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Review Paper

A Priori Error Estimates for Biharmonic Eigenvalue Problems with Simply Supported Boundary Conditions

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ABSTRACT: *The biharmonic eigenvalue problem is a classical fourth-order partial differential equation and a subject of significant research interest, particularly in applied fields such as elasticity, fluid mechanics, and quantum mechanics. Specifically, the biharmonic eigenvalue problem under simply supported boundary conditions finds wide applications in thin plate vibration modeling. To accurately solve such problems, numerical methods play a crucial role. Among them, the discontinuous Galerkin finite element method offers high mesh flexibility and adaptability, enabling arbitrary high-order accuracy and flexible handling of complex boundary conditions. As a result, DG methods have become an essential numerical tool for solving various partial differential equations and practical problems. In the context of simply supported boundary conditions, a priori error estimates for the biharmonic eigenvalue problem serve as a vital tool for assessing the discrepancy between numerical and exact solutions. These estimates provide deeper insights into the convergence and stability of numerical methods. This paper investigates a priori error estimates for the biharmonic eigenvalue problem with simply supported boundary conditions. By employing the discontinuous Galerkin method, we analyze its error behavior in solving the biharmonic eigenvalue problem and derive corresponding error estimates. These estimates not only help evaluate the accuracy of numerical schemes but also provide theoretical support for practical applications.*

KEYWORDS: *Biharmonic eigenvalue problem, Discontinuous Galerkin method, A priori error estimates*

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I. INTRODUCTION

The biharmonic eigenvalue problem, which investigates the eigenvalues and eigenfunctions of the fourth-order differential operator Δ^2, represents a significant research topic in mathematical physics and engineering mechanics. This problem finds extensive applications across multiple disciplines, including thin plate vibration analysis, structural stability assessment, image processing, and materials science. Over the years, numerous scholars have dedicated substantial efforts to developing numerically efficient solutions for the biharmonic equation, with continuous optimization and innovation in solution methodologies.

In reference [1-3], the finite difference method was employed to solve the biharmonic equation, and a compact difference scheme for the biharmonic equation was presented [3]. The finite difference method is easier to comprehend and apply when dealing with simple or lower-order problems. However, it becomes more challenging for general domains or equations with complex boundary conditions, whereas the finite element method offers greater flexibility. Huang Xuehai et al. [4] proposed the modified Argyris element, a conforming finite element method for Kirchhoff plate bending problems. Using standard techniques, they derived a priori error estimates for both the modified Argyris element and its corresponding finite element method. In reference [5], a mixed finite element method [5-7] was employed to solve the biharmonic equation. This approach introduces an intermediate variable to reduce the fourth-order equation into two second-order equations. The mixed variational problem is then discretized using mixed finite element spaces satisfying certain conditions, yielding numerical solutions for both the original variable and the intermediate variable that satisfy the original equation. On the other hand, the discontinuous Galerkin method [8-10] adopts a finite element approach with completely discontinuous basis functions. This method is capable of handling complex boundary problems and allows for flexible local mesh refinement and polynomial degree variation across elements. Therefore, the discontinuous Galerkin method is often used to solve various eigenvalue problems, such as the Steklov eigenvalue problem [11], the Laplace eigenvalue problem [12], and the biharmonic eigenvalue problem [13-16], among others. Emmanuil et al. [10] derived a discontinuous Galerkin finite element scheme for the biharmonic equation. In reference [16], Xi Yingxia et al. proposed a multilevel correction scheme based on nonconforming finite elements for solving biharmonic eigenvalue problems. This method transforms the eigenvalue problem on a fine grid into an eigenvalue problem on a coarse grid along with a series of source problems on fine grids. The Ciarlet-Raviart mixed method [17-22] is also applicable to biharmonic eigenvalue problems. In reference [23], Yang Yidu devised an adaptive Ciarlet-Raviart mixed method employing piecewise polynomials of degree less than or equal to m. When the eigenfunctions are sufficiently smooth, this approach enables the numerical eigenvalues for the corresponding biharmonic eigenvalue problem to achieve optimal convergence order. The interior penalty discontinuous Galerkin (IPDG) method imposes penalty terms on the jumps of approximate solutions across interelement edges/faces. Compared with conforming finite element methods, the IPDG approach offers significantly greater flexibility. In reference [24], an hp-version interior penalty discontinuous Galerkin finite element method was developed for the biharmonic equation, where both stability analysis and a priori error estimates were rigorously established. The derived error bounds were shown to be optimal with respect to the mesh size h and slightly suboptimal in terms of the polynomial degree p.

Focusing on the biharmonic eigenvalue problem with simply supported boundaries, this work develops a discontinuous Galerkin finite element approximation using the interior penalty discontinuous Galerkin (IPDG) framework. A rigorous error analysis is presented for the eigenvalue problem, including the derivation of a priori error estimates. These theoretical results offer fundamental guidance for assessing the numerical method's accuracy and its practical implementation.

II. IDENTIFICATION OF SEGMENTED AUTOREGRESSIVE

$L^p(\omega)$ to represent a standard Lebesgue space, where $1 \leqslant p \leqslant \infty, \omega \subset \mathbb{R}^2$, The corresponding norm is expressed by $\|\cdot\|_{L^p(\omega)}$. In this paper, the norm of $L^2(\omega)$ is represented by $\|\cdot\|_{\omega}$. We also use $H^s(\omega)$ to express the standard Hilbert Sobolev space of real functions defined at $\omega \subset \mathbb{R}^2$ with index $s \geqslant 0$, and the corresponding norm and semi-norm are $\|\cdot\|_{s,\omega}$ and $|\cdot|_{s,\omega}$. Let Ω be the bounded open polygon region of R^2, and let $\partial\Omega$ represent its boundary. Consider the simply supported boundary condition eigenvalue problem: find $\lambda \in C$ and $u \in H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$\begin{cases} \Delta^2 u = \lambda u, & in\ \Omega \\ u = \Delta u = 0, & on\ \partial\Omega. \end{cases} \tag{2-1}$$

Denote

$$(u,v) = \int_\Omega uv dx,$$

$$a(u,v) = (\nabla u, \nabla v) + (u,v), u, v \in H^1(\Omega),$$

and define a continuous bilinear form

$$a(u,v) = (\Delta u, \Delta v), u, v \in H_0^1(\Omega) \cap H^2(\Omega).$$

Then, there exists two positive constants A and B independent of u and v, such that the bilinear form $a(\cdot,\cdot)$ is satisfied

$$|a(u,v)| \leqslant A\|u\|\|v\|, \quad u, v \in H_0^1(\Omega) \cap H^2(\Omega)$$

$$|a(u,v)| \geqslant B\|v\|^2, \quad v \in H_0^1(\Omega) \cap H^2(\Omega)$$

.

The weak form of (2.1) is to find $(\lambda, u) \in R \times H_0^1(\Omega) \cap \Omega), u \neq 0$, such that

$$a(u,v) = \lambda(u,v), v \in H_0^1(\Omega) \cap H^2(\Omega).$$

Let $\mathcal{T}$ be a conforming subdivision of Ω into disjoint triangular or quadrilateral elements $\kappa \in \mathcal{T}$, on this assumption that the subdivision is shape regular and constructed by affine mapping $\mathcal{F}_\kappa$, where $\mathcal{F}_\kappa: \hat{\kappa} \to \kappa$, with nonsingular Jacobin, where $\hat{\kappa}$ is the reference triangle or quadrilateral. It is assumed that the mapping is constructed to ensure that $\overline{\Omega} = \cup_{\kappa \in \mathcal{T}} \bar{\kappa}$ and the elemental edges are straight line segments.

The broken Laplacian $\Delta_h u$ is defined by

$$(\Delta_h u)|_\kappa = \Delta(u|_\kappa), \quad \kappa \in \mathrm{T}$$

For a non-negative integer r, $\mathcal{P}_r(\hat{\kappa})$ is used to represent the set of all polynomials of degree at most r if $\hat{\kappa}$ is a reference triangle, and $\mathcal{P}_r(\hat{\kappa})$ is used to represent the set of polynomials of tensor product if $\hat{\kappa}$ is a reference quadrilateral. For $r = 2$, consider its finite element space

$$S := \{v \in L^2(\Omega) : v|_K \circ F_K \in \mathrm{P}_2(\hat{\kappa}), \kappa \in \mathrm{T}\},$$

We use Γ_h to represent the union (including the boundary) of all one-dimensional unit edges associated with the subdivision $\mathcal{T}$. In addition, we decompose Γ_h into two disjoint subsets, i.e. $\Gamma_h = \Gamma_\partial \cup \Gamma_{\text{int}}$, where $\Gamma_{\text{int}} := \Gamma_h \setminus \Gamma_\partial$.

Let $\kappa^+$and κ^- be two elements of the shared edge $e := \partial\kappa^+ \cap \partial\kappa^- \subset \Gamma_{\text{int}}$. Define the outward normal unit vectors on e corresponding to $\partial\kappa^+$ and $\partial\kappa^-$, respectively, as $\mathbf{n}^+$ and $\mathbf{n}^-$. For functions $v: \Omega \to \mathbb{R}$ and $\mathbf{q}: \Omega \to \mathbb{R}^2$, these functions may be discontinuous in Γ_h, the following is defined for $v^+ := v|_{e\subset\partial\kappa^+}, v^- := v|_{e\subset\partial\kappa^-}, \mathbf{q}^+ := \mathbf{q}|_{e\subset\partial\kappa^+}$,

$$\{v\} := \frac{1}{2}(v^+ + v^-), \quad [v] := v^+\mathbf{n}^+ + v^-\mathbf{n}^-$$

$$\{\mathbf{q}\} := \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-), \quad [\mathbf{q}] := \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-.$$

If $e \in \partial\kappa \cap \Gamma_\partial$, then these definitions are changed as follows:

$$\{v\} := v^+, \quad \{\mathbf{q}\} := \mathbf{q}^+, \quad [v] := v^+\mathbf{n}, \quad [\mathbf{q}] := \mathbf{q}^+ \cdot \mathbf{n}.$$

With the above definition, it can be verified

$$\sum_{\kappa\in\mathrm{T}} \int_{\partial\kappa} v\mathbf{q}\cdot\mathbf{n} \mathrm{d}s = \int_{\Gamma_h} [v]\cdot\{\mathbf{q}\}\mathrm{d}s + \int_{\Gamma_{\text{int}}} \{v\}[\mathbf{q}]\mathrm{d}s.$$

To define $h_\kappa := \mathrm{diam}(\kappa)$, and collect them into the elementwise constant function $\mathbf{h} : \Omega \to \mathbb{R}$, with $\mathbf{h}|_\kappa = h_\kappa, \kappa \in \mathcal{T}$, and $\mathbf{h}|_e = \{\mathbf{h}\}, e \subset \Gamma_h$. We always assume that the families of meshes considered are locally quasi-uniform, there are constants $c \geqslant 1$ independent of $\mathbf{h}$, for any pair of elements κ^+ and κ^- in $\mathcal{T}$, that share an edge, we have

$$c^{-1} \leqslant \frac{h_{\kappa^+}}{h_{\kappa^-}} \leqslant c.$$

1 Poincar´e inequality [19]

Let $\Omega \subset R^2$ be a bounded set, for any function $v \in H_0^1(\Omega)$, there exists a constant C such that the following inequality holds

$$\|v\|_\Omega \leq C \|\nabla v\|_\Omega . \tag{2-6}$$

2 Cauchy-Schwarz inequality [20]

Let $f(x)$, $g(x)$ be integrable on $[a,b]$, then we have

$$[\int_a^b f(x)g(x)dx]^2 \leq [\int_a^b f(x)^2 dx][\int_a^b g(x)^2 dx]. \tag{2-7}$$

Let $a_i, b_i \in \mathrm{R}(i = 1,2,...,n)$,then

$$\sum_{i=1}^n |a_i b_i| \leq (\sum_{i=1}^n a_i^2)^{\frac{1}{2}} (\sum_{i=1}^n b_i^2)^{\frac{1}{2}}, \tag{2-8}$$

which holds with equality if and only if $b_i = ka_i (i = 1,2,...,n)$, ($k$ is a constant).

3 Young inequality [21]

$a,b > 0, p,q > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$ab \leq \frac{\delta a^p}{p} + \frac{\delta b^q}{q\delta^{\frac{q}{p}}}, \quad \forall \delta > 0, \tag{2-9}$$

in particular, when p = q = 2, we have

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2. \tag{2-10}$$

4 Inverse inequality [22]

For any $\forall v \in V_h(\Omega_0)$, there is

$$\|v\|_{1,\Omega_0} \leq Ch^{-1} \|v\|_{0,\Omega_0} . \tag{2-11}$$

5 Trace inequality

Set $1< p \leq \infty, v \in W^{1,p}(\kappa)$, we have

$$\| v \|_{0,p,\partial\kappa} \lesssim h_{\kappa}^{-\frac{1}{p}} (\| v \|_{0,p,\kappa} + h_{\kappa} \| \nabla v \|_{0,p,\kappa}). \tag{2-12}$$

On the unit circle κ, multiply both sides of the equation by $\tilde{v} \in H_0^1(\kappa) \cap H^2(\kappa)$ and integrate to obtain

$$\int_{\kappa} \Delta^2 u\tilde{v}dx = \int_{\kappa} \lambda u\tilde{v}dx. \tag{2-13}$$

By Green's integral

$$\int_{\kappa} \Delta^2 u\tilde{v}dx = -\int_{\kappa} \nabla\Delta u \nabla\tilde{v}dx + \int_{\partial\kappa} \nabla\Delta u \cdot \tilde{v}\mathbf{n}ds.$$

S

Thus

$$\int_{\kappa} \lambda u\tilde{v}dx = \int_{\kappa} \Delta u\Delta\tilde{v}dx + \int_{\partial\kappa} \nabla\Delta u \cdot \tilde{v}\mathbf{n}ds - \int_{\partial\kappa} \Delta u \cdot \nabla\tilde{v}\mathbf{n}ds, \tag{2-14}$$

when $u = \Delta u = 0$ and (2-14), we have

$$\sum_{\kappa\in T} \int_{\kappa} \Delta_h u\Delta_h \tilde{v}dx + \int_{\Gamma_h} \{\nabla\Delta u\}[\tilde{v}]ds - \int_{\Gamma_{int}} \{\Delta u\}[\nabla\tilde{v}]ds = \int_{\Omega} \lambda u\tilde{v}dx. \tag{2-15}$$

Add to the left end of (2-15)

$$\int_{\Gamma_h} \sigma[u][\tilde{v}]\mathrm{d}s, \int_{\Gamma_{\mathrm{int}}} \tau[\nabla u][\nabla\tilde{v}]\mathrm{d}s. \tag{2-16}$$

We first introduce the lifting operator $\mathcal{L}: \mathcal{S} := S^2 + (H_0^1(\Omega) \cap H^2(\Omega)) \to S^2$ by

$$\int_{\Omega} \mathrm{L}(\tilde{v})u\mathrm{d}x = \int_{\Gamma_h} [\tilde{v}] \cdot \{\nabla u\}\mathrm{d}s - \int_{\Gamma_{\mathrm{int}}} \{u\}[\nabla\tilde{v}]\mathrm{d}s \quad u \in S. \tag{2-17}$$

And the lifting operator $\mathcal{L}$ has stability: for $w \in \mathcal{S}$, there is

$$\| \mathrm{L}(w) \|_{\Omega}^2 \leq C(\| \sqrt{\sigma}[w] \|_{\Gamma_h}^2 + \| \sqrt{\tau}[\nabla w] \|_{\Gamma_{\mathrm{int}}}^2).$$

Where $\sigma = C_{\sigma}\mathbf{h}^3$, $\tau = C_{\tau}\mathbf{h}$.

Proof. See [5].

Define bilinear form as $a_h: \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ by

$$\begin{aligned} a_h(u_h, v_h) &= \int_{\Omega} (\Delta_h u_h \Delta_h v_h + \mathrm{L}(u_h)\Delta_h v_h + \Delta_h u_h \mathrm{L}(v_h))\mathrm{d}x \\ &+ \int_{\Gamma_h} \sigma[u_h][v_h]\mathrm{d}s + \int_{\Gamma_{\mathrm{int}}} \tau[\nabla u_h][\nabla v_h]\mathrm{d}s \end{aligned} \quad , . \tag{2-18}$$

here the internal penalty parameter $\sigma: \Gamma_h \to \mathbb{R}$, $\tau: \Gamma_{\mathrm{int}} \to \mathbb{R}$ of the segmentation constant is defined as

$$\sigma|_{\Gamma_h} = \sigma_0(\mathbf{h}|_e)^{-3}, \quad \tau|_{\Gamma_{\mathrm{int}}} = \tau_0(\mathbf{h}|_e)^{-1},$$

where $\sigma_0 > 0, \tau_0 > 0$, in order to guarantee the stability of the IPDG method defined in (2.7), σ, τ must be selectively large enough.

The finite element approximation of (2.4) is to find $(\lambda_h, u_h) \in R \times S^2$, such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h), v_h \in S. \tag{2-19}$$

The source problem of (2.4) is to find $w \in H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$a(w, v) = (f, v), v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2-20}$$

The DG approximation of (2-20) is to find $w_h \in S^2$, such that

$$a_h(w_h, v_h) = (f, v_h), v_h \in S. \tag{2-21}$$

Define the linear bounded operator $T: L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$a(Tf, v) = (f, v), f \in L^2(\Omega), v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2-22}$$

The equivalent operator from of (2.4) is

$$Tu = \frac{1}{\lambda} u. \tag{2-23}$$

By using (2-20), the corresponding discrete solution operator $T_h: L^2(\Omega) \to S$ can be defined:

$$a_h(T_h f, v) = (f, v), f \in L^2(\Omega), v \in S. \tag{2-24}$$

The equivalent operator from of (2-20) is

$$T_h u_h = \frac{1}{\lambda_h} u_h. \tag{2-25}$$

From the consistency of discontinuous finite element method, let w be the solution of (2-22), and $f \in L^2(\Omega)$, then

$$w = Tf, w_h = T_h f. \tag{2-26}$$

From (2-21) and (2-26), we obtain

$$a_h(w - w_h, v_h) = 0, v_h \in S. \tag{2-27}$$

For any function $w \in \mathcal{S}$, introduce sum space $S = S^2 + (H_0^1(\Omega) \cap H^2(\Omega))$, that assigns a locally discontinuous finite element norm, where the energy norm is defined as follows:

$$\| w \|_G = (\| \Delta_h w \|_\Omega^2 + \| \sqrt{\sigma}[w] \|_{\Gamma_h}^2 + \| \sqrt{\tau}[\nabla w] \|_{\Gamma_{int}}^2)^{\frac{1}{2}}. \tag{2-28}$$

There is $a_h(\cdot,\cdot)$ is continuous and coercive：

$$| a_h(w,v) | \leq C_1 \| w \|_G \| v \|_G, \quad w, v \in S \tag{2-29}$$

$$a_h(w,w) \geq C_2 \| w \|_G^2, \quad w \in S \tag{2-30}$$

where $\sigma: \Gamma_h \to \mathbb{R}, \tau: \Gamma_{\text{int}} \to \mathbb{R}$ is a piecewise continuous function, C_1 and C_2 are positive constants depending only on the mesh parameters.

Proof. For $w, v \in S^2$, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&| a_h(w,v) | \\
&\leq \left| \int_\Omega (\Delta_h w \Delta_h v + \mathrm{L}(w)\Delta_h v + \mathrm{L}(v)\Delta_h w) \mathrm{d}x + \int_{\Gamma_h} \sigma [w][v] \mathrm{d}s + \int_{\Gamma_{int}} \tau [\nabla w][\nabla v] \mathrm{d}s \right| \\
&\leq \| \Delta_h w \|_\Omega \| \Delta_h v \|_\Omega + \| \mathrm{L}(w) \|_\Omega \| \Delta_h v \|_\Omega + \| \mathrm{L}(v) \|_\Omega \| \Delta_h w \|_\Omega \\
&+ \| \sqrt{\sigma}[w] \|_{\Gamma_h} \| \sqrt{\sigma}[v] \|_{\Gamma_h} + \| \sqrt{\tau}[\nabla w] \|_{\Gamma_{int}} \| \sqrt{\tau}[\nabla v]] \|_{\Gamma_{int}} \\
&\leq C \| w \|_G \| v \|_G .
\end{aligned}$$

continuity is valid.

$$\begin{aligned}
&\| w \|_G^2 \| v \|_G^2 \\
&= \left(\| \Delta_h w \|_\Omega^2 + \| \sqrt{\sigma} [w] \|_{\Gamma_h}^2 + \| \sqrt{\tau} [\nabla w] \|_{\Gamma_{int}}^2 \right) \\
&\cdot \left(\| \Delta_h v \|_\Omega^2 + \| \sqrt{\sigma} [v] \|_{\Gamma_h}^2 + \| \sqrt{\tau} [\nabla v] \|_{\Gamma_{int}}^2 \right) \\
&= \| \Delta_h w \|_\Omega^2 \| \Delta_h v \|_\Omega^2 + \| \Delta_h w \|_\Omega^2 \left(\| \sqrt{\sigma} [v] \|_{\Gamma_h}^2 + \| \sqrt{\tau} [\nabla v] \|_{\Gamma_{int}}^2 \right) \\
&+ \| \Delta_h v \|_\Omega^2 \left(\| \sqrt{\sigma} [w] \|_{\Gamma_h}^2 + \| \sqrt{\tau} [\nabla w] \|_{\Gamma_{int}}^2 \right) \\
&+ \left(\| \sqrt{\sigma} [w] \|_{\Gamma_h}^2 + \| \sqrt{\tau} [\nabla w] \|_{\Gamma_{int}}^2 \right) \left(\| \sqrt{\sigma} [v] \|_{\Gamma_h}^2 + \| \sqrt{\tau} [\nabla | v] \|_{\Gamma_{int}}^2 \right),
\end{aligned}$$

from the definition of norm and the Young inequality that

$$\begin{aligned}
a_h(w,w) &= \| w \|_G^2 + 2 \int_\Omega \mathrm{L}(w) \Delta_h w \mathrm{d}x \\
&\geq \| w \|_G^2 - 2 \| L(w) \|_\Omega^2 - \frac{1}{2} \| \Delta_h w \|_\Omega^2 .
\end{aligned}$$

And because

$$\begin{aligned}
&\| w \|_G^2 - 2 \| L(w) \|_\Omega^2 - \frac{1}{2} \| \Delta_h w \|_\Omega^2 \\
&= \| \Delta_h w \|_\Omega^2 + \| \sqrt{\sigma} [w] \|_{\Gamma_\text{h}}^2 + \| \sqrt{\tau} [\nabla w] \|_{\Gamma_\text{int}}^2 \\
&\quad - 2c (\| \sqrt{\sigma} [w] \|_{\Gamma_\text{h}}^2 + \| \sqrt{\tau} [\nabla w] \|_{\Gamma_\text{int}}^2) - \frac{1}{2} \| \Delta_h w \|_\Omega^2 \\
&= \frac{1}{2} \| \Delta_h w \|_\Omega^2 + \sqrt{1 - 2c} (\| \sqrt{\sigma} [w] \|_{\Gamma_\text{h}}^2 + \| \sqrt{\tau} [\nabla w] \|_{\Gamma_\text{int}}^2).
\end{aligned}$$

When $0 < c < \frac{1}{2}$, the proof is completed.

Let $w \in H^{2+r}(\Omega)(1 < r \leqslant 2)$ be the solution of (2-22), and $f \in L^2(\Omega)$, assuming the following regularity estimate holds:

$$\|w\|_{2+r} \lesssim \|f\|_{\Omega}. \tag{2-31}$$

Let w^I be the quadratic interpolation of w, then:

$$\|w - w^I\|_G \lesssim h\|w\|_{3,\Omega}, \tag{2-32}$$

also $[w - w^I] = 0$.

Lemma 2.1. (Proposition 4.9 in [6]) Let $\kappa \in \mathcal{T}$ and $v \in H^{s_\kappa}(\kappa), s_\kappa > 3$, then there exists the polynomial $\Pi v \in S^{h_\kappa}$, satisfying $(0 \leqslant m \leqslant s_\kappa)$

$$\|v - \Pi v\|_{m,\kappa} \lesssim h_\kappa^{s_\kappa - m}\|v\|_{s_\kappa,\kappa}, \tag{2-33}$$

$$\|v - \Pi v\|_{0,\partial\Omega} \lesssim h_\kappa^{s_\kappa - \frac{1}{2}}\|v\|_{s_\kappa,\kappa}. \tag{2-34}$$

Introduce the global interpolation operator $\Pi: (H_0^1(\Omega) \cap H^2(\Omega)) \to S$, such that $\Pi(u)|_\kappa = \Pi(u|_\kappa)$, for the vector-value function $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_d)$, define $\Pi(\mathbf{r})|_\kappa = (\Pi\mathbf{r}_1, \Pi\mathbf{r}_2, \cdots, \Pi\mathbf{r}_d)$.

Lemma 2.3. (lemma 2.1 in [7]) Let $\kappa \in \mathcal{T}$, $e \subset \partial\kappa$, and $0 < \xi < \frac{1}{2}$, for any $v \in H^{1+\xi}(\kappa)$ with $\Delta v \in L^2(\kappa)$, there exists a positive constant C independent of v such that

$$\|\nabla v \cdot \mathbf{n}\|_{\xi - \frac{1}{2}, e} \leq C\left(\|\nabla v\|_{\xi,\kappa} + h_\kappa^{1-\xi}\|\Delta v\|_{0,\kappa}\right), \kappa \in \mathrm{T}, e \in \partial\kappa. \tag{2-35}$$

Theorem 2.1. Let w and w_h be the solution of (2-20) and (2-21), for all $\kappa \in \mathcal{T}$, and $s_\kappa > 3, 0 < \xi < \frac{1}{2}$, then there holds

$$\|w - w_h\|_G \lesssim \inf_{v_h \in S}\|w - v_h\|_G + h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega} + h\|f\|_{\Omega}, \tag{2-36}$$

$$\|w - w_h\|_G \lesssim h\|w\|_3 + h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega} + h\|f\|_{\Omega}.. \tag{2-37}$$

Proof. Firstly, we prove (2-35) by utilizing (2-27), (2-29) and (2-30), we obtain

$$\begin{aligned}
&\|v_h - w_h\|_G^2 \\
&\lesssim |a_h(v_h - w_h, v_h - w_h)| \\
&\lesssim a_h(w - w_h, v_h - w_h) + a_h(v_h - w, v_h - w_h) \\
&\lesssim \|v_h - w\|_G\|v_h - w_h\|_G \\
&+\int_{\Gamma_h}(\{\nabla\Delta(v_h - w)\}[v_h - w_h] + \{\nabla\Delta(v_h - w_h)\}[v_h - w])\mathrm{d}s \\
&+\int_{\Gamma_{int}}(\{\Delta(v_h - w)\}[\nabla(v_h - w_h)] + \{\Delta(v_h - w_h)\}[\nabla(v_h - w)])\mathrm{d}s.
\end{aligned}$$

By (2-35), the inverse estimate, and the definition of the energy norm, we have

$$\begin{aligned}
&\int_{\Gamma_h}\{\nabla\Delta(v_h - w)\}[v_h - w_h])\mathrm{d}s \\
&\lesssim \sum_{e \partial\Gamma_h}\|\{\nabla\Delta(v_h - w)\} \cdot \mathbf{n}\|_{\xi - \frac{1}{2}, e}\|[v_h - w_h]\|_{\frac{1}{2} - \xi, e} \\
&\lesssim \sum_\kappa h^{\xi+1}\|\nabla\Delta(v_h - w)\|_{\xi,\kappa}\left(\|h^{-\frac{3}{2}}[v_h - w]\|_{0,e}^2\right)^{\frac{3}{2}} \\
&+\sum_\kappa h_\kappa^{1-\xi}\|\Delta^2(v_h - w)\|_{0,\kappa} h^{\xi+1}\left(\|h^{-\frac{3}{2}}[v_h - w]\|_{0,e}^2\right)^{\frac{1}{2}} \\
&\lesssim \left(h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega} + h^2\|f\|_{\Omega}\right)\|v_h - w_h\|_G.
\end{aligned} \tag{2-38}$$

It can also be known that

$$\int_{\Gamma_h}\{\nabla\Delta(v_h - w_h)\}[v_h - w])\mathrm{d}s = 0. \tag{2-39}$$

By the trace inequality, the definition of the energy norm and (2-31) we have

$$
\begin{aligned}
&\int_{\Gamma_{int}} \left\{\Delta\left(v_h - w\right)\right\}[\nabla\left(v_h - w_h\right)]\mathrm{d}s \\
&\lesssim \sum_{e\in\Gamma_{int}} \|\Delta(v_h - w)\|_{0,e}\|[\nabla(v_h - w_h)]\|_{0,e} \\
&\lesssim \sum_{\kappa} h^{-\frac{1}{2}} \|\Delta(v_h - w)\|_{0,k}\ h^{\frac{1}{2}}(\|h^{-\frac{1}{2}}[\nabla(v_h - w_h)]\|_{0,e}^2)^{\frac{1}{2}} \\
&+\sum_{\kappa} h_{\kappa}^{\frac{1}{2}} \|\nabla\Delta(v_h - w)\|_{0,k}\ h^{\frac{1}{2}}(\|h^{-\frac{1}{2}}[\nabla(v_h - w_h)]\|_{0,e}^2)^{\frac{1}{2}} \\
&\lesssim (\|v_h - w\|_G + h\|\nabla\Delta w\|_{\Omega})\|v_h - w_h\|_G \\
&\lesssim (\|v_h - w\|_G + h\|f\|_{\Omega})\|v_h - w_h\|_G .
\end{aligned}
$$

(2-40)

Similarly

$$
\begin{aligned}
&\int_{\Gamma_{int}} \{\Delta\left(v_h - w_h\right)\}[\nabla\left(v_h - w\right)]\mathrm{d}s \\
&\lesssim \sum_{e\in\Gamma_{int}} \|\Delta(v_h - w_h)\|_{0,e}\|[\nabla(v_h - w)]\|_{0,e} \\
&\lesssim \sum_{\kappa} h^{-\frac{1}{2}} \|\Delta(v_h - w_h)\|_{0,k}\ h^{\frac{1}{2}}(\|h^{-\frac{1}{2}}[\nabla(v_h - w)]\|_{0,e}^2)^{\frac{1}{2}} \\
&+\sum_{\kappa} h_{\kappa}^{\frac{1}{2}} \|\nabla\Delta(v_h - w_h)\|_{0,k}\ h^{\frac{1}{2}}(\|h^{-\frac{1}{2}}[\nabla(v_h - w)]\|_{0,e}^2)^{\frac{1}{2}} \\
&\lesssim \|v_h - w_h\|_G\|v_h - w\|_G .
\end{aligned}
$$

(2-41)

From (2-38), (2-39), (2-40), and (2-41)

$$
\|v_h - w_h\|_G \lesssim \|v_h - w\|_G + h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega} + h\|f\|_{\Omega} . \tag{2-42}
$$

Using the triangle inequality we obtain (2-36).

To prove (2-37), by (2-28), let $E_h(w) = w - \Pi w$ have

$$
\begin{aligned}
\|E_h(w)\|_G^2 &\lesssim \sum_{\kappa\in\mathrm{T}} \|\Delta_h E_h(w)\|_{0,\kappa}^2 + \sum_{e\in\Gamma_h} \|h^{-\frac{3}{2}}[E_h(w)]\|_{0,\Gamma_h}^2 \\
&+ \sum_{e\in\Gamma_{int}} \|h^{-\frac{1}{2}}[\nabla E_h(w)]\|_{0,\Gamma_{int}}^2 \\
&\lesssim \sum_{\kappa\in\mathrm{J}} \|\Delta_h E_h(w)\|_{0,\kappa}^2 + \sum_{e\in\Gamma_h} \|h^{-\frac{3}{2}}[E_h(w)]\|_{0,\Gamma_h}^2 \\
&+ \sum_{e\in\Gamma_h} \|h^{-\frac{1}{2}}[\nabla E_h(w)]\|_{0,\Gamma_h}^2 . \\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

Estimate of I_1 , from (2-33) and $s_k = 3$

$$
\|\Delta_h E_h(w)\|_{0,\kappa}^2 \lesssim (h\|w\|_{3,\kappa})^2. \tag{2-43}
$$

Estimate I_2 , by (2-34), the trace inequality and the inverse estimate, we have

$$\begin{aligned}
&\|h^{-\frac{3}{2}}[E_h(w)]\|_{0,e}^2 \\
&=\|h^{-\frac{3}{2}}((E_h(w))_{\kappa^+}-(E_h(w))_{\kappa^-})\cdot\mathbf{n}\|_{0,e}^2 \\
&\lesssim h^{-3}(h_e^{-\frac{1}{2}}\|E_h(w)\|_{0,\kappa^+\cup\kappa^-}+h_e^{\frac{1}{2}}\|\nabla E_h(w)\|_{0,\kappa^+\cup\kappa^-})^2 \\
&\lesssim h^{-3}(h_e^{-\frac{1}{2}}\|E_h(w)\|_{0,\kappa^+\cup\kappa^-})^2 \\
&\lesssim (h\|w\|_{3,\kappa})^2.
\end{aligned} \tag{2-44}$$

Estimate I_3, and by the same token

$$\begin{aligned}
\|h^{-\frac{1}{2}}[\nabla E_h(w)]\|_{0,e}^2 &= \|h^{-\frac{1}{2}}((\nabla E_h(w))_{\kappa^+}-(\nabla E_h(w))_{\kappa^-})\cdot\mathbf{n}\|_{0,e}^2 \\
&\lesssim h^{-1}(h_e^{-\frac{1}{2}}\|\nabla E_h(w)\|_{0,\kappa^+\cup\kappa^-})^2 \\
&\lesssim (h\|w\|_{3,\kappa})^2.
\end{aligned} \tag{2-45}$$

By the definition of norm, (2-43), (2-44), and (2-45)

$$\|w-\Pi w\|_G \lesssim \left(\sum_{\kappa\in J}(h\|w\|_{3,\kappa})^2\right)^{\frac{1}{2}}. \tag{2-46}$$

By using the error estimate and the interpolation estimate $\inf_{v_h\in S}\|w-v_h\|_G \lesssim \|w-\Pi w\|_G$, we obtained

$$\begin{aligned}
&\inf_{v_h\in S}\|w-v_h\|_G + h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega} + h\|f\|_\Omega \\
&\lesssim h\|w\|_{3,\Omega} + h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega} + h\|f\|_\Omega .
\end{aligned} \tag{2-47}$$

Then from (2-36) and (2-47)

$$\|w-w_h\|_G \lesssim h\|w\|_3 + h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega} + h\|f\|_\Omega,$$

the proof is completed.

Theorem 2.2. Let w and w_h be the solution of (2-20) and (2-21), then there holds:

$$\|w-w_h\|_\Omega \lesssim h\|w-w_h\|_G + h^2\|f\|_\Omega, \tag{2-48}$$

$$\|w-w_h\|_\Omega \lesssim h^2\|w\|_{2+r,\Omega} . \tag{2-49}$$

Proof. w^I is the quadratic interpolation of w, form (2-27) and (2-32), we have

$$\begin{aligned}
(w-w_h, f) &= a_h\left(w-w_h, w\right) = a_h\left(w-w_h, w-w^I\right) \\
&\lesssim \|w-w_h\|_G \|w-w^I\|_G \\
&+\int_{\Gamma_h}\left(\{\nabla\Delta\left(w-w_h\right)\}\left[w-w^I\right]+\{\nabla\Delta\left(w-w^I\right)\}[w-w_h]\right)\mathrm{d}s \\
&+\int_{\Gamma_{int}}\left(\{\Delta\left(w-w_h\right)\}\left[\nabla\left(w-w^I\right)\right]+\{\Delta\left(w-w^I\right)\}[\nabla\left(w-w_h\right)]\right)\mathrm{d}s.
\end{aligned} \tag{2-50}$$

From $[w-w^I]=0$, we derive

$$\int_{\Gamma_h}\left\{\nabla\Delta\left(w-w_h\right)\right\}[w-w^I]\mathrm{d}s = 0. \tag{2-51}$$

From lemma 2.2, the inverse estimate, definition of energy norm, (2-31) and taking $\xi = r-1$, we deduce

$$
\begin{aligned}
&\int_{\Gamma_h}\{\nabla\Delta\left(w-w^I\right)\}[w-w_h]\mathrm{d}s\\
&\lesssim \sum_{e\subset\Gamma_h}\|\{\nabla\Delta w\}\cdot\mathbf{n}\|_{\xi-\frac{1}{2},e}\|[w-w_h]\|_{\frac{1}{2}-\xi,e}\\
&\lesssim \sum_{\kappa}\left(\nabla\Delta w_{\xi,\kappa}+h_\kappa^{1-\xi}\Delta^2 w_{0,\kappa}\right)h^{\xi+1}\left(h^{-\frac{3}{2}}[w-w_h]_{0,e}^2\right)^{\frac{1}{2}}\\
&\lesssim \left(h^{\xi+1}\|\nabla\Delta w\|_{\xi,\Omega}+h^2\|f\|_{\Omega}\right)\|w-w_h\|_G\\
&\lesssim \left(h^{\xi+1}w_{2+r}+h^2 f_{\Omega}\right)\|w-w_h\|_G\\
&\lesssim h^r\|f\|_{\Omega}\|w-w_h\|_G\,.
\end{aligned}
\tag{2-52}
$$

By the trace inequality with $\frac{1}{2}<\beta\leqslant 1$, the interpolation estimates and the definition of energy norm, we get

$$
\begin{aligned}
&\int_{\Gamma_{int}}\left\{\Delta\left(w-w^I\right)\right\}\left[\nabla\left(w-w_h\right)\right]\mathrm{d}s\\
&\lesssim \sum_{e\in\Gamma_{int}}\|\left\{\Delta\left(w-w^I\right)\right\}\|_{0,e}\|\left[\nabla\left(w-w_h\right)\right]\|_{0,e}\\
&\lesssim \sum_{\kappa}\|\Delta\left(w-w^I\right)\|_{0,\kappa}\left(\|h^{-\frac{1}{2}}[\nabla\left(w-w_h\right)]\|_{0,e}^2\right)^{\frac{1}{2}}\\
&+\sum_{\kappa}h^{\frac{1}{2}}h_\kappa^{\beta-\frac{1}{2}}|\Delta\left(w-w^I\right)|_{\beta,\kappa}\left(\|h^{-\frac{1}{2}}[\nabla\left(w-w_h\right)]\|_{0,e}^2\right)^{\frac{1}{2}}\\
&\lesssim \left(\|w-w^I\|_G+h^{\beta}\|w-w^I\|_{2+\beta,\kappa}\right)\|w-w_h\|_G\\
&\lesssim (h\|w\|_3+h^r\|w\|_{2+r})\|w-w_h\|_G\\
&\lesssim h\|w\|_{2+r}\|w-w_h\|_G\lesssim h\|f\|_{\Omega}\|w-w_h\|_G\,.
\end{aligned}
\tag{2-53}
$$

From the trace inequality, (2-31), (2-32) and the definition of energy norm, we derive

$$
\begin{aligned}
&\int_{\Gamma_{int}}\{\Delta\left(w-w_h\right)\}[\nabla\left(w-w^I\right)]\mathrm{d}s\\
&\lesssim \sum_{e\in\Gamma_{int}}h^{\frac{1}{2}}\|\Delta\left(w-w_h\right)\|_{0,e}\left(\|h^{-\frac{1}{2}}[\nabla\left(w-w^I\right)]\|_{0,e}^2\right)^{\frac{1}{2}}\\
&+\sum_{\kappa}h^{\frac{1}{2}}h_\kappa^{\beta-\frac{1}{2}}|\Delta\left(w-w^I\right)|_{\beta,\kappa}\left(\|h^{-\frac{1}{2}}[\nabla\left(w-w_h\right)]\|_{0,e}^2\right)^{\frac{1}{2}}\\
&\lesssim h^{\frac{1}{2}}\|w-w^I\|_G\left(h^{-\frac{1}{2}}\|(w-w_h)\|_G+h_\kappa^{\frac{1}{2}}\|\nabla\Delta w\|_{0,\kappa}\right)\\
&\lesssim h^{\frac{3}{2}}\|w\|_3\left(h^{-\frac{1}{2}}\|(w-w_h)\|_G+h_\kappa^{\frac{1}{2}}\|\nabla\Delta w\|_{0,\kappa}\right)\\
&\lesssim h\|w\|_{2+r}\|w-w_h\|_G+h^2\|w\|_{2+r}^2\\
&\lesssim h\|f\|_{\Omega}\|w-w_h\|_G+h^2\|f\|_{\Omega}^2\,.
\end{aligned}
\tag{2-54}
$$

by (2-50), (2-51), (2-52), (2-53) and (2-5)

$$w - w_{h\Omega} \lesssim hw - w_{hG} + h^2 f_\Omega.$$

Next, we prove (2-49). From (2-37), (2-54), and (2-48), obtain

$$\begin{aligned} \| w - w_h \|_\Omega &\lesssim h^{r+3} \| w \|_{2+r,\Omega} + h^2 \| \nabla \Delta w \|_\Omega \\ &\lesssim h^{r+3} \| w \|_{2+r,\Omega} + h^2 \| w \|_{2+r,\Omega} \\ &\lesssim h^2 \| w \|_{2+r,\Omega} . \end{aligned} \tag{2-55}$$

which proves (2-49).

Taking $s = 2 + r(1 < r,, 2)$ in (2-37) and the regularity estimate the following stability estimate

$$\begin{aligned} \| T_h f \|_G &\lesssim \| T_h f - Tf \|_G + \| Tf \|_G \\ &\lesssim \| T_h f - Tf \|_G + \| Tf \|_2 \\ &\lesssim h^r \| Tf \|_{2+r} + \| Tf \|_2 \lesssim \| f \|_\Omega . \end{aligned} \tag{2-56}$$

III. A PRIORI ERROR ANALYSIS

Let λ be the jth eigenvalue of (2.4), with algebraic multiplicities q and the ascent $\alpha = 1$,where $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+q-1}$. When $\| T_h - T \|_{0,\Omega} \to 0$, q eigenvalue $\lambda_{j,h}, \cdots \lambda_{j+q-1,h}$ of (2.9) will converge to λ. Let $M(\lambda)$ be the generalized eigenvector space of (2.4) related to λ, $M_h(\lambda)$ be the direct sum of the generalized eigenvector space of (2.9) related to λ_h, and λ_h converge to λ.

The subspacegap between the two closed subspaces V and U is denoted as

$$\delta(U,V) = \sup_{u \in V, u_\Omega = 1} \inf_{v \in U} u - v_\Omega, \hat{\delta}(U,V) = \max\{\delta(U,V), \delta(V,U)\}.$$

$\hat{\lambda}_h = \frac{1}{2} \sum_{i=i}^{j+q-1} \lambda_{i,h}$ denotes the arithmetic mean。

Theorem 3.1 The following inequality holds

$$\hat{\delta}\big(M(\lambda), M_h(\lambda)\big) \lesssim h^2, \tag{3-1}$$

$$\left| \lambda_h - \lambda \right| \lesssim h^2, \tag{3-2}$$

$$| \lambda_h - \lambda | \lesssim h^2. \tag{3-3}$$

Let $u_h \in M_h(\lambda)$ be the direct sum of the generalized eigenspaces of (2-19), and $0 < \xi < \frac{1}{2}$. Then there exists an eigenfunction u of (2-4) such that

$$\| u - u_h \|_{0,\Omega} \lesssim h^2, \tag{3-4}$$

$$\| u - u_h \|_G \lesssim h \| u \|_{2+r,\Omega} + h^{\xi+1} \| \nabla \Delta u \|_{\xi,\Omega} + h^4. \tag{3-5}$$

Proof Let $Tf = w$ and $T_h f = w_h$. Combining the operator form, regularity estimates, and equation (2-49), we obtain

$$\begin{aligned} \| T - T_h \|_\Omega &= \sup_{0 \neq f \in L^2(\Omega)} \frac{\| Tf - T_h f \|_\Omega}{\| f \|_\Omega} = \sup_{0 \neq f \in L^2(\Omega)} \frac{\| w - w_h \|_\Omega}{\| f \|_\Omega} \\ &\lesssim \sup_{0 \neq f \in L^2(\Omega)} \frac{h^2 \| f \|_\Omega}{\| f \|_\Omega} \lesssim h^2 \to 0, (h \to 0). \end{aligned} \tag{3-6}$$

From Theorems (7.1), (7.2), (7.3), and (7.4) in reference [44], we have

$$\hat{\delta}\big(M(\lambda), M_h(\lambda)\big) \lesssim (T - T_h)|_{M(\lambda)\Omega}, \tag{3-7}$$

$$\left| \lambda - \hat{\lambda}_h \right| \lesssim \sum_{i,l=j}^{j+q-1} | ((T - T_h)\varphi_i, \varphi_l) | + \| (T - T_h)|_{M(\lambda)} \|_\Omega^2, \tag{3-8}$$

$$|\lambda-\lambda_h| \lesssim \sum_{n,i=j}^{j+q-1} |((T-T_h)\varphi_i,\varphi_l)| + \|(T-T_h)|_{M(\lambda)}\|_{\Omega}^2|, \tag{3-9}$$

$$\|u-u_h\|_{\Omega} \lesssim \|(T-T_h)|_{M(\lambda)}\|_{\Omega}. \tag{3-10}$$

where $\{\varphi_i\}_{i=j}^{j+q-1}$ forms a basis for $M(\lambda)$.

It can be inferred from Theorem 2.2 and Theorem 2.1 that

$$\begin{aligned}\|(T-T_h)|_{M(\lambda)}\|_{\Omega} &= \sup_{f\in M(\lambda),\|f\|_{\Omega}=1} \|Tf-T_h f\|_{\Omega} \\ &\lesssim \sup_{f\in M(\lambda),\|f\|_{\Omega}=1} h^2 \|Tf\|_{r+2,\Omega}.\end{aligned} \tag{3-11}$$

By inserting (3-11) into (3-7), we obtain (3-1).
Inserting (3-11) into (3-10) gives (3-4).

Using the operator properties and regularity estimates, the Galerkin orthogonality relations (2-27) and (2-29) yield

$$\begin{aligned}((T-T_h)\varphi_i,\varphi_l) &= a_h\left(T\varphi_i - T_h\varphi_i, T\varphi_l\right) \\ &= a_h\left(T\varphi_i - T_h\varphi_i, T\varphi_l - T_h\varphi_l\right) \\ &\lesssim \|T\varphi_i - T_h\varphi_i\|_G \|T\varphi_l - T_h\varphi_l\|_G \\ &\lesssim h\|T\varphi_i\|_3\, h\|T\varphi_l\|_3 \\ &\lesssim h^2.\end{aligned} \tag{3-12}$$

By inserting (3-12) into (3-8), we obtain (3-2).
Inserting (3-12) into (3-9) gives (3-3).

From $u = \lambda T u$ and $u_h = \lambda_h T_h u_h$, the triangle inequality combined with (2-56), (3-3), and (3-4) yields

$$\begin{aligned}\|u-u_h\|_G - \|u-\lambda T_h u\|_G &\lesssim \|u_h - \lambda T_h u\|_G = \|T_h(\lambda_h u_h - \lambda u)\|_G \\ &\lesssim \|\lambda_h u_h - \lambda u\|_{\Omega} \lesssim h^4.\end{aligned} \tag{3-13}$$

Based on (2-36) and (2-37), we derive

$$\begin{aligned}u-\lambda T_h u_G &= \lambda T u - \lambda T_h u_G \\ &\lesssim \inf_{v_h\in S^2} \|\lambda T u - v_h\|_G + h^{\xi+1}\|\nabla\Delta u\|_{\xi,\Omega} + h\|u\|_{\Omega} \\ &\lesssim h^r u_{2+r} + h^{\xi+1}\nabla\Delta u_{\xi,\Omega} + h u_{\Omega} \\ &\lesssim h u_{2+r,\Omega} + h^{\xi+1}\nabla\Delta u_{\xi,\Omega}.\end{aligned} \tag{3-14}$$

By combining equations (3-13) and (3-14), we obtain equation (3-5).

IV. NUMERICAL EXPERIMENTS

In this section, we conducted computational experiments on the Matlab 2017a platform to validate the effectiveness of our method. Considering problem (2-1), our program was compiled under the iFEM software package, and we performed calculations using the SIPG method (with penalty coefficients$\sigma = 70, \tau = 70$). We considered two test domains: the square domain Ω_S with vertices at (0,0), (1,0), (1,1), (0,1), and the hexagonal

domain Ω_H with vertices at (1,7,2), (2,7,3), (3,7,4), (7,5,4), (7,6,5), (1,6,7). Since the exact eigenvalues are unknown, we took the reference eigenvalue $\lambda = 389.6365$ for the square domain. For the hexagonal domain, we used a previous reference eigenvalue
$\lambda = 51.198878119786$.

Table 1 Numerical Eigenvalue Results for the Domain Ω_S

$Domin$	l	dof	$\lambda = 389.6365$	$Error$
Ω_S	1	768	1.0e+02*4.804360813618129	90.7995813618128
	2	1056	1.0e+02*4.0162473161865	11.98823162
	6	3888	1.0e+02*3.92258114095587	2.62161409558774
	10	18564	1.0e+02*3.90122271091472	0.485771091471861
	14	87588	1.0e+02*3.89736006335979	0.099506810507876
	16	206172	1.0e+02*3.89680850296493	0.044593973099722

Table 2 Numerical Eigenvalue Results for the Domain Ω_H

$Domin$	l	dof	$\lambda = 51.198878119786$	$Error$
Ω_H	1	2304	56.681054076591394	5.482175956805392
	2	2616	54.063730945910883	2.864852826124881
	6	8334	52.153313504680995	0.954435384894992
	10	28248	51.509294035452548	0.310415915666546
	12	53400	51.376378012289926	0.177499892503924
	15	136656	51.266349425181744	0.067471305395742

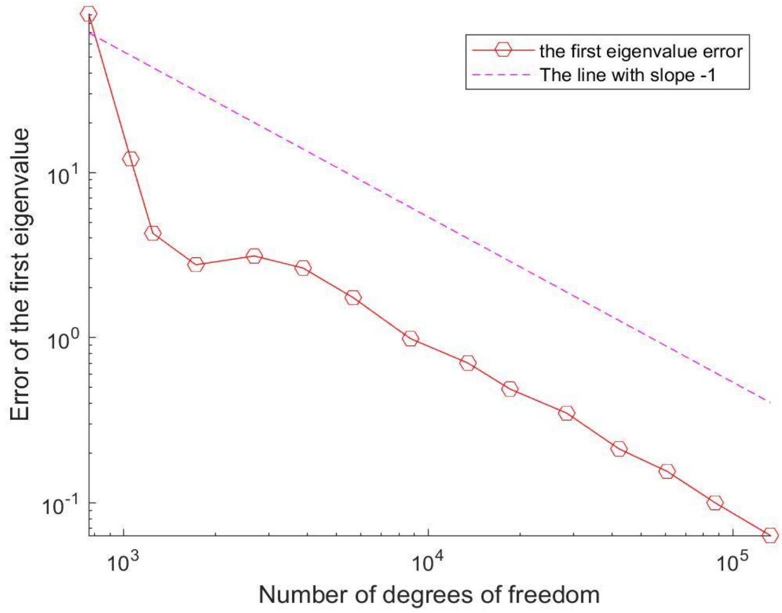

Figure 1 Error Curve of the Second Eigenvalue on the Test Domain Ω_S

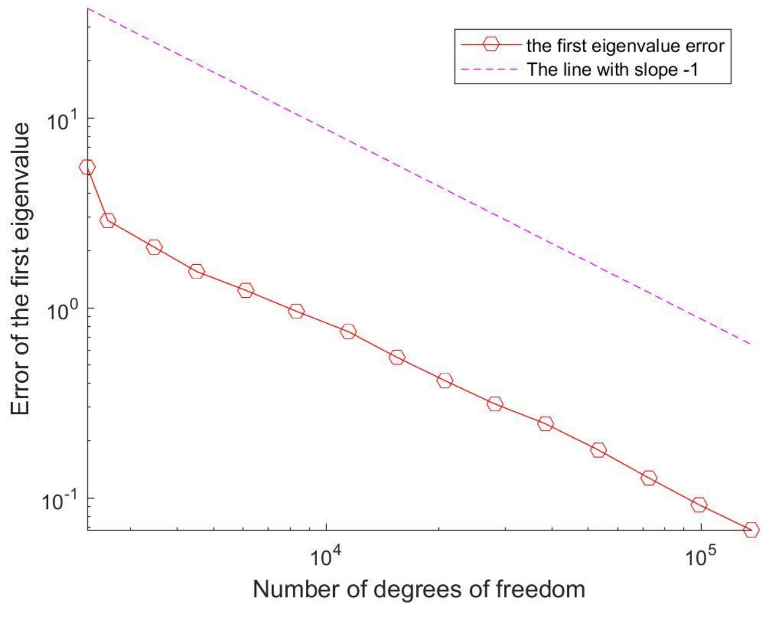

Figure 2 Error Curve of the Second Eigenvalue on the Test Domain Ω_H

V. CONCLUSION

The biharmonic eigenvalue problem has a wide range of applications in fields such as elastic mechanics, including thin plate vibration models, fluid mechanics, and quantum mechanics. This paper presents a discontinuous Galerkin method for solving the biharmonic eigenvalue problem under simply supported boundary conditions and derives the a priori error estimate for the biharmonic equation under these conditions. The most significant aspect is proving the convergence of the discrete solution operator $T_h$in the $L^2(\Omega)$ norm to the Dirichlet operator T, that is, $\| T - T_h \|_{\Omega} \rightarrow 0, (h \rightarrow 0)$. In conclusion, we conducted numerical experiments using the discontinuous Galerkin method to obtain the eigenvalue numerical solutions. From the error curves, it can be observed that our method achieves the optimal convergence rate for the eigenvalues and provides the optimal error estimates for the eigenfunction. This numerical experiment demonstrates the effectiveness of the algorithm. Additionally, this study provides theoretical support and numerical methods for practical problems such as thin plate vibrations and structural stability analysis, which helps to enhance the precision and efficiency of engineering design.

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