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Review Paper

The Three Minimal Techniques of Variational Method

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Abstract
The variational method, as the core tool of functional extremum analysis, has a wide and profound impact on mathematical theory and engineering applications through its three basic operations - chain rule, partial integration, and variable substitution. This article systematically reviews the historical development of these three methods and their key roles in classical mechanics and partial differential equations. The research results indicate that the chain rule provides an effective approach for the variation of complex functionals by handling composite dependency relationships; Divisional integration is not only indispensable in high-order derivative reduction and boundary condition treatment, but also lays the foundation for weak solution theory of partial differential equations; Variable substitution reveals the conservation laws and structural characteristics of physical systems (such as Noether's theorem) through symmetry utilization and dimensionality reduction simplification. Through typical cases such as Dirichlet energy functionals, this paper verifies the synergistic effect of compulsion, convexity, and weak lower semi continuity in proving the existence and uniqueness of minima.

Keywords: *Variational method, Chain rule, Divisional points, Variable substitution, coercivity, convexity.*

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I. Research background

1.1 The Historical Development and Mathematical Needs of Variational Method

Variational method is a mathematical branch that studies functional extremum. It is not only closely related to many branches of mathematics, but also provides important principles for physics, and has a wide range of applications[1-2]. Driving classic problems: Since the proposal of the steepest descent line problem and the isopach problem in the 17th century, variational methods have gradually developed into the core branch of functional extremum analysis. The establishment of the Euler Lagrange equation relies on the reduction of higher-order derivatives by partial integration[3], while optimization problems under complex constraints (such as geodesics) require the use of variable substitution to simplify geometric structures, transforming functional extremum problems into differential equations, and laying the foundation for the solution paradigm of minimal problems.

Variational calculus is a science that studies functional extremum problems and is a branch of classical mathematics This book starts with typical problems in mathematics, physics, and mechanics to establish a model for finding the extremum of a functional. It uses line algebra, numerical calculus, and functional comparison, as well as the comparison between Fermat's principle, Hamilton's principle, and variational principle in physics and mechanics. It introduces the methods and steps for finding the extremum of a functional in a clear and concise manner, and concludes that the necessary condition for a functional to obtain an extremum is that its first-order variation is zero. It also derives the Euler equation satisfied by the extremum curves (surfaces) of various types of functionals, and solves some practical problems by solving the Euler equation[4].

The introduction of the chain rule: when the functional involves composite functions (such as $J[u(\phi(x))]$) or multivariate coupling in field theory, the chain rule becomes a necessary tool for handling the dependencies between variables[5].

1.2 Theoretical support of physics

The principle of minimum action: Hamilton's principle requires functional analysis of the action $S = \int L(q, \dot{q}, t)\, dt$ perform variational analysis, Partial integration is used to eliminate boundary terms, while variable substitution (such as Legendre transform) transforms Lagrangian mechanics into Hamiltonian form.

Field theory and gauge invariance: in electromagnetic field and quantum field theory, gauge transformation (variable substitution) requires the use of the chain rule to maintain the invariance of the action, reflecting the unity of physical symmetry and mathematical tools.

Elasticity and static equilibrium state: In the problem of minimizing the potential energy of elastic bodies, variable substitution (such as strain stress relationship) simplifies complex constitutive equations into solvable forms.

1.3 The Promotion of Modern Mathematics and Computational Science

The establishment of Sobolev space and weak solution theory in the 20th century relied on the derivation of the weak form of PDE through partial integration[6].

Innovation in computational methods: In the finite element method (FEM), variable substitution is used for grid generation, and the chain rule supports automatic differentiation (AD) technology, becoming the core of machine learning optimization.

The intersection of geometry and topology: the extension of geodesic problems on manifolds (such as Riemannian geometry) relies on variable substitution (such as exponential mapping) to simplify local coordinate systems.

II. Research meaning

2.1 Theoretical significance

The collaborative use of chain rule, partial integration, and variable substitution provides a unified methodology for solving problems ranging from classical problems (such as catenary lines) to modern nonlinear problems (such as minimal surfaces).

Promoting the development of mathematical analysis: The application of fractional integration in Sobolev space laid the foundation for PDE weak solution theory (such as the existence proof of Dirichlet problem).

2.2 Application value

Engineering and Physical Modeling: (1) Finite element analysis: Partial integration is used to construct weak form equations and support static equilibrium simulation in structural mechanics; (2) Optimal control: Variable substitution (such as state control variable separation) simplifies the rocket trajectory optimization problem.

Computational Science and Artificial Intelligence: (1) Deep learning optimization: Backpropagation of chain rule to minimize the loss function of neural networks; (2) Image processing: The TV denoising model (Total Variation) processes the image gradient field through partial integration.

III. Modeling issues

The typical problem in variational research is finding the minimum:

$$\inf_{u \in \mathcal{M}} I(u), \tag{3.1}$$

where

$$I(u) = \int_{\Omega} L(x, u(x), \nabla u(x))\mathrm{d}x, \tag{3.2}$$

and $\Omega \in \mathbb{R}^d$ is an open area, $\mathcal{M}$ It is a set of functions defined on Ω, L is called Lagrangian, abbreviated as Lagrangian.

When considering one-dimensional problems, we usually write the independent variable as t, let $\Omega = (a, b)$, $L = L(t, x(t), \dot{x}(t))$, we have

$$I(x) = \int_a^b L(t, x(t), x'(t))\, dt. \tag{3.3}$$

Variational problem is a minimization problem or a general optimization problem (finding the minimum, maximum, or saddle point), which has many similarities with the optimization problems of functions in calculus below:

$$\inf_{u \in \mathcal{M}} f(x), \tag{3.4}$$

The difference is that in ordinary optimization, the variable is a point, while in variational problems, the variable is a function.

Ordinary optimization: $x \longmapsto f(x)$;

Variational problem: $x \mapsto (u(x), \nabla u(x)) \mapsto I(u)$.

IV. The role of calculus tools

The independent variable in function $f(x)$ is a number $x \in \mathbb{R}$ or vector $x \in \mathbb{R}^d$, We have calculus as a tool that can be used. The independent variable in functional $I(\cdot)$ is a function u, itself is a function of x in some euclidean space $\mathbb{R}^d$. The function of a function is called a functional. Therefore, functional analysis is an essential foundational knowledge for variational methods. On the other hand, variational problems are also a place where functional analysis can shine.

The subset $\mathcal{M}$ in variational problems exists in a certain function space, for example $H^1(\Omega)$, while $\mathcal{M}$ in calculus optimization problems is a subset of $\mathbb{R}^d$, which is finite dimensional. Function spaces are usually infinite dimensional, so many results or properties of finite dimensional linear spaces may not necessarily hold in infinite dimensional spaces and need to be reexamined.

In this case: compactness of sets. In a finite dimensional norm vector space, a set is compact if and only if it is bounded and closed. Think back, if $\mathcal{M}$ is tight, $f(x)$ is continuous on $\mathcal{M}$, then $f(x)$ can reach its extremum. In infinite dimensional space, boundedness and closure are necessary conditions for compactness, but not sufficient conditions. For example, the unit closed ball in L^2 is clearly a bounded closed set, but not a sequence tight one. For example, $\{e_n\}, e_n = (0, 0, ..., 1, 0, ...)$, The sequence with 1 at position n and 0 at other positions. If there is a convergent subsequence, it is a Cauchy sequence, But $\| e_n - e_m \| = \sqrt{2}$, it is impossible to approach infinitely. This is because with infinite coordinates, there is room for infinite elements to maintain social distance between each other.

The solution is to introduce a weaker topology in the function space, so that the unit sphere is compact relative to that topology. This ensures that the existence of subsequences converges in a weak sense, thereby proving the existence of solutions to variational problems. From the perspective of open sets, the weaker the topology, the fewer open sets it has, and therefore the fewer open covers it has, making it easier to satisfy the definition of compactness: every open cover has finite sub covers.

Suppose $f: U \to V$ is a function between two topological spaces. When any open set in V remains an open set in U under the original image of f, then f is continuous. Due to the fewer open sets in weak topology, it is not conducive to satisfying the definition of continuity. To solve this problem, the continuity can be relaxed to the lower half continuity under weak topology, thereby reducing the open set while maintaining the fitness of the variational problem. However, in order to ensure the existence of solutions to variational problems, it is usually necessary to introduce coercivity and convexity conditions for functionals.

4.1 Coercivity

In variational methods, coercivity is a key property in functional analysis that ensures the existence of minima for functionals in certain function spaces. The specific definition is as follows: let X be a reflexive Banach space (such as Sobolev space $W^{1,p}(\Omega)$), functional $L: X \to \mathbb{R} \cup \{+\infty\}$, it is called coercivity if:

$$\lim_{\| u \|_X \to +\infty} L(u) = +\infty. \tag{4.1}$$

that is to say, when the norm of function u tends to infinity, the functional value $L(u)$ also tends to infinity.

4.1.1 The proof framework of mandatory theory

This article takes the classic Dirichlet energy functional as an example to illustrate the proof steps of the coercivity theory, suppose:

$$L(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^{2}dx-\int_{\Omega}f\,u dx, \tag{4.2}$$

where $\Omega\in\mathbb{R}^{n}$ is a bounded region, $f\in L^{2}(\Omega)$ and $u\in H_{0}^{1}(\Omega)$.

Step1(Choose the appropriate function space): let $X=H_{0}^{1}(\Omega)$, its norm is

$$\|u\|_{H_{0}^{1}}=\left(\int_{\Omega}|\nabla u|^{2}dx\right)^{1/2}. \tag{4.3}$$

According to the Poincaré inequality, this norm is equivalent to the L^{2} norm, that is to say , there exists $C>0$, such that:

$$\|u\|_{L^{2}}\leq C\|\nabla u\|_{L^{2}}. \tag{4.4}$$

Step2(Proof of coercivity): estimate the functional $L(u)$:

$$L(u)=\frac{1}{2}\|\nabla u\|_{L^{2}}^{2}-\int_{\Omega}f\,u dx. \tag{4.5}$$

Using the Cauchy-Schwarz inequality and the Poincaré inequality:

$$\left|\int_{\Omega}f\,u dx\right|\leq\|f\|_{L^{2}}\|u\|_{L^{2}}\leq C\|f\|_{L^{2}}\|\nabla u\|_{L^{2}}. \tag{4.6}$$

Therefore,

$$L(u)\geq\frac{1}{2}\|\nabla u\|_{L^{2}}^{2}-C\|f\|_{L^{2}}\|\nabla u\|_{L^{2}}. \tag{4.7}$$

Let $\|\nabla u\|_{L^{2}}=R$, when $R\to+\infty$, the dominant term on the right is $\frac{1}{2}R^{2}$, therefore:

$$\lim_{R\to+\infty}L(u)\geq\lim_{R\to+\infty}\left(\frac{1}{2}R^{2}-C\|f\|_{L^{2}}R\right)=+\infty. \tag{4.8}$$

This indicates that $L(u)$ is coercive on $H_{0}^{1}(\Omega)$.

Step3(Combining weak lower semi continuity): The gradient term $\|\nabla u\|_{L^{2}}^{2}$ in the Dirichlet energy functional is weakly lower semi continuous (due to convexity and differentiability). Therefore, the functional $L(u)$ satisfies: coercivity (proven); weak lower half continuity.

If a functional is forced and weakly lower semi continuous on a reflexive Banach space, then it has a global minimum $u_{0}\in X$.

Note that, Correction method for non coercive functionals

(1) Add penalty term: such as introducing $\epsilon\|u\|_{L^{2}}^{2}$ term to force the functional to grow at infinity.

(2) Constrained optimization: Analyze on subspaces (such as $\int_{\Omega}u\,dx=0$) and use compactness to restore coercivity.

4.2 Convexity

In variational and optimization theory, convexity is an important property of functionals, which directly relates to the existence, uniqueness, and stability of extremum solutions. The specific definition is as follows: let X be a real normed linear space(such as Sobolev space $W^{1,p}(\Omega)$), functional $L:X\to\mathbb{R}\cup\{+\infty\}$ is called convex if:

$$L(\lambda u+(1-\lambda)v)\leq\lambda L(u)+(1-\lambda)L(v). \tag{4.9}$$

If the above inequality strictly holds for $u\neq v$, then L is called a strictly convex functional.

4.2.1 The Judgment Theorem for Convex functionals

Theorem 1 (First Order Condition): If the functional L is $Fr\acute{e}chet$ differentiable on the Banach space X, then L is a convex functional if and only if:

$$L(v) \geq L(u) + \langle L'(u), v-u \rangle, \quad \forall u, v \in X, \tag{4.10}$$

where, $L'(u)$ is the $Fr\acute{e}chet$ derivative of L at u.

Theorem 2 (Second Order Condition) If L quadratic $Fr\acute{e}chet$ differentiable, then L is a convex functional if and only if its $Hessian$ operator $L''(u)$ is non negatively definite, i.e.

$$\langle L''(u)h, h \rangle \geq 0, \quad \forall u, h \in X. \tag{4.11}$$

4.2.2 The proof framework of convex functionals

Firstly, considering the convexity of linear functionals, we also take the Dirichlet energy functional as an example to prove its convexity, consider the following Dirichlet energy functional:

$$L(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx, \quad u \in H_0^1(\Omega). \tag{4.12}$$

Step 1 (Decompose the functional):

$$L(u) = L_1(u) - L_2(u)$$

where $L_1(u) = \frac{1}{2} \| \nabla u \|_{L^2}^2, L_2(u) = \int f u dx.$

Step 2 (Prove the convexity of $L_1(u)$): for any $u, v \in H_0^1(\Omega)$, we have

$$L_1(\lambda u + (1-\lambda)v) = \frac{1}{2} \| \nabla(\lambda u + (1-\lambda)v) \|_{L^2}^2 \leq \lambda L_1(u) + (1-\lambda) L_1(v), \tag{4.13}$$

It can be obtained from the linearity of the gradient and the square convexity of the norm.

Step 3 (Prove the linearity of $L_2(u)$): Since $L_2(u)$ is a linear functional, it is naturally a convex functional.

Step 4 (The linear combination of convex functionals maintains convexity) : the difference between convex functionals and linear functionals remains convex functionals.

Next, we will discuss the convexity of general nonlinear functionals. Consider functional:

$$L(u) = \int_{\Omega} \left(\frac{1}{p} |\nabla u|^p + V(x) u \right) dx, \quad p \geq 1, \tag{4.14}$$

where $V(x) \in L^{\infty}(\Omega)$, the proof steps are as follows:

(1) Convexity of gradient term: $|\nabla u|^p$ is a convex function on u (since $p \geq 1$ causes $t \mapsto |t|^p$ to be convex on $\mathbb{R}^n$).

(2) Integral maintains convexity: The integral of a convex function remains a convex functional (defined by integral linearity and convexity).

(3) Convexity of linear terms: $V(x)u$ is linear with u, therefore it is a convex functional.

4.2.3 Strict convexity determination

Theorem 3: If the functional L satisfies:

$$L(\lambda u + (1-\lambda)v) < \lambda L(u) + (1-\lambda) L(v), \quad \forall u \neq v, \lambda \in (0,1), \tag{4.15}$$

It is called L strictly convex.

Common strict convex functionals include:

(1) Dirichlet energy functional: when $|L(u) = \frac{1}{2}\int |\nabla u|^2 dx$, the quadratic convexity of the gradient term is strictly convex.

(2) Functional with strong convex terms: such as $L(u) = \frac{1}{2} \| \nabla u \|_{L^2}^2 + \frac{\epsilon}{2} \| u \|_{L^2}^2$, where $\epsilon > 0$.

4.2.4 The relationship between convexity and other variational properties

(1) convexity + coercivity $\rightarrow$ minimum existence uniqueness: If L is coercive and strictly convex, then there exists a unique minimum point in the reflexive Banach space X.

(2) convexity + weak lower semi continuity $\rightarrow$ existence of minimum value: if a convex functional is weakly lower semi continuous in Banach space, its minimum exists.

V. The core role of chain rule, partial integration, and variable substitution in minimization problems

5.1 Chain rule: handling composite dependency relationships

When calculating derivatives, we must be careful about the dependencies between variables. Its mathematical form can be expressed as:

Functional $J[u] = \int_a^b F(x, u, u')\,dx$, if there is an intermediate variable $u = g(v)$, its variation needs to be passed through the chain rule:

$$\delta J = \int_a^b \left(\frac{\partial F}{\partial u} g'(v) + \frac{\partial F}{\partial u'} g''(v) \right) \delta v dx. \tag{5.1}$$

5.2 Partial integration: eliminating high-order terms and boundary treatment

Functional $I(\cdot)$ involves derivatives ∇u and integrals $\int_\Omega$. The interaction between these two is reflected in the fractional integral. When the region Ω is not smooth (such as in polyhedra), it is necessary to use partial integration in segments, and jump conditions will occur at the edges and vertices of the polyhedron. As for the derivation of the Euler-Lagrange equation: the high-order derivative term $\int u''(x)\,\delta u dx$ in the functional variation is transformed into boundary terms and first-order derivative terms through partial integration, for example:

$$\delta J = \left[\frac{\partial F}{\partial u'} \delta u \right]_a^b + \int_a^b \left(\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} \right) \delta u dx. \tag{5.2}$$

Generation of natural boundary conditions: in the free endpoint problem, forcing the boundary term to be zero(such as $\left. \frac{\partial F}{\partial u'} \right|_{x=a,b} = 0$).

5.3 Variable substitution: dimensionality reduction and symmetry utilization

The most magical move can derive the deepest level results of variational calculus. Here are two examples: Legendre transformation and Noether's theorem. The Legendre transform transforms Lagrangian into Hamiltonian and Euler Lagrange equation into Hamiltonian system. If a scalar potential function is introduced, the Hamilton-Jacobi equation can be derived. Different variables and equations reveal different structures of the same physical system.

If the Lagrangian is invariant under certain transformations, there exists a conservation law. This is the most profound result in variational calculus: Noether's theorem.

In fact, the more accurate term for 'invariance' is' symmetry ', while' variable transformation 'can be understood as' group interaction'.

The Noether theorem demonstrates a profound connection between symmetry and conservation laws, such as:

(1) Time translation symmetry $\rightarrow$ conservation of energy;

(2) Spatial translational symmetry $\rightarrow$ conservation of momentum;

(3) Rotational symmetry $\rightarrow$ conservation of angular momentum.

VI. Application examples

Case 1 (Elliptical Partial Differential Equations)

Consider the existence of weak solutions to the Poisson equation:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{6.1}$$

the corresponding energy functional is:

$$L(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx. \tag{6.2}$$

(1) Coercivity: obtained directly from $\alpha = 1$.

(2) Existence of minimal solution: there exists a unique $u_0 \in H_0^1(\Omega)$ such that $L(u_0) = minL(u)$, the weak solution of the equation.

Case 2 (uniqueness of solutions to elliptic equations)

Consider equation:

$$-\Delta u + |u|^{p-2}u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad p \geq 2. \tag{6.3}$$

the corresponding energy functional is:

$$L(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega f u dx. \tag{6.4}$$

(1) Convexity: the gradient term $|\nabla u|^2$ and the power term $|u|^p (p \geq 2)$ are both convex functions, and the overall functional is convex after integration.

(2) Uniqueness guarantee: strict convexity (because $|u|^p$ is strictly convex when $p > 2$) ensures that the solution is unique.

Case 3 (one dimensional variational minimization problem)

For one-dimensional problems, we use $t \in \mathbb{R}$ instead of x, and $\dot{x}$ instead of ∇u. Note that although $t \in \mathbb{R}$, the function $x(t)$ can be a curve in space, i.e. x can be a vector, typically representing the motion trajectory of an object, then the first derivative $\dot{x}$ is velocity and the second derivative $\ddot{x}$ is acceleration.

Consider the variational problem:

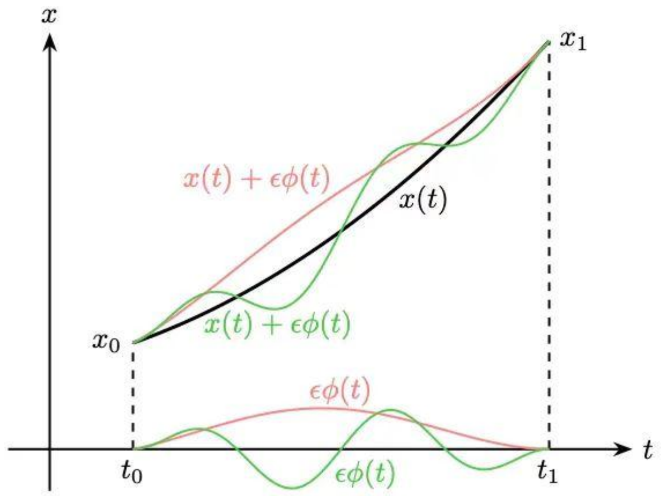
$$\inf_{x \in \mathcal{M}} \int_0^1 L(t, x(t), \dot{x}(t)) \mathrm{d}t, \tag{6.5}$$

A typical example is:

(1) $\mathcal{M} = \{x \in C^1(0,1), x(0) = x_0, x(1) = x_1\}$;

(2) $L(t, x, v) = \frac{1}{2}m|v|^2 - U(x)$, it is the difference between kinetic energy $T(v) = \frac{1}{2}m|v|^2$ and potential energy $U(x)$.

Recall that the first-order condition for min $f(x)$ is $f'(x) = 0$. But now $x = x(t)$ itself is a function that can change at every point $t \in (0,1)$. How to take the derivative of a function?



The idea is to introduce changes in the function. Let ϕ be the testing function in $\mathcal{M}_0$, satisfy certain boundary conditions, if $x \in \mathcal{M}$, So $x+\epsilon\phi \in \mathcal{M}$. For the example above, $\mathcal{M}_0 = \{\phi \in C^1(0,1): \phi(0) = \phi(1) = 0\}$.

Let $f(\epsilon) := I(x+\epsilon\phi)$, so we can take the derivative of $f(\epsilon)$. Obviously, x is the solution of $\inf I(x)$ if and only if 0 is the minimum value of $f(\epsilon)$. The corresponding first-order condition is

$$f'(0) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} I(x+\epsilon\phi)|_{\epsilon=0} = 0, \tag{6.6}$$

Chain rule: by taking the derivative of the chain rule, we obtain the variational form of the Euler-Lagrange equation:

$$\int_0^1 L_x(t,x,\dot{x})\phi + L_v(t,x,\dot{x})\dot{\phi} = 0 \quad \forall \phi \in \mathcal{M}_0. \tag{6.7}$$

Chain rule: by applying partial points, we obtain:

$$\int_0^1 \left[L_x(t,x,\dot{x}) - \frac{\mathrm{d}}{\mathrm{dt}} L_v(t,x,\dot{x})\right]\phi = 0 \quad \forall \phi \in \mathcal{M}_0. \tag{6.8}$$

Because $\phi \in \mathcal{M}_0$, the boundary term in the partial integral has disappeared. Now make $\mathcal{M}_0$ is dense in $L^2(0,1)$, and we obtain a strong form of the Euler-Lagrange equation:

$$-\frac{\mathrm{d}}{\mathrm{dt}} L_v(t,x(t),\dot{x}(t)) + L_x(t,x(t),\dot{x}(t)) = 0. \tag{6.9}$$

for $L(t,x,v) = \frac{1}{2}m|v|^2 - U(x)$, the equation above becomes Newton's equation:

$$m\ddot{x} = F, \quad F = -\nabla_x U. \tag{6.10}$$

The method of deriving Newton's equations of motion by solving variational problems and using Euler-Lagrange equations is called the "principle of minimum action".

Variable substitution: consider the following variable substitution:

$$\begin{cases} q = x, \\ p = L_v(t,q,\dot{q}) \end{cases} \tag{6.11}$$

To derive the Hamiltonian system, we obtain the total differential on both sides of $H(p,q,t) = p\cdot\dot{q} - L(t,q,\dot{q})$ by:

$$\begin{aligned} &\mathrm{d}H = H_p\mathrm{d}p + H_q\mathrm{d}q + H_t\mathrm{d}t \\ &p\,\mathrm{d}\dot{q} + \dot{q}\,\mathrm{d}p - L_t\mathrm{d}t - L_q\mathrm{d}q - L_v d\dot{q} = \dot{q}\,\mathrm{d}p - \dot{p}\,\mathrm{d}q - L_t\mathrm{d}t. \end{aligned} \tag{6.12}$$

The final step is due to definition of $p = L_v$ and using new variables $L_q = \dot{p}$ to write the Euler-Lagrange equation.

When the Hamiltonian $H(p,q)$ is independent of t, for the solution $(p(t),q(t))$ of the Hamiltonian system, we obtain the conservation of the Hamiltonian, which is:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(p(t),q(t)) = H_p \dot{p} + H_q \dot{q} = 0. \tag{6.13}$$

An important example is $L = T - U$, for $T(v) = \frac{1}{2} m|v|^2$, the variable $p = mv$ is momentum, the Hamiltonian is the total energy

$$H = p\dot{q}(p) - L = pv - T(v) - U(q) = \frac{1}{2} m|v|^2 + U(q) = T + U. \tag{6.14}$$

So we have obtained the law of conservation of energy. The underlying reason is due to the symmetry of Lagrangian towards time translation.

Symmetry implies conservation laws, and conservation quantities themselves can generate corresponding symmetries. This bidirectional connection greatly inspired the development of theoretical physics. In practical research, scientists often first discover a certain conserved quantity in experiments, and then go back to search for the symmetry behind it, in order to derive a beautiful theoretical framework.

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