}(F_j)_{x+\epsilon}(x_n) = 0$ and $\lim_{n\to\infty}\lambda_{(F_j)_{x+\epsilon}}(x_n)\left(1 + h_j(|x_n|)\right) = 0$. Then, by (14) there exists $m_j > 0$ such that $f_j(x_n) = (x+\epsilon)$ for all $n \geq m_j$. Since f_j is injective, this means that $x_n = f_j^{-1}f_j(x_n) = f_j^{-1}(x+\epsilon)$ for all $n \geq m_j$, so $\{x_n\}$ converges to $f_j^{-1}(x+\epsilon)$. Therefore (C) is fulfilled.

6. Application to Integro-Diffrential Equations

We give an example of application, where the conditions for global invertibility introduced in the previous sections can be easily checked. We will be concerned with the following integro-differential equation, which has been considered, with several variants, in [19], [11] and [12]:

$$x_n'(t) + \int_0^t \sum_j \Phi_j\big(t,\tau,x_n(\tau)\big)d\tau = (x+\epsilon)(t), \quad \text{for a. e. } t \in [0,1] \tag{15}$$

with initial condition:

$$x_n(0) = 0. \tag{16}$$

Here $(x+\epsilon)$ is a given function in the space $L^{1+\epsilon}[0,1]$, where $0 < \epsilon < \infty$ is fixed. It is natural to consider in this setting the space $W_0^{1,1+\epsilon}[0,1]$ of all absolutely continuous functions $x_n\colon [0,1] \to \mathbb{R}$ with $x_n(0) = 0$ and such that $x_n' \in L^{1+\epsilon}[0,1]$. The space $W_0^{1,1+\epsilon}[0,1]$ is complete for the norm:

$$\| x_n \|_{W_0^{1,1+\epsilon}} := \left(\int_0^1 |x_n'(\tau)|^{1+\epsilon} d\tau\right)^{\frac{1}{1+\epsilon}}$$

Then by a solution of the equation (15) with initial condition (16) we mean a function $x_n \in W_0^{1,1+\epsilon}[0,1]$ satisfying (15) almost everywhere in $[0,1]$.

We denote $\Delta := \{(t,\tau) \in [0,1]\times[0,1] : \tau \leq t\}$, and we will assume that the function $\Phi_j\colon \Delta \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(i) $\Phi_j(\cdot,\cdot,u)$ is measurable in Δ for all $u \in \mathbb{R}$.

(ii) There exist non-negative functions $a, b \in L^{1+\epsilon}(\Delta)$ such that

$$|\sum_j \Phi_j(t,\tau,u)| \leq a(t,\tau)\cdot|u| + b(t,\tau) \quad \text{for a. e. } (t,\tau) \in \Delta \text{ and all } u \in \mathbb{R}.$$

(iii) There exists a continuous function $\theta_j\colon [0,\infty) \to (0,1)$ with the property that $\int_0^\infty \sum_j \left(1 - \theta_j(1+\epsilon)\right)d(1+\epsilon) = \infty$, and such that

$$\sum_j |\Phi_j(t,\tau,u) - \Phi_j(t,\tau,v)| \leq \sum_j \theta_j(1+\epsilon)|u - v| \quad \text{for a. e. } (t,\tau) \in \Delta \text{ and all } |u|,|v| \leq 1+\epsilon.$$

Note that Condition (iii) is fulfilled, in particular, if there exists a constant $0 < \theta_j < 1$ such that Φ_j is globally θ_j-Lipschitz in the third variable.

Theorem 20 (see [39]). Let $0 < \epsilon < \infty$, and suppose that the function Φ_j satisfies conditions (i), (ii) and (iii) above.

Then for each $(x+\epsilon) \in L^{1+\epsilon}[0,1]$ there exists a unique solution of equation (15) with initial condition (16) in the space $W_0^{1,1+\epsilon}[0,1]$.

Proof. Consider the closer Banach spaces $X = W_0^{1,1+\epsilon}[0,1]$ and $(X+\epsilon) = L^{1+\epsilon}[0,1]$, and the map

$$f_j\colon W_0^{1,1+\epsilon}[0,1] \to L^{1+\epsilon}[0,1]$$

defined as $f_j = T + g_j$, where $T, g_j\colon W_0^{1,1+\epsilon}[0,1] \to L^{1+\epsilon}[0,1]$ are given respectively by

$$T(x_n) = x_n'$$

and

$$g_j(x_n)(t) = \int_0^t \sum_j \Phi_j\big(t,\tau,x_n(\tau)\big)d\tau.$$

It is clear that T is a linear isomorphism which is, in fact, an isometry; that is,

$$\| T(x_n) \|_{L^{1+\epsilon}} = \|x_n'\|_{L^{1+\epsilon}} = \left(\int_0^1 |x_n'(\tau)|^{1+\epsilon} d\tau\right)^{\frac{1}{1+\epsilon}} = \| x_n \|_{W_0^{1,1+\epsilon}}.$$

Thus we have that $C(T) = \|T^{-1}\|^{-1} = 1$.

On the other hand, we are next going to check that g_j is well-defined and Lipschitz on bounded subsets of $W_0^{1,1+\epsilon}[0,1]$. First note that, given $x_n \in W_0^{1,1+\epsilon}[0,1]$, for every $t \in [0,1]$ we have:

$$|x_n(t)| = \left|\int_0^t x_n'(\tau)d\tau\right| \leq \int_0^1 |x_n'(\tau)|d\tau = \|x_n'\|_{L^1} \leq \|x_n'\|_{L^{1+\epsilon}} = \| x_n \|_{W_0^{1,1+\epsilon}},$$

and therefore $\| x_n \|_\infty \leq \| x_n \|_{W_0^{1,1+\epsilon}}$. Now let us check that $g_j(x_n) \in L^{1+\epsilon}[0,1]$. Note that, from Hölder inequality, we have that

$$\left|\int_0^t \sum_j \Phi_j(t,\tau,x_n(\tau))d\tau\right| \leq \sum_j \left(\int_0^t |\Phi_j(t,\tau,x_n(\tau))|^{1+\epsilon}d\tau\right)^{\frac{1}{1+\epsilon}}.$$

Since $|a+b|^{1+\epsilon} \leq 2^{1+\epsilon}(|a|^{1+\epsilon}+|b|^{1+\epsilon})$, we obtain that

$$\int_0^1 \sum_j |g_j(x_n)(t)|^{1+\epsilon}dt \leq \int_0^1 \int_0^t \sum_j |\Phi_j(t,\tau,x_n(\tau))|^{1+\epsilon}d\tau dt$$
$$\leq 2^{1+\epsilon} \int_\Delta a(t,\tau)^{1+\epsilon}|x_n(\tau)|^{1+\epsilon} + b(t,\tau)^{1+\epsilon}dt < +\infty.$$

Now let $\epsilon \geq -1$ and consider $u, v \in W_0^{1,1+\epsilon}[0,1]$ with $\| u \|_{W_0^{1,1+\epsilon}} \leq 1+\epsilon$ and $\| v \|_{W_0^{1,1+\epsilon}} \leq 1+\epsilon$. For each $t \in [0,1]$:

$$\sum_j |g_j(u)(t) - g_j(v)(t)| \leq \int_0^t \sum_j |\Phi_j(t,\tau,u(\tau)) - \Phi_j(t,\tau,v(\tau))|d\tau$$
$$\leq \int_0^1 \sum_j \theta_j(1+\epsilon)\cdot|u(\tau)-v(\tau)|d\tau \leq \sum_j \theta_j(1+\epsilon)\cdot\| u-v \|_\infty \leq \sum_j \theta_j(1+\epsilon)\cdot \| u-v \|_{W_0^{1,1+\epsilon}}.$$

Then

$$\sum_j \| g_j(u) - g_j(v) \|_{L^{1+\epsilon}} = \left(\int_0^1 \sum_j |g_j(u)(t)-g_j(v)(t)|^{1+\epsilon}dt\right)^{\frac{1}{1+\epsilon}} \leq \sum_j \theta_j(1+\epsilon)\cdot \| u-v \|_{W_0^{1,1+\epsilon}}.$$

This implies in particular that, for every $x_n \in W_0^{1,1+\epsilon}[0,1]$ with $\| x_n \|_{W_0^{1,1+\epsilon}} \leq 1+\epsilon$ we have that $\operatorname{Lip} g_j(x_n) \leq \theta_j(1+2\epsilon)$ for every $\epsilon > 0$. By the continuity of θ_j we deduce that $\operatorname{Lip} g_j(x_n) \leq \theta_j(1+\epsilon)$ whenever $\| x_n \|_{W_0^{1,1+\epsilon}} \leq 1+\epsilon$.

From Example 8 we have that $Jf_j(x_n) := T + \operatorname{Lip} g_j(x_n)\cdot \bar{B}_{L(X,X+\epsilon)}$ is a pseudo-Jacobian of f_j, satisfying the strong chain rule condition. Let us see that Jf_j is also regular at every $x_n \in W_0^{1,1+\epsilon}[0,1]$. Indeed, suppose that $\| x_n \|_{W_0^{1,1+\epsilon}} \leq 1+\epsilon$. For each $R \in \mathcal{L}(X, X+\epsilon)$ with $\| R \| \leq \operatorname{Lip} g_j(x_n)$ we have $\|R\circ T^{-1}\| \leq \operatorname{Lip} g_j(x_n) \leq \theta_j(1+\epsilon) < 1$. In particular, the operator $\mathrm{Id}_{X+\epsilon} + R\circ T^{-1}$ is an isomorphism on $(X+\epsilon)$. In this way we obtain that $T+R$ is an isomorphism. On the other hand, also using Example 8, we have

$$\begin{aligned}\operatorname{Reg} Jf_j(x_n) &= \operatorname{Sur} Jf_j(x_n) = \inf\{C(T+R) : \| R \| \leq \operatorname{Lip} g_j(x_n)\}\\ &\geq \inf\{C(T+R) : \| R \| \leq \theta_j(1+\epsilon)\} \geq 1-\theta_j(1+\epsilon) = m_j(1+\epsilon) > 0,\end{aligned}$$

where the continuous function $m_j(1+\epsilon) := 1-\theta_j(1+\epsilon)$ satisfies that $\int_0^\infty \sum_j m_j(1+\epsilon)d(1+\epsilon) = \infty$. Therefore, condition (A') and all the requirements of Theorem 14 are satisfied, and the desired conclusion follows. Also, from Remark 19 we see that, for each $(x+\epsilon) \in L^{1+\epsilon}[0,1]$, the functional $(F_j)_{x+\epsilon}$ defined in (2) satisfies the weighted Chang-Palais-Smale condition (C) for the weight

$$h_j(1+\epsilon) = \frac{\theta_j(1+\epsilon)-\theta_j(0)}{1-\theta_j(1+\epsilon)}.$$

Thus Theorem 18 also applies in this case.

To finish with, we give an explicit example of function Φ_j for which Theorem 20 applies.

Example 21. For each $0 < \epsilon < \infty$, the function

$$\Phi_j(t,\tau,u) := \max\left\{\frac{t^3}{2}|\sin(\tau u)|, |tu| - \log(1+|tu|)\right\}$$

satisfies conditions (i), (ii) and (iii) above. Then for each $(x+\epsilon) \in L^{1+\epsilon}[0,1]$ there exists a unique solution of equation (15) with initial condition (16) in the space $W_0^{1,1+\epsilon}[0,1]$.

Proof. Condition (i) is clear. If we define $\eta(u) := |u| - \log(1+|u|)$, it is easy to check that, for $|u| \le 1+\epsilon$ and $|v| \le 1+\epsilon$ we have that

$$|\eta(u) - \eta(v)| \le \frac{1+\epsilon}{2+\epsilon}|u-v|.$$

Now using that $|\max\{a_1,a_2\} - \max\{b_1,b_2\}| \le \max\{|a_1-b_1|, |a_2-b_2|\}$, we obtain that whenever $0 \le \tau \le t \le 1$ and $|u| \le 1+\epsilon, |v| \le 1+\epsilon$:

$$\sum_j |\Phi_j(t,\tau,u) - \Phi_j(t,\tau,v)| \le \max\left\{\frac{1}{2}|u-v|, \frac{1+\epsilon}{2+\epsilon}|u-v|\right\} = \sum_j \theta_j(1+\epsilon)|u-v|,$$

where $\theta_j(1+\epsilon) := \max\left\{\frac{1}{2}, \frac{1+\epsilon}{2+\epsilon}\right\}$. In particular, Φ_j satisfies condition (ii) with $a=1$ and $b=0$. Furthermore,

$$\int_0^\infty \sum_j \left(1-\theta_j(1+\epsilon)\right) d(1+\epsilon) = \int_0^1 \left(1-\frac{1}{2}\right) d(1+\epsilon) + \int_1^\infty \left(1-\frac{1+\epsilon}{2+\epsilon}\right) d(1+\epsilon)$$
$$= \frac{1}{2} + \int_1^\infty \frac{1}{2+\epsilon} d(1+\epsilon) = \infty,$$

so condition (iii) is also fulfilled.

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