

On Mean Li-Yorke Chaos in Banach Spaces

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Abstract

The pioneers authors in [43] investigate the notion of mean Li-Yorke chaos for operators on Banach spaces. They show that it differs from the notion of distributional chaos of type 2, contrary to what happens of topological dynamics on compact metric spaces. They prove that an operator is mean Li-Yorke chaotic if and only if it has an absolutely mean irregular vector. So, absolutely Cesàro bounded operators are never mean Li-Yorke chaotic. Dense mean Li-Yorke chaos is shown to be equivalent to the existence of a dense (or residual) set of absolutely mean irregular vectors. As a consequence, every mean Li-Yorke chaotic operator is densely mean Li-Yorke chaotic on some infinite-dimensional closed invariant subspaces. They obtained a sufficient condition for the existence of a dense absolutely mean irregular manifold, and construct an example of others of an invertible hypercyclic operator such that every nonzero vector is absolutely mean irregular for both the operator and its inverse. Hence, mean Li-Yorke chaos is imbedded for C_0 -semigroups of operators on Banach spaces. On the way of [43] we apply an application.

Keywords: Banach spaces, operators, mean Li-Yorke chaos, absolute Cesàro boundedness, absolutely mean irregular vector, distributional chaos, hypercyclicity.

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I. Introduction

We investigate in the area of dynamical systems notions related to averages involving orbits or pseudo-orbits, such as mean Li-Yorke chaos [24],[29], mean equicontinuity and mean sensitivity [31], and notions of shadowing with average error in tracing [39]. [43] investigate the notion of mean Li-Yorke chaos for operators on Banach spaces. It turns out that this notion is intimately related to the notion of absolutely mean irregular vector. Moreover, we also establish some results on absolutely Cesàro bounded operators.

We state the relevant definitions.

Definition 1. An operator T on a Banach space X is said to be mean Li-Yorke chaotic if there is an uncountable subset S of X (a mean Li-Yorke set for T) such that every pair (x^r, y^r) of distinct points in S is a mean Li-Yorke pair for T , in the sense that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r - T^j y^r\| = 0 \quad \text{and} \quad \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r - T^j y^r\| > 0.$$

If S can be chosen to be dense (resp. residual) in X , then we say that T is densely (resp. generically) mean Li-Yorke chaotic.

Definition 2. Given an operator T and a vector x^r , we say that x^r is an absolutely mean irregular (resp. absolutely mean semi-irregular) vector for T if

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \quad \text{and} \quad \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = \infty \quad (\text{resp. } > 0).$$

Definition 3. [28] An operator T is said to be absolutely Cesàro bounded if there exists a constant $\epsilon > -1$ such that

$$\sup_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| \leq (1 + \epsilon) \sum_r \|x^r\| \text{ for all } x^r \in X.$$

The notion of distributional chaos has closer relations. For at least four different notions of distributional chaos, namely DC1, DC2, DC2 $_{\frac{1}{2}}$ and DC3. For (nonlinear) dynamical systems on compact metric spaces, mean Li-Yorke chaos is equivalent to DC2 [20], which is not equivalent to DC1 [36]. Here The situation is different. [13] show that DC1 and DC2 are always equivalent for operators on Banach spaces, but we will show that DC2 is not equivalent to mean Li-Yorke chaos.

As it show before no absolutely Cesàro bounded operator is mean Li-Yorke chaotic. We also establish a Mean Li-Yorke Chaos Criterion and show several examples, including an example of a DC1 (= DC2) operator which is not mean Li-Yorke chaotic. Finally, we show that the Frequent Hypercyclicity Criterion implies mean Li-Yorke chaos.

We also establish a Dense Mean Li-Yorke Criterion and several sufficient conditions for dense mean Li-Yorke chaos. As an application, we show an example of a densely mean Li-Yorke chaotic operator which is not Cesàro Hypercyclic.

We establish a sufficient condition for the existence of a dense absolutely mean irregular manifold. As an application, we obtain a dichotomy for unilateral weighted backward shifts $(B_r)_w$, which says that either $\frac{1}{N} \sum_{r,j=1}^N \|((B_r)_w)^j x^r\| \mapsto 0$ for all $x^r \in X$ or $(B_r)_w$ admits a dense absolutely mean irregular manifold. So, we give an example of a densely mean Li-Yorke chaotic operator which is not hypercyclic.

We give some characterizations for generic mean Li-Yorke chaos and construct an example of an invertible hypercyclic operator T such that both T and T^{-1} are completely absolutely mean irregular.

We study the mean Li-Yorke chaos for C_0 -semigroups of operators on Banach spaces.

II. Notations and Preliminaries

Let X be an arbitrary Banach space and $L(X)$ is the space of all bounded linear operators on X . Let $T \in L(X)$ be Li-Yorke chaotic if there exists an uncountable set $\Gamma \subset X$ such that for every pair $(x^r, y^r) \in \Gamma \times \Gamma$ of distinct points, we have

$$\liminf_{\epsilon \rightarrow \infty} \sum_r \|T^{(1+2\epsilon)} x^r - T^{(1+2\epsilon)} y^r\| = 0 \quad \text{and} \quad \limsup_{\epsilon \rightarrow \infty} \sum_r \|T^{(1+2\epsilon)} x^r - T^{(1+2\epsilon)} y^r\| > 0.$$

Γ is called a scrambled set for T and each such pair (x^r, y^r) is called a Li-Yorke pair for T .

The lower and the upper densities of a set $A_r \subset \mathbb{N}$ are defined as

$$\underline{\text{dens}}(A_r) := \liminf_{\epsilon \rightarrow \infty} \frac{\text{card}(A_r \cap [1, (1 + 2\epsilon)])}{(1 + 2\epsilon)} \quad \text{and} \quad \overline{\text{dens}}(A_r) := \limsup_{\epsilon \rightarrow \infty} \frac{\text{card}(A_r \cap [1, (1 + 2\epsilon)])}{(1 + 2\epsilon)},$$

respectively. Given $T \in L(X)$, $x^r, y^r \in X$ and $\delta > 0$, the lower and the upper distributional functions of x^r, y^r associated to T are defined by

$$F_{x^r, y^r}(\delta) := \underline{\text{dens}}(\{j \in \mathbb{N} : \|T^j x^r - T^j y^r\| < \delta\}),$$

$$F_{x^r, y^r}^*(\delta) := \overline{\text{dens}}(\{j \in \mathbb{N} : \|T^j x^r - T^j y^r\| < \delta\}),$$

respectively. If the pair (x^r, y^r) satisfies

(DC1) $F_{x^r, y^r}^* \equiv 1$ and $F_{x^r, y^r}(\epsilon) = 0$ for some $\epsilon > 0$, or

(DC2) $F_{x^r, y^r}^* \equiv 1$ and $F_{x^r, y^r}(\epsilon) < 1$ for some $\epsilon > 0$, or

(DC2 $_{\frac{1}{2}}$) There exist $c > 0$ and $\epsilon \geq 0$ such that $F_{x^r, y^r}(\delta) < c < F_{x^r, y^r}^*(\delta)$ for all $0 < \delta < 1 + \epsilon$, or

(DC3) $F_{x^r, y^r}(\delta) < F_{x^r, y^r}^*(\delta)$ for all δ in a nondegenerate interval $J \subset (0, \infty)$,

then (x^r, y^r) is called a distributionally chaotic pair of type $(1 + \epsilon) \in \{1, 2, 2\frac{1}{2}, 3\}$ for T . The operator T is said to be distributionally chaotic of type $(1 + \epsilon)$ (DC $(1 + \epsilon)$) if there exists an uncountable set $\Gamma \subset X$ such that every pair (x^r, y^r) of distinct points in Γ is a distributionally chaotic pair of type $(1 + \epsilon)$ for T . In this case, Γ is a distributionally scrambled set of type $(1 + \epsilon)$ for T . Distributional chaos of type 1 is often called simply distributional chaos.

For operators on Banach spaces, DC1 and DC2 are always equivalent [13], and imply Li-Yorke chaos. Li-Yorke chaos and distributional chaos for linear operators have been studied in [6,8,10,11,12,13,14,27,28,32,33,38,40,41,42], for instance.

We also recall that $T \in L(X)$ is frequently hypercyclic (FH), upper-frequently hypercyclic (UFH), reiteratively hypercyclic (RH) or hypercyclic (H) if there exists $x^r \in X$ such that for every nonempty open subset U of X , the set $\{(1+2\epsilon) \in \mathbb{N} : T^{(1+2\epsilon)}x^r \in U\}$ has positive lower density, has positive upper density, has positive upper Banach density or is nonempty, respectively. T is Cesàro hypercyclic if there exists $x^r \in X$ such that the sequence $\left(\frac{1}{(1+2\epsilon)} \sum_{r,j=0}^{2\epsilon} T^j x^r\right)_{(1+2\epsilon) \in \mathbb{N}}$ is dense in X . Moreover, T is mixing if for every nonempty open sets $U, V \subset X$, there exists $n_0 \in \mathbb{N}$ such that $T^{(1+2\epsilon)}(U) \cap V \neq \emptyset$ for all $(1+2\epsilon) \geq n_0$. T is weakly-mixing if $T \oplus T$ is hypercyclic, and T is Devaney chaotic if it is hypercyclic and has a dense set of periodic points. See [4,5,15,16,25,26], for instance.

Finally, the orbit of x^r is distributionally near to 0 if there is $A_r \subset \mathbb{N}$ with $\overline{\text{dens}}(A_r) = 1$ such that $\lim_{(1+2\epsilon) \in A_r} T^{(1+2\epsilon)}x^r = 0$. We say that x^r has a distributionally unbounded orbit if there is $B_r \subset \mathbb{N}$ with $\overline{\text{dens}}(B_r) = 1$ such that $\lim_{(1+2\epsilon) \in B_r} T^{(1+2\epsilon)}x^r = \infty$. If the orbit of x^r has both properties, then x^r is a distributionally irregular vector for T . It was proved in [11] that T is distributionally chaotic if and only if T has a distributionally irregular vector.

III. Mean Li-Yorke Chaotic Operators and Absolutely Mean Irregular Vectors

We begin with some useful characterizations of absolutely Cesàro bounded operators.

Theorem 4 (see [43]). For every $T \in L(X)$, the following assertions are equivalent:

- (i) T is not absolutely Cesàro bounded;
- (ii) There is a vector $x^r \in X$ such that

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = \infty;$$

- (iii) The set of all vectors $y^r \in X$ such that

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|T^j y^r\| = \infty$$

is residual in X .

Proof. The implications (iii) \Rightarrow (ii) \Rightarrow (i) are trivial.

(i) \Rightarrow (ii): Since T is not absolutely Cesàro bounded, given $\delta > 0$ and $\epsilon \geq 0$, there exist $x^r \in X$ and $N \in \mathbb{N}$ so that

$$\|x^r\| < \delta \text{ and } \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| > 1 + \epsilon.$$

Let us assume that (ii) is false, that is,

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| < \infty \text{ for all } x^r \in X.$$

Then, we can define inductively sequences $(x_{(1+2\epsilon)}^r)$ in X and $(N_{(1+2\epsilon)})$ in \mathbb{N} so that $\|x_{(1+2\epsilon)}^r\| < 2^{-(1+2\epsilon)}$ for all $(1+2\epsilon)$, and

$$\frac{1}{N_{1+\epsilon}} \sum_{r,j=1}^{N_{1+\epsilon}} \|T^j (x_1^r + \dots + x_{(1+2\epsilon)}^r)\| > 1 + \epsilon \text{ whenever } \epsilon \geq 0.$$

Let $x^r := \sum_{\epsilon=0}^{\infty} x_{(1+2\epsilon)}^r \in X$. Then

$$\frac{1}{N_{1+\epsilon}} \sum_{r,j=1}^{N_{1+\epsilon}} \|T^j x^r\| \geq 1 + \epsilon \text{ for all } (1+\epsilon) \in \mathbb{N}.$$

This is a contradiction, since we are assuming that (ii) is false.

(ii) \Rightarrow (iii): Let A_r denote the set considered in (ii). Since

$$A_r = \bigcap_{\epsilon=0}^{\infty} \bigcup_{N=1}^{\infty} \left\{ y^r \in X : \frac{1}{N} \sum_{r,j=1}^N \|T^j y^r\| \right\} > 1 + \epsilon,$$

we have that A_r is a G_δ set. If $z^r \in X \setminus A_r$ then

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{j=1}^N \|T^j z^r\| < \infty.$$

Let $x^r \in X$ be a vector given by (ii). Then

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{j=1}^N \|T^j(z^r + \lambda x^r)\| = \infty \quad \text{whenever } \lambda \neq 0$$

which implies that $z^r \in \overline{A_r}$. Thus, A_r is dense in X .

We show that mean Li-Yorke chaos is equivalent to the existence of an absolutely mean irregular vector.

Theorem 5 (see [43]). For every $T \in L(X)$, the following assertions are equivalent:

- (i) T is mean Li-Yorke chaotic;
- (ii) T has a mean Li-Yorke pair;
- (iii) T has an absolutely mean semi-irregular vector;
- (iv) T has an absolutely mean irregular vector;
- (v) The restriction of T to some infinite-dimensional closed T -invariant subspace Y has a residual set of absolutely mean irregular vectors.

Proof. The implications (v) \Rightarrow (iv) \Rightarrow (iii) are trivial.

(iii) \Rightarrow (ii): If x^r is an absolutely mean semi-irregular vector for T , it is clear that $(x^r, 0)$ is a mean Li-Yorke pair for T .

(ii) \Rightarrow (i): If (x^r, y^r) is a mean Li-Yorke pair for T , then $u^r := x^r - y^r$ is an absolutely mean semi-irregular vector for T . Hence, it follows easily that $\{\lambda u^r : \lambda \in \mathbb{K}\}$ is an uncountable mean Li-Yorke set for T .

(i) \Rightarrow (v): Let (x^r, y^r) be a mean Li-Yorke pair for T and put $u^r := x^r - y^r$. Let

$$Y := \overline{\text{span}}(\text{Orb}(u^r, T)),$$

which is an infinite-dimensional closed T -invariant subspace of X . Consider the operator $S \in L(Y)$ obtained by restricting T to Y . We claim that S is not absolutely Cesàro bounded. Indeed, suppose that this is not the case and let $\epsilon \geq 0$ be such that

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|S^j z^r\| \leq (1 + \epsilon) \sum_r \|z^r\| \quad \text{for all } z^r \in Y$$

Since (x^r, y^r) is a mean Li-Yorke pair for T , we have that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|S^j u^r\| = 0 \quad \text{and} \quad \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|S^j u^r\| > \epsilon,$$

for some $\epsilon > 0$. Let $\delta > 0$ be so small that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|S^j u^r\| > \delta + \epsilon.$$

There are $N \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\frac{1}{N} \sum_{r,j=1}^N \|S^j u^r\| < \delta \quad \text{and} \quad \frac{1}{N + \epsilon} \sum_{r,j=1}^{N+\epsilon} \|S^j u^r\| > \delta + \epsilon.$$

Let $K \in \{1, \dots, N\}$ be the largest integer such that $\|S^K u^r\| < \delta$. Then

$$\frac{1}{K} \sum_{r,j=1}^K \|S^j u^r\| < \delta.$$

Since

$$\delta + \epsilon < \frac{1}{N + \epsilon} \sum_{r,j=1}^{N+\epsilon} \|S^j u^r\|$$

$$\begin{aligned}
 &= \frac{K}{(N+\epsilon)^2} \sum_{r,j=1}^{N+\epsilon} \|S^j u^r\| + \frac{1}{N+\epsilon} \sum_{r,j=K+1}^{N+\epsilon} \|S^j u^r\| \\
 &< \delta + \frac{1}{(N+\epsilon)-K} \sum_{r,j=K+1}^{N+\epsilon} \|S^j u^r\|,
 \end{aligned}$$

we have that

$$\epsilon < \frac{1}{(N+\epsilon)-K} \sum_{r,j=K+1}^{N+\epsilon} \|S^j u^r\| = \frac{1}{(N+\epsilon)-K} \sum_{r,j=K+1}^{(N+\epsilon)-K} \|S^j S^K u^r\| \leq (1+\epsilon) \sum_r \|S^K u^r\| < (1+\epsilon)\delta.$$

Hence, $\epsilon > \frac{\epsilon}{\delta} - 1$. Since $\delta > 0$ can be chosen arbitrarily close to 0, we have a contradiction. This proves that S is not absolutely Cesàro bounded. Therefore, the set

$$A_r := \left\{ z^r \in Y : \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|S^j z^r\| = \infty \right\}$$

is residual in Y by Theorem 4. On the other hand, let

$$B_r := \left\{ z^r \in Y : \inf_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|S^j z^r\| = 0 \right\}$$

Since

$$B_r = \bigcap_{\epsilon=0}^{\infty} \bigcup_{N=1}^{\infty} \left\{ z^r \in Y : \frac{1}{N} \sum_{r,j=1}^N \|S^j z^r\| < \frac{1}{1+\epsilon} \right\},$$

we have that B_r is a G_δ set. Since there is an increasing sequence $(N_{1+\epsilon})$ in \mathbb{N} such that

$$\lim_{\epsilon \rightarrow \infty} \frac{1}{N_{1+\epsilon}} \sum_{r,j=1}^{N_{1+\epsilon}} \|S^j u^r\| = 0,$$

it follows that $\text{span}(\text{Orb}(u^r, S))$ is contained in B_r . Hence, B_r is a residual set in Y . Our conclusion is that $A_r \cap B_r$ is a residual set in Y , which is formed by absolutely mean irregular vectors for S (hence for T).

Corollary 6. No absolutely Cesàro bounded operator on a Banach space is mean Li-Yorke chaotic.

The proof of Theorem 5 clearly implies the following result:

Theorem 7. For every $T \in L(X)$, the set of all absolutely mean irregular vectors for T is dense in the set of all absolutely mean semi-irregular vectors for T .

Definition 8. We say that $T \in L(X)$ satisfies the Mean Li-Yorke Chaos Criterion (MLYCC) if there exists a subset X_0 of X with the following properties:

- (a) $\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j u^r\| = 0$ for every $x^r \in X_0$;
- (b) there are sequences $(y_{1+\epsilon}^r)$ in $\overline{\text{span}(X_0)}$ and $(N_{1+\epsilon})$ in \mathbb{N} such that

$$\frac{1}{N_{1+\epsilon}} \sum_{r,j=1}^{N_{1+\epsilon}} \|T^j y_{1+\epsilon}^r\| > \sum_r \|y_{1+\epsilon}^r\| \quad \text{for every } (1+\epsilon) \in \mathbb{N}$$

We show that this criterion characterizes mean Li-Yorke chaos.

Theorem 9 (see [43]). An operator $T \in L(X)$ is mean Li-Yorke chaotic if and only if it satisfies the MLYCC.

Proof. (\Rightarrow): By Theorem 5, T has an absolutely mean irregular vector x^r . So, it is enough to consider $X_0 := \{x^r\}$.

(\Leftarrow): If T has an absolutely mean semi-irregular vector, then T is mean Li-Yorke chaotic by Theorem 5. So, let us assume that this is not the case. Then, (a) implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \quad \text{for every } x^r \in X_0.$$

Let

$$Y := \overline{\left\{ x^r \in X : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}},$$

which is a closed T -invariant subspace of X . Consider the operator $S \in L(Y)$ obtained by restricting T to Y . Since the sequence $(y_{1+\epsilon}^r)$ lies in Y , $(1 + \epsilon)$ implies that S is not absolutely Cesàro bounded. Hence, by Theorem 4, the set

$$A_r := \left\{ y^r \in Y : \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|S^j y^r\| = \infty \right\}$$

is residual in Y . On the other hand, since the set

$$B_r := \left\{ y^r \in Y : \inf_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|S^j y^r\| = 0 \right\}$$

is obviously dense in Y , it is residual in Y . Thus, $A_r \cap B_r$ is residual in Y , which proves the existence of an absolutely mean irregular vector for T . This contradicts our assumption that there are no absolutely mean semi-irregular vector for T .

Example 10 [43]. There are mixing operators that are not mean Li-Yorke chaotic. Indeed, in [13] it was given some examples of mixing operators that are absolutely Cesàro bounded. One of them was taken from [33], and appears in the PhD Thesis of Beltrán Meneu with a proof provided by the third author (Theorem 3.7.3 in [7]). By Corollary 6, these operators are not mean Li-Yorke chaotic.

Example 11 [43]. There are Devaney chaotic operators that are not mean Li-Yorke chaotic. Indeed, let T be the Devaney chaotic operator constructed by [34]. Suppose that $x^r \neq 0$ and

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0.$$

For every $\delta > 0$, since

$$\frac{\text{card}(\{1 \leq j \leq N : \|T^j x^r\| \geq \delta\})}{N} \leq \frac{1}{N} \sum_{r,j=1}^N \frac{\|T^j x^r\|}{\delta} = \frac{1}{\delta} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\|,$$

we have that

$$\underline{\text{dens}} \left(\left\{ \epsilon \geq -\frac{1}{2} : \|T^{1+2\epsilon} x^r\| \geq \delta \right\} \right) = 0.$$

This contradicts Claim 10 in [34]. Hence, T is not mean Li-Yorke chaotic.

Example 12 [43]. There are frequently hypercyclic operators that are not mean Li-Yorke chaotic.

Indeed, [6] proved that there exists a frequently hypercyclic operator (thus this operator is DC2 $\frac{1}{2}$ by [13]) such that the orbit of no $x^r \neq 0$ is distributionally near of 0. Hence, it is not DC1 and not mean Li-Yorke chaotic [13].

Example 13 [43]. There are distributionally chaotic operators that are not mean Li-Yorke chaotic.

We recall the following result that was obtained in [13]: If $X = c_0(\mathbb{N})$ ($X = c_0(\mathbb{Z})$) or $X = \ell^{1+\epsilon}(\mathbb{N})$ ($X = \ell^{1+\epsilon}(\mathbb{Z})$) for some $0 \leq \epsilon \leq \infty$, then there exists an (invertible) operator $T \in L(X)$ which is distributionally chaotic and satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = \infty \quad \text{for all } x^r \in X \setminus \{0\}.$$

Since these operators do not have an absolutely mean irregular vector, Theorem 5 guarantees that they are not mean Li-Yorke chaotic. Since DC1 and DC2 are equivalent for operators on Banach spaces [13], this implies that the notions of mean Li-Yorke chaos and DC2 are not equivalent for operators on Banach spaces, contrary to what happens in the context of topological dynamics on compact metric spaces [20].

Remark 14. The above examples are Li-Yorke chaotic, but not mean Li-Yorke chaotic.

We recall that an operator T on a separable Banach space X is said to satisfy the Frequent Hypercyclicity Criterion (FHC) if there exist a dense subset X_0 of X and a map $S : X_0 \rightarrow X_0$ such that, for any $x^r \in X_0$,

- $\sum_{r,1+2\epsilon=0}^{\infty} T^{1+2\epsilon} x^r$ converges unconditionally,
- $\sum_{r,1+2\epsilon=0}^{\infty} S^{1+2\epsilon} x^r$ converges unconditionally,
- $TSx^r = x^r$.

If T satisfies this criterion, then T is frequently hypercyclic, Devaney chaotic, mixing and distributionally chaotic [11],[16]. Since we have just seen that there are examples of frequently hypercyclic operators, Devaney chaotic operators, mixing operators and distributionally chaotic operators that are not mean Li-Yorke chaotic, it is natural to ask if the Frequent Hypercyclicity Criterion implies mean Li-Yorke chaos. We now see that the answer is yes.

Proposition 15 (see [43]). If $T \in L(X)$ satisfies the Frequent Hypercyclicity Criterion, then T has a residual set of absolutely mean irregular vectors.

Proof. If T satisfies the Frequent Hypercyclicity Criterion, then the set

$$\{x^r \in X : \lim_{\epsilon \rightarrow \infty} \|T^{1+2\epsilon} x^r\| = 0\}$$

is dense in X and T is distributionally chaotic. Hence, we can apply [13].

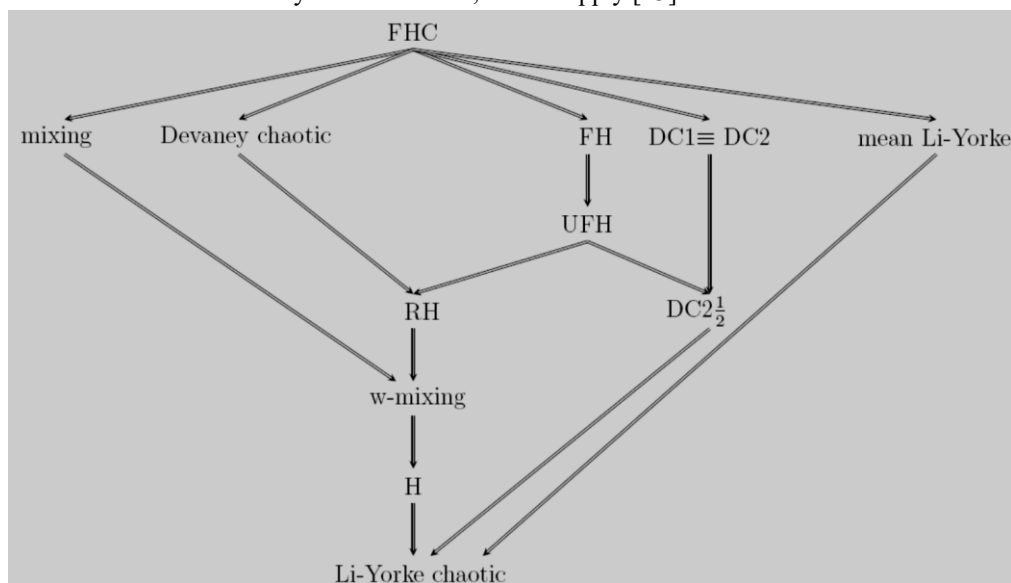


Figure 1 (see [43]). Implications between different definitions related with hypercyclicity and chaos for operators on Banach spaces.

Concerning Figure 1, it is easy to construct a mean Li-Yorke chaotic operator which is not hypercyclic. Indeed, let $T \in L(X)$ be any mean Li-Yorke chaotic operator and consider the operator $T \oplus I \text{ on } X \oplus X$, where I denotes the identity operator on X . However, the following question remains open.

Question 16. Is there a Banach (or Hilbert) space operator which is mean Li-Yorke chaotic but is not distributionally chaotic?

Related to Question 16, we have the following result from [13]: If $X = c_0(\mathbb{N})$ ($X = c_0(\mathbb{Z})$) or $X = \ell^{1+\epsilon}(\mathbb{N})$ ($X = \ell^{1+\epsilon}(\mathbb{Z})$) for some $0 \leq \epsilon \leq \infty$, then there exists an (invertible) operator $T \in L(X)$ which admits an absolutely mean irregular vector whose orbit is not distributionally unbounded.

In particular, this shows that an absolutely mean irregular vector is not necessarily distributionally irregular. However, we don't know if the operator admits other vectors with distributionally unbounded orbit, or if it has a distributionally irregular vector.

IV. Densely Mean Li-Yorke Chaotic Operators

We show that, in separable spaces, dense mean Li-Yorke chaos is equivalent to the existence of a residual set of absolutely mean irregular vectors.

Theorem 17 (see [43]). Assume X separable. For every $T \in L(X)$, the following assertions are equivalent

- T is densely mean Li-Yorke chaotic;
- T has a dense set of mean Li-Yorke pairs;

- (iii) T has a residual set of mean Li-Yorke pairs;
- (iv) T has a dense set of absolutely mean semi-irregular vectors;
- (v) T has a dense set of absolutely mean irregular vectors;
- (vi) T has a residual set of absolutely mean irregular vectors.

Proof. (ii) \Leftrightarrow (iv): It follows easily from the fact that (x^r, y^r) is a mean Li-Yorke pair for T if and only if $x^r - y^r$ is an absolutely mean semi-irregular vector for T .

(iv) \Leftrightarrow (v): It follows from Theorem 7.

(i) \Leftrightarrow (vi): The set R of all absolutely mean irregular vectors for T is the intersection of the sets

$$A_r := \left\{ x^r \in X : \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = \infty \right\} \text{ and } B_r := \left\{ x^r \in X : \inf_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}.$$

It follows from Theorem 4 that A_r is residual in X whenever it is nonempty. And we saw in the proof of Theorem 5 that B_r is residual in X whenever it is dense in X . Thus, R is residual in X whenever it is dense in X .

(vi) \Rightarrow (iii): If (U_j) is a sequence of dense open sets in X such that every vector in $\cap U_j$ is absolutely mean irregular for T , then the sets $V_j := \{(x^r, y^r) \in X \times X : x^r - y^r \in U_j\}$ are open and dense in $X \times X$, and every point in $\cap V_j$ is a mean Li-Yorke pair for T .

The implications (iii) \Rightarrow (ii) and (i) \Rightarrow (ii) are obvious.

(vi) \Rightarrow (i): Let R be the set of all absolutely mean irregular vectors for T and let (y_j^r) be a dense sequence in X . Let $D := \mathbb{Q}$ or $\mathbb{Q} + i\mathbb{Q}$, depending on whether the scalar field \mathbb{K} is \mathbb{R} or \mathbb{C} , respectively. Since R is residual in X , we can choose inductively linearly independent vectors $x_1^r, x_2^r, x_3^r, \dots \in X$ such that $x_1^r \in B_r(y_1^r; 1) \cap R$ and

$$x_{2(1+\epsilon)}^r \in B_r\left(y_{2(1+\epsilon)}^r; \frac{1}{2(1+\epsilon)}\right) \bigcap_{(\alpha_1, \dots, \alpha_{1+2\epsilon}) \in D^{1+2\epsilon}} (\alpha_1 x_1^r + \dots + \alpha_{1+2\epsilon} x_{1+2\epsilon}^r + R) (1+2\epsilon) \in \mathbb{N}$$

Hence,

$$N + \epsilon := \{\alpha_1 x_1^r + \dots + \alpha_{1+\epsilon} x_{1+\epsilon}^r : \epsilon \geq 0 \text{ and } \alpha_1, \dots, \alpha_{1+\epsilon} \in D\}$$

is a dense D -vector subspace of X consisting (up to 0) of absolutely mean irregular vectors for T . In particular, $N + \epsilon$ is a dense mean Li-Yorke set for T . Since $N + \epsilon$ is countable, we need to enlarge $N + \epsilon$ in order to obtain an uncountable dense mean Li-Yorke set for T . Let

$$N := \{\alpha_2 x_2^r + \dots + \alpha_{1+\epsilon} x_{1+\epsilon}^r : \epsilon \geq 0 \text{ and } \alpha_2, \dots, \alpha_{1+\epsilon} \in D\}$$

For each $y^r \in N \setminus \{0\}$, let $(A_r)_{y^r} := \{\lambda \in \mathbb{K} : y^r - \lambda x_1^r \text{ is absolutely mean irregular for } T\}$. Since $(A_r)_{y^r}$ is a G_δ set in \mathbb{K} containing D , $(A_r)_{y^r}$ is residual in \mathbb{K} . Thus, $A_r := \bigcap_{y^r \in N \setminus \{0\}} (A_r)_{y^r}$ is also a residual set in \mathbb{K} containing D . By Zorn's Lemma, there is a maximal D -vector subspace H of \mathbb{K} such that $D \subset H \subset A_r$. If H were countable, then

$$B_r := \bigcap_{\beta \in D \setminus \{0\}} \bigcap_{\alpha \in H} \beta(\alpha + A_r)$$

would be residual in \mathbb{K} . By choosing $\gamma \in B_r \setminus H$, we would have that $H' := H + \{\beta \gamma : \beta \in D\}$ is a D -vector subspace of \mathbb{K} satisfying $D \subset H' \subset A_r$ and $H \subsetneq H'$, which would contradict the maximality of H . Thus, $N + \epsilon' := \{\alpha x_1^r : \alpha \in H\} + N$ is the uncountable mean Li-Yorke set for T we were looking for.

Remark 18. Note that in the previous theorem the separability of X was used only in the proof that (vi) \Rightarrow (i). The equivalences

$$(ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$$

are valid for any Banach space.

In view of Theorem 5, it is true that an operator $T \in L(X)$ is mean Li-Yorke chaotic if and only if there is a mean Li-Yorke set for T which is a one-dimensional subspace of X . We shall now show that the chaotic behaviour always occurs in a much larger subspace of X .

Theorem 19 (see [43]). Every mean Li-Yorke chaotic operator $T \in L(X)$ is densely mean Li-Yorke chaotic on some infinite-dimensional closed T -invariant subspace of X .

Proof. By Theorem 5, the restriction of T to a certain infinite-dimensional closed T -invariant subspace Y of X has a residual set of absolutely mean irregular vectors. The proof of Theorem 5 actually constructs a separable such Y . Hence, we can apply Theorem 17 and conclude that T is densely mean Li-Yorke chaotic on Y .

Definition 20. We say that $T \in L(X)$ satisfies the Dense Mean Li-Yorke Chaos Criterion (DMLYCC) if there exists a dense subset X_0 of X with properties (a) and (b) of Definition 8.

Theorem 21 (see [43]). Assume X separable. An operator $T \in L(X)$ is densely mean Li-Yorke chaotic if and only if it satisfies the DMLYCC.

Proof. (\Rightarrow): By Theorem 17, T has a dense set X_0 of absolutely mean irregular vectors. Clearly, X_0 satisfies properties (a) and (b) of Definition 8.

(\Leftarrow): Since X_0 is dense in X , condition (a) implies that the set

$$B_r := \left\{ x^r \in X : \inf_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}$$

is residual in X . By (b), T is not absolutely Cesàro bounded. Hence, by Theorem 4, the set

$$A_r := \left\{ x^r \in X : \inf_{N \in \mathbb{N}} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = \infty \right\}$$

is also residual in X . Thus, T has a residual set of absolutely mean irregular vectors. By Theorem 17, we conclude that T is densely mean Li-Yorke chaotic.

Theorem 22 (see [43]). If $T \in L(X)$ and

$$X_0 := \left\{ x^r \in X : \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}$$

is dense in X , then the following assertions are equivalent:

- (i) T is mean Li-Yorke chaotic;
- (ii) T has a residual set of absolutely mean irregular vectors;
- (iii) T is not absolutely Cesàro bounded;
- (iv) $\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j y_0^r\| > 0$ for some $y_0^r \in X$

Proof. (i) \Rightarrow (iii): It follows from Corollary 6.

(iii) \Rightarrow (ii): Since X_0 is dense in X by hypothesis, it is residual in X . Hence, (ii) follows from Theorem 4.

The implication (ii) \Rightarrow (i) follows from Theorem 5.

The implication (i) \Rightarrow (iv) is trivial.

(iv) \Rightarrow (i): Suppose that T is not mean Li-Yorke chaotic. By Theorem 5, there is no absolutely mean semi-irregular vector for T , and so

$$X_0 := \left\{ x^r \in X : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}.$$

Thus, X_0 is a residual subspace of X , which implies that $X_0 = X$ and contradicts (iv).

Remark 23. In the case that the space X is separable, Theorem 17 shows that condition (ii) in the above theorem can be replaced by

(ii') T is densely mean Li-Yorke chaotic.

As an immediate consequence of the previous theorem, we have the following dichotomy for unilateral weighted backward shifts on Banach sequence spaces.

Corollary 24 [43]. Let X be a Banach sequence space in which $(e_{1+2\epsilon})_{(1+2\epsilon) \in \mathbb{N}}$ is a basis [26]. Suppose that the unilateral weighted backward shift

$$(B_r)_w(x_1^r, x_2^r, x_3^r, \dots) := (w_2 x_2^r, w_3 x_3^r, w_4 x_4^r, \dots)$$

is an operator on X . Then either

- (a) $(B_r)_w$ is mean Li-Yorke chaotic, or
- (b) $(B_r)_w$ is absolutely Cesàro bounded.

If $\{x^r : T^{1+2\epsilon} x^r \rightarrow 0\}$ is a dense set in X , then:

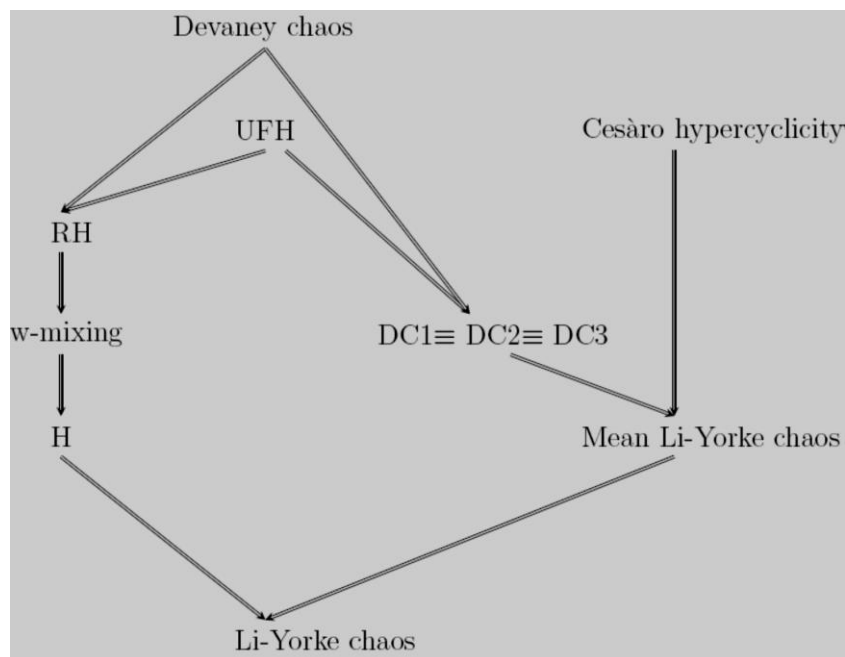


Figure 2 (see [43]). Implications between different definitions related with hypercyclicity and chaos for operators on Banach spaces when $\{x^r : T^{1+2\epsilon}x^r \rightarrow 0\}$ is a dense set in X .

Another consequence of Theorem 22 is given in the following corollary. Items (a) and (b) improve Theorems 27 and 28 of [13], respectively.

Corollary 25 [43]. Let $T \in L(X)$ be such that

$$\left\{ x^r \in X : \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}$$

is dense in X . If any of the following conditions is true:

- (a) T is distributionally chaotic,
- (b) T is Cesàro hypercyclic,
- (c) X is a Banach space and $\limsup_{\epsilon \rightarrow \infty} \frac{\|T^{1+2\epsilon}\|}{1+2\epsilon} > 0$,
- (d) X is a Hilbert space and $\limsup_{\epsilon \rightarrow \infty} \frac{\|T^{1+2\epsilon}\|}{(1+2\epsilon)^{\frac{1}{2}}} > 0$,
- (e) T has an eigenvalue λ with $|\lambda| \geq 1$,

then there is a residual set of absolutely mean irregular vectors for T . If, in addition, X is separable, then T is densely mean Li-Yorke chaotic.

Proof. Any of these conditions implies that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j y_0^r\| > 0 \text{ for some } y_0^r \in X$$

(for items (c) and (d) this follows from Theorem 2.4 and Corollary 2.6 of [9]). Thus, the result follows from Theorem 22.

Example 26 [43]. There are densely mean Li-Yorke chaotic operators that are not Cesàro hypercyclic.

Indeed, let T be the weighted backward shift on $\ell^1(\mathbb{N})$ defined by

$$Te_1 = 0 \text{ and } Te_{1+\epsilon} = \left(\frac{1+\epsilon}{\epsilon}\right) e_\epsilon \text{ for } \epsilon > 0.$$

Since $\|T^{1+2\epsilon}\| = 2(1+\epsilon)$ for all $(1+2\epsilon) \in \mathbb{N}$, Corollary 25(c) implies that T is densely mean Li-Yorke chaotic. Moreover, the equalities $\|T^{1+2\epsilon}\| = 2(1+\epsilon)$, $(1+2\epsilon) \in \mathbb{N}$ also imply that $\left(\frac{T^{1+2\epsilon}x^r}{1+2\epsilon}\right)$ is not dense in $\ell^1(\mathbb{N})$, for every $x^r \in \ell^1(\mathbb{N})$. Thus, T is not Cesàro hypercyclic [30].

Question 27. Is the operator defined in the above example distributionally chaotic?

5 Dense Lineability of Absolutely Mean Irregular Vectors

Definition 28. An absolutely mean irregular manifold for $T \in L(X)$ is a vector subspace Y of X such that every nonzero vector in Y is absolutely mean irregular for T .

Such a manifold is clearly a mean Li-Yorke set for T . The following dichotomy gives us a sufficient condition for the existence of a dense absolutely mean irregular manifold.

Theorem 29 (see [43]). Assume X separable. If $T \in L(X)$ and

$$X_0 := \left\{ x^r \in X : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}$$

is dense in X , then either

- (a) $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0$ for every $x^r \in X$, or
- (b) T admits a dense absolutely mean irregular manifold.

Proof. Suppose that (a) is false and let us prove (b). By Theorem 22, T has an absolutely mean irregular vector. Let $\epsilon := \|T\| > 0$. Then, we can construct a sequence (x_m^r) of normalized vectors in X_0 and an increasing sequence (N_m) of positive integers so that

$$\frac{1}{N_m} \sum_{r,i=1}^{N_m} \|T^i x_m^r\| > m(2(1+\epsilon))^m \text{ and } \frac{1}{N_m} \sum_{r,i=1}^{N_m} \|T^i x_k^r\| < \frac{1}{m} \text{ for } k = 1, \dots, m-1.$$

Given $\alpha, \beta \in \{0, 1\}^{\mathbb{N}}$, we say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i \in \mathbb{N}$. Let (r_j) be a sequence of positive integers such that $r_{j+1} \geq 1 + r_j + N_{r_j+1}$ for all $j \in \mathbb{N}$. Let $\alpha \in \{0, 1\}^{\mathbb{N}}$ be defined by $\alpha_{1+2\epsilon} = 1$ if and only if $1 + 2\epsilon = r_j$ for some $j \in \mathbb{N}$. For each $\beta \in \{0, 1\}^{\mathbb{N}}$ such that $\beta \leq \alpha$ and β contains an infinite number of 1's, we define

$$x_\beta^r := \sum_{r,i} \frac{\beta_i}{(2(1+\epsilon))^i} x_i^r = \sum_{r,j} \frac{\beta_{r_j}}{(2(1+\epsilon))^{r_j}} x_{r_j}^r.$$

Take $(1+\epsilon) \in N$ with $\beta_{r_{1+\epsilon}} = 1$. Since $\frac{1}{N_{r_{1+\epsilon}}} \sum_{r,i=1}^{N_{r_{1+\epsilon}}} \|T^i x_{r_{1+\epsilon}}^r\| > r_{1+\epsilon} (2(1+\epsilon))^{r_{1+\epsilon}}$ and $\frac{1}{N_{r_{1+\epsilon}}} \sum_{r,i=1}^{N_{r_{1+\epsilon}}} \|T^i x_{1+\epsilon}^r\| < \frac{1}{r_{1+\epsilon}}$ for each $1+\epsilon < r_{1+\epsilon}$, we have that

$$\begin{aligned} \frac{1}{N_{r_{1+\epsilon}}} \sum_{r,i=1}^{N_{r_{1+\epsilon}}} \|T^i x_\beta^r\| &\geq \frac{1}{(2(1+\epsilon))^{r_{1+\epsilon}}} \frac{1}{N_{r_{1+\epsilon}}} \sum_{r,i=1}^{N_{r_{1+\epsilon}}} \|T^i x_{r_{1+\epsilon}}^r\| - \frac{1}{N_{r_{1+\epsilon}}} \sum_{r,i=1}^{N_{r_{1+\epsilon}}} \sum_{r,j \neq 1+\epsilon} \frac{\beta_{r_j}}{(2(1+\epsilon))^{r_j}} \|T^i x_{r_j}^r\| \\ &> r_{1+\epsilon} - \frac{1}{r_{1+\epsilon}} \sum_{r,j < 1+\epsilon} \frac{1}{(2(1+\epsilon))^{r_j}} - \sum_{r,j > 1+\epsilon} \frac{\|x_{r_j}^r\|}{2^{r_j}} \\ &\geq r_{1+\epsilon} - 1. \end{aligned}$$

On the other hand, since $\frac{1}{N_{r_{1+\epsilon}+1}} \sum_{r,i=1}^{N_{r_{1+\epsilon}+1}} \|T^i x_{1+\epsilon}^r\| < \frac{1}{r_{1+\epsilon}+1}$ for each $\epsilon < r_{1+\epsilon}$, then

$$\begin{aligned} \frac{1}{N_{r_{1+\epsilon}+1}} \sum_{r,i=1}^{N_{r_{1+\epsilon}+1}} \|T^i x_\beta^r\| &\leq \frac{1}{N_{r_{1+\epsilon}+1}} \sum_{r,i=1}^{N_{r_{1+\epsilon}+1}} \sum_{j \leq 1+\epsilon} \frac{\beta_{r_j} \|T^i x_{r_j}^r\|}{(2c)^{r_j}} + \frac{1}{N_{r_{1+\epsilon}+1}} \sum_{r,i=1}^{N_{r_{1+\epsilon}+1}} \sum_{j > 1+\epsilon} \frac{\beta_{r_j} \|T^i x_{r_j}^r\|}{(2c)^{r_j}} \\ &\leq \frac{1}{r_{1+\epsilon}+1} \sum_{j \leq 1+\epsilon} \frac{1}{(2c)^{r_j}} + \frac{1}{r_{1+\epsilon}+1} \sum_{r,i=1}^{N_{r_{1+\epsilon}+1}} \sum_{j > 1+\epsilon} \frac{\|x_{r_j}^r\|}{2^{r_j}} \\ &< \frac{1}{r_{1+\epsilon}+1}. \end{aligned}$$

Thus, x_β^r is an absolutely mean irregular for T .

Now, let $(w_{1+2\epsilon})$ be a dense sequence in X_0 and choose $\gamma_{1+2\epsilon} \in \{0, 1\}^{\mathbb{N}}$, $(1+2\epsilon) \in \mathbb{N}$ such that each $\gamma_{1+2\epsilon}$ contains an infinite number of 1's, $\gamma_{1+2\epsilon} \leq \alpha$ for every $(1+2\epsilon) \in \mathbb{N}$, and the sequences $\gamma_{1+2\epsilon}$ have mutually disjoint supports. Define $v_{1+2\epsilon} := \sum_{r,i} \frac{\gamma_{1+2\epsilon,i}}{(2(1+\epsilon))^i} x_i^r$ and $y_{1+2\epsilon}^r := w_{1+2\epsilon} + \frac{1}{1+2\epsilon} v_{1+2\epsilon}$, $(1+2\epsilon) \in \mathbb{N}$.

Then $Y := \text{span}\{y_{1+2\epsilon}^r : (1+2\epsilon) \in \mathbb{N}\}$ is a dense subspace of X . Moreover, if $y^r \in Y \setminus \{0\}$, then we can write $y^r = w_0 + \sum_{r,1+\epsilon} \frac{\rho_{1+\epsilon}}{(2(1+\epsilon))^{1+\epsilon}} x_{1+\epsilon}^r$, where $w_0 \in X_0$ and the sequence of scalars $(\rho_{1+\epsilon})$ takes only a finite number of values (each of them infinitely many times). As in the above proof we can show that the vector $v := \sum_{r,1+\epsilon} \frac{\rho_{1+\epsilon}}{(2(1+\epsilon))^{1+\epsilon}} x_{1+\epsilon}^r$ is absolutely mean irregular for T . Since $y^r = w_0 + v$ and $w_0 \in X_0$, we conclude that y^r is also absolutely mean irregular for T .

Here is an application of the previous theorem.

Corollary 30 [43]. If $(B_r)_w$ is a unilateral weighted backward shift on a Banach sequence space X in which $(e_{1+2\epsilon})_{(1+2\epsilon) \in \mathbb{N}}$ is a basis, then either

- (a) $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|((B_r)_w)^j x^r\| = 0$ for every $x^r \in X$, or
- (b) $(B_r)_w$ admits a dense absolutely mean irregular manifold.

Example 31 [43]. There are densely mean Li-Yorke chaotic operators that are not hypercyclic.

Indeed, let $X = c_0(\mathbb{N})$ or $X = \ell^{1+\epsilon}(\mathbb{N})$ for some $0 \leq \epsilon < \infty$. Consider the unilateral weighted backward shift $(B_r)_w : X \rightarrow X$ whose weight sequence is given by

$$w := \left(\frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2}, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, \dots \right)$$

with successive blocks of $\frac{1}{2}$'s and 2's. Since $\sup_{1+2\epsilon \in \mathbb{N}} \prod_{j=1}^{1+2\epsilon} |w_j| = 1 < \infty$, $(B_r)_w$ is not hypercyclic [26]. On the other hand, if we define $x^r \in X$ by putting $\frac{1}{2^{1+2\epsilon}}$ in the position of the last 2 in the $(1+2\epsilon)^{\text{th}}$ block of 2's, for each $(1+2\epsilon) \in \mathbb{N}$, and 0 otherwise, then

$$\|((B_r)_w)^{1+2\epsilon} x^r\| \geq 1 \text{ for all } (1+2\epsilon) \in \mathbb{N}.$$

Hence, Corollary 30 guarantees that $(B_r)_w$ is densely mean Li-Yorke chaotic.

Remark 32. In Corollary 25, if X is separable and we assume the stronger property that

$$\left\{ x^r \in X : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|T^j x^r\| = 0 \right\}$$

is dense in X , then we can conclude that T has a dense absolutely mean irregular manifold.

6 Generically Mean Li-Yorke Chaotic Operators

We have the following characterizations of generic mean Li-Yorke chaos.

Theorem 33 [43]. For every $T \in L(X)$, the following assertions are equivalent:

- (i) T is generically mean Li-Yorke chaotic;
- (ii) Every non-zero vector is absolutely mean semi-irregular for T ;
- (iii) X is a mean Li-Yorke set for T .

The proof is analogous to that of [12].

Definition 34. We say that an operator $T \in L(X)$ is completely absolutely mean irregular if every vector $x^r \in X \setminus \{0\}$ is absolutely mean irregular for T .

Thus, every completely absolutely mean irregular operator is generically mean LiYorke chaotic. The converse is not true in general (see Remark 40). Our next goal is to construct an invertible hypercyclic operator T such that both T and T^{-1} are completely absolutely mean irregular. The construction is a modification of the type of examples of completely distributionally irregular operators provided in [33]. We first recall one of the main results in [33].

Theorem 35.[33] Let $v = (v_j)_{j \in \mathbb{Z}}$ be a weight sequence that satisfies the following conditions:

- (a) there are sequences of integers $(n_j)_{j \in \mathbb{Z}}$ and $(m_j)_{j \in \mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$, and $\epsilon > 0$ such that $(1+\epsilon)v_{m_{-(1+\epsilon)}} \geq v_j$ for every $j \in [m_{-(1+\epsilon)}, m_\epsilon]$, $(1+\epsilon) \in \mathbb{N}$, and if we consider

$$S_{1+\epsilon} := \sup \left\{ \frac{v_j}{v_{j-1}} ; j \notin [(1+\epsilon)_{-(1+\epsilon)}, (1+\epsilon)_\epsilon] \right\}, \quad (1+\epsilon) \in \mathbb{N},$$

then for every $\epsilon > 0$ we find $(1+\epsilon) \in \mathbb{N}$ with $v_{(1+2\epsilon)_{1+\epsilon}} < \epsilon$ and

$$S_{1+\epsilon}^{(1+\epsilon)((1+2\epsilon)_{1+\epsilon} - (1+\epsilon)_{-(1+\epsilon)})} \leq \min \left\{ 1 + \epsilon, \frac{\min\{v_i ; (1+\epsilon)_{-(1+\epsilon)} \leq i \leq (1+\epsilon)_\epsilon\}}{v_{(1+2\epsilon)_{1+\epsilon}}} \right\},$$

(b) for every $N \in \mathbb{N}$, there exists $(1 + \epsilon) \in \mathbb{N}$ such that $v_j > N$, for $1 + \epsilon \leq j \leq N_{1+\epsilon}$.

Then the forward shift $T : \ell^{1+\epsilon}(v, \mathbb{Z}) \rightarrow \ell^{1+\epsilon}(v, \mathbb{Z})$ is completely distributionally irregular. Actually, it was shown that, given an arbitrary non-zero vector $x^r \in \ell^{1+\epsilon}(v, \mathbb{Z})$, and an arbitrary $\delta > 0$, there is $(1 + \epsilon) \in \mathbb{N}$ as big as we want such that

$$(I1) \quad \|B_r^l x^r\|^{1+\epsilon} < \delta \text{ for any } l \in [(1 + 2\epsilon)_{1+\epsilon} - (1 + \epsilon)_{-(1+\epsilon)}, (1 + \epsilon)((1 + 2\epsilon)_{1+\epsilon} - (1 + \epsilon)_{-(1+\epsilon)})].$$

The type of examples that we will consider involve the inverse too, and we also need to recall the following result:

Corollary 36. [33] Let $v = (v_j)_{j \in \mathbb{Z}}$ be a weight sequence that satisfies the following conditions:

(a) there are sequences of integers $(n_j)_{j \in \mathbb{Z}}$ and $(m_j)_{j \in \mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$, and $\epsilon > 0$ such that $(1 + \epsilon)v_{m_{(1+\epsilon)}} \geq v_j$ for every $j \in [m_{-(1+\epsilon)}, m_{1+\epsilon}]$, $(1 + \epsilon) \in \mathbb{N}$, and if we consider

$$S_{1+\epsilon} := \inf \left\{ \frac{v_j}{v_{j-1}} ; j \notin [m_{-(1+\epsilon)}, m_{1+\epsilon}] \right\}, \quad (1 + \epsilon) \in \mathbb{N},$$

then for every $\epsilon > 0$ we find $(1 + \epsilon) \in \mathbb{N}$ with $v_{(1+2\epsilon)_{-(1+\epsilon)}} < \epsilon$ and

$$S_{1+\epsilon}^{(1+\epsilon)((1+2\epsilon)_{1+\epsilon} - (1+\epsilon)_{-(1+\epsilon)})} \leq \min \left\{ 1 + \epsilon, \frac{\min\{v_i ; (1 + \epsilon)_{-(1+\epsilon)} \leq i \leq (1 + \epsilon)_{1+\epsilon}\}}{v_{(1+2\epsilon)_{-(1+\epsilon)}}} \right\},$$

(b) for every $N \in \mathbb{N}$, there exists $(1 + \epsilon) \in \mathbb{N}$ such that $v_j > N$, for $-(1 + \epsilon) \leq j \leq -N(1 + \epsilon)$. Then the backward shift $B_r = T^{-1} : \ell^{1+\epsilon}(v, \mathbb{Z}) \rightarrow \ell^{1+\epsilon}(v, \mathbb{Z})$ is completely distributionally irregular.

In this case, it turns out that, given an arbitrary non-zero vector $x^r \in \ell^{1+\epsilon}(v, \mathbb{Z})$, and an arbitrary $\delta > 0$, there is $(1 + \epsilon) \in \mathbb{N}$ as big as we want such that

$$\|B_r^l x^r\|^{1+\epsilon} < \delta \text{ for any } l \in [(1 + \epsilon)_{1+\epsilon} - (1 + 2\epsilon)_{-(1+\epsilon)}, (1 + \epsilon)((1 + \epsilon)_{1+\epsilon} - (1 + 2\epsilon)_{-(1+\epsilon)})]. \quad (I2)$$

These results allow us to provide examples of completely absolutely mean irregular operators, which are a modification of Example 3.5 of [33]. The sequences of integers $(n_j)_{j \in \mathbb{Z}}$ and $(m_j)_{j \in \mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$, are such that for every $(1 + \epsilon) \in \mathbb{Z}$ we have

$$v_{j-1} \leq v_j \text{ when } (1 + 2\epsilon)_{1+\epsilon} < j \leq (1 + \epsilon)_{1+\epsilon}, \text{ and } v_{j-1} \geq v_j \text{ when } (1 + \epsilon)_{1+\epsilon} < j \leq (1 + 2\epsilon)_{1+\epsilon}.$$

In other words, the positions $v_{(1+\epsilon)_{1+\epsilon}}$ represent “hills” of the weight sequence, and the positions v_n are “valleys”.

Theorem 37 (see [43]). There exists a bilateral shift T on $\ell^{1+\epsilon}(v, \mathbb{Z})$ such that both T and T^{-1} are completely absolutely mean irregular, completely distributionally irregular and hypercyclic.

Proof. We first consider

$$(a) \quad n_0 = -1, m_0 = 1, n_1 = 4, m_{-1} = -4, v_{n_0} = 1, v_{m_0} = 2^{\frac{1}{4}}, v_{n_1} = 2^{-\frac{1}{3}}, v_{m_{-1}} = 2^{1/4}, \text{ and}$$

$$(b) \quad (1 + \epsilon)_{1+\epsilon} = -(1 + 2\epsilon)_{-(1+\epsilon)}, (1 + \epsilon) \in \mathbb{Z}, v_{(1+2\epsilon)_{1+\epsilon}} = (21 + \epsilon)^{-\frac{1}{3}}, v_{(1+\epsilon)_{1+\epsilon}} = (3 + \epsilon)^{\frac{1}{4}},$$

$$v_{(1+2\epsilon)_{-(1+\epsilon)}} = (22 + \epsilon)^{-\frac{1}{3}}, v_{(1+\epsilon)_{-(1+\epsilon)}} = (2 + \epsilon)^{\frac{1}{4}}, (1 + \epsilon) \in \mathbb{N}, v_i/v_{i-1} = v_j/v_{j-1} \text{ if } i, j \in [(1 + 2\epsilon)_{(1+\epsilon)}, (1 + \epsilon)_{1+\epsilon}], \text{ or if } i, j \in [(1 + \epsilon)_{1+\epsilon}, (1 + 2\epsilon)_{1+\epsilon}], (1 + \epsilon) \in \mathbb{Z}, \text{ and}$$

$$(c) \quad (1 + \epsilon)_{1+\epsilon} - (1 + 2\epsilon)_{1+\epsilon} > 2((1 + \epsilon)_\epsilon - (1 + 2\epsilon)_\epsilon), (1 + 2\epsilon)_{2+\epsilon} - (1 + \epsilon)_{1+\epsilon} > 2((1 + 2\epsilon)_{1+\epsilon} - (1 + \epsilon)_\epsilon), (1 + \epsilon) \in \mathbb{N}.$$

We will check that the hypotheses of Theorem 35 and Corollary 36 are satisfied.

Condition (b) gives $\frac{\min\{v_i ; (1+\epsilon)_{-1+\epsilon} \leq i \leq (1+\epsilon)_\epsilon\}}{v_{(1+2\epsilon)_{1+\epsilon}}} = \frac{v_{(1+2\epsilon)_{-(1+\epsilon)}}}{v_{(1+2\epsilon)_{1+\epsilon}}} = \left(\frac{21+\epsilon}{20+\epsilon}\right)^{1/3}$ for every $(1 + \epsilon) \in \mathbb{N}$, and the supremum of the slope of v outside the interval $[(1 + \epsilon)_{-(1+\epsilon)}, (1 + \epsilon)_\epsilon]$ is $S_{1+\epsilon} = v_j/v_{j-1}$ for any $(1 + 2\epsilon)_{1+\epsilon} < j \leq (1 + \epsilon)_{1+\epsilon}$, $(1 + \epsilon) \in \mathbb{N}$. We set $\epsilon = 1$ and, since $S_{1+\epsilon}^{(1+\epsilon)_{1+\epsilon} - (1+2\epsilon)_{1+\epsilon}} = \frac{v_{(1+\epsilon)_{1+\epsilon}}}{v_{(1+2\epsilon)_{1+\epsilon}}} = (21 + \epsilon)^{1/3} (3 + \epsilon)^{1/4}$, we need that

$$\begin{aligned} S_{1+\epsilon}^{(1+\epsilon)((1+2\epsilon)_{1+\epsilon} - (1+\epsilon)_{-(1+\epsilon)})} &= ((2(1 + \epsilon))^{2/3} (3 + \epsilon)^{1/2})^{\frac{(1+\epsilon)((1+2\epsilon)_{1+\epsilon} - (1+\epsilon)_{-(1+\epsilon)})}{((1+\epsilon)_{1+\epsilon} - (1+2\epsilon)_{1+\epsilon})}} \\ &\leq \frac{\min\{v_i ; (1 + \epsilon)_{-(1+\epsilon)} \leq i \leq (1 + \epsilon)_\epsilon\}}{v_{(1+2\epsilon)_{1+\epsilon}}} \\ &= \left(\frac{2(1 + \epsilon)}{(1 + 2\epsilon)}\right)^{1/3}. \end{aligned}$$

This can be obtained for, e.g., $(1 + \epsilon)_{1+\epsilon} = (161 + \epsilon^3 + 1)_{(1+2\epsilon)_{1+\epsilon}}$, $(1 + \epsilon) \in \mathbb{N}$. Indeed, for this selection and for any $\epsilon \geq 1$ (the case $\epsilon = 0$ trivially satisfies the above inequality), we have

$$S_{1+\epsilon}^{(1+\epsilon)((1+2\epsilon)_{1+\epsilon} - (1+\epsilon)_{-(1+\epsilon)})} < (2(1 + \epsilon)(3 + \epsilon))^{1/241 + \epsilon^2} \leq (2(1 + \epsilon))^{1/121 + \epsilon^2} \leq \left(\frac{2(1 + \epsilon)}{(1 + 2\epsilon)} \right)^{1/3}.$$

Thus, T satisfies (I1). If $l \leq (1 + 2\epsilon)_{1+\epsilon} - (1 + \epsilon)_{-(1+\epsilon)}$, then

$$\|T^l x^r\| \leq \frac{v_{(1+\epsilon)_{1+\epsilon}}}{v_{(1+2\epsilon)_{1+\epsilon}}} \|x^r\| \leq (2(1 + \epsilon))^{2/3} \|x^r\|,$$

which yields that, given an arbitrary non-zero vector $x^r \in \ell^{1+\epsilon}(v, \mathbb{Z})$ and an arbitrary $\delta > 0$, there is $1 + \epsilon \in \mathbb{N}$ as big as we want such that

$$\frac{1}{N} \sum_{r,j=1}^N \|T^l x^r\| < \frac{(2(1 + \epsilon))^{2/3}}{(2 + \epsilon)} \sum_r \|x^r\| + \delta,$$

for $N = (1 + \epsilon)((1 + 2\epsilon)_{1+\epsilon} - (1 + \epsilon)_{-(1+\epsilon)})$, and we obtain that T is completely absolutely mean irregular as soon as we show that condition (b) in Theorem 35 is satisfied. For this, we notice that

$$\begin{aligned} v_{r_0} &= v_{(1+\epsilon)_{1+\epsilon}} (1 + \epsilon)_{1+\epsilon}^{r_0 - (1+\epsilon)_{1+\epsilon}} \geq (3 + \epsilon)^{1/4} (1 + \epsilon)_{1+\epsilon}^{2(1+\epsilon)(1+\epsilon)_{1+\epsilon}} \\ &> (3 + \epsilon)^{1/4} \left(\frac{2(1 + \epsilon)}{(3 + 2\epsilon)} \right)^{1/3} \\ &\text{if } (1 + \epsilon)_{1+\epsilon} \leq r_0 \leq (3 + 2\epsilon)(1 + \epsilon)_{1+\epsilon}, \end{aligned}$$

which yields that T is completely absolutely mean irregular.

Analogously, $\frac{\min\{v_i : (1+\epsilon)_{-(1+\epsilon)} \leq i \leq (1+\epsilon)_{1+\epsilon}\}}{v_{(1+2\epsilon)_{1+\epsilon}}} = \frac{v_{(1+2\epsilon)_{1+\epsilon}}}{v_{(1+2\epsilon)_{-(1+\epsilon)}}} = \left(\frac{2(1+\epsilon)}{(1+2\epsilon)} \right)^{1/3}$ for every $(1 + \epsilon) \in \mathbb{N}$, and the infimum of the slope of v outside the interval $[(1 + \epsilon)_{-(1+\epsilon)}, (1 + \epsilon)_{1+\epsilon}]$ is $(1 + \epsilon)_{1+\epsilon} = v_j/v_{j-1}$ for any $(1 + \epsilon)_{1+\epsilon} < j \leq (1 + 2\epsilon)_{1+\epsilon}$, $(1 + \epsilon) \in \mathbb{N}$. Again, we set $\epsilon = 1$ and, since $(1 + \epsilon)_{1+\epsilon}^{(1+\epsilon)_{1+\epsilon} - (1+2\epsilon)_{1+\epsilon}} = ((2(2 + \epsilon))^{1/3} (3 + \epsilon)^{1/4})$, we need that

$$\begin{aligned} (1 + \epsilon)_{1+\epsilon}^{(1+\epsilon)((1+2\epsilon)_{-(1+\epsilon)} - (1+\epsilon)_{1+\epsilon})} &= (1 + \epsilon)_{1+\epsilon}^{-2(1+\epsilon)(1+\epsilon)_{1+\epsilon}} \\ &= ((2(2 + \epsilon))^{2/3} (3 + \epsilon)^{1/2})^{\frac{(1+\epsilon)(1+\epsilon)_{1+\epsilon}}{((1+2\epsilon)_{1+\epsilon} - (1+\epsilon)_{1+\epsilon})}} \leq \left(\frac{(3 + 2\epsilon)}{2(1 + \epsilon)} \right)^{1/3}, \end{aligned}$$

which is easily satisfied if we set, e.g., $(1 + 2\epsilon)_{1+\epsilon} = (161 + \epsilon^3 + 1)(1 + \epsilon)_{1+\epsilon}$, $(1 + \epsilon) \in \mathbb{N}$. Thus, $B_r = T^{-1}$ satisfies (I2). If $l \leq (1 + \epsilon)_{1+\epsilon} - (1 + 2\epsilon)_{-(1+\epsilon)}$, then

$$\|B_r^l x^r\| \leq \frac{v_{(1+\epsilon)_{1+\epsilon}}}{v_{(1+2\epsilon)_{1+\epsilon}}} \|x^r\| \leq (2(1 + \epsilon))^{2/3} \|x^r\|.$$

As before, we obtain that $B_r = T^{-1}$ is completely absolutely mean irregular in case that condition (b) in Corollary 36 is satisfied. Indeed, we easily have that

$$v_{r_0} > ((2 + \epsilon))^{1/4} \left(\frac{2(1 + \epsilon)}{(3 + 2\epsilon)} \right)^{1/3} \text{ if } (3 + 2\epsilon)(1 + \epsilon)_{-(1+\epsilon)} \leq r_0 \leq (1 + \epsilon)_{-(1+\epsilon)},$$

which implies condition (b).

For the hypercyclicity of T , since it is invertible, it suffices to show that there is an increasing sequence $(j_{1+\epsilon})_{1+\epsilon}$ in \mathbb{N} such that $\lim_{1+\epsilon} v_{j_{1+\epsilon}} = \lim_{1+\epsilon} v_{-j_{1+\epsilon}} = 0$ (See Theorem 3.2 in [22]). Let $j_{1+\epsilon} := ((1 + \epsilon)_{1+\epsilon} + (1 + 2\epsilon)_{1+\epsilon})/2 = (81 + \epsilon^3 + 1)(1 + 2\epsilon)_{1+\epsilon}$. We have

$$v_{j_{1+\epsilon}} = S_{1+\epsilon}^{j_{1+\epsilon} - (1+2\epsilon)_{1+\epsilon}} v_{(1+2\epsilon)_{1+\epsilon}} = ((21 + \epsilon)^{1/6} (3 + \epsilon)^{1/8}) \frac{1}{(21 + \epsilon)^{1/3}},$$

that tends to 0 as $1 + \epsilon$ goes to infinity. Note that $R_{1+\epsilon} = v_j/v_{j-1}$ has the same value for any $j \in [(1 + 2\epsilon)_{-(1+\epsilon)}, (1 + \epsilon)_{-(1+\epsilon)}]$, and thus for all $(1 + \epsilon) \in \mathbb{N}$ we have $R_{1+\epsilon} < S_{1+\epsilon}$ and

$$v_{-j_{1+\epsilon}} = R_{1+\epsilon}^{-j_{1+\epsilon} - (1+2\epsilon)_{-(1+\epsilon)}} v_{(1+2\epsilon)_{-(1+\epsilon)}} < S_{1+\epsilon}^{(1+\epsilon)_{1+\epsilon} - j_{1+\epsilon}} \frac{1}{(3 + 2\epsilon)^{1/3}}.$$

Therefore, $\lim_{1+\epsilon} v_{j_{1+\epsilon}} = \lim_{1+\epsilon} v_{-j_{1+\epsilon}} = 0$, which concludes the hypercyclicity of T .

Corollary 38 [43]. There exists an operator T on a Banach space X such that the whole space X is a mean Li-Yorke set for T .

Remark 39. Since mean Li-Yorke chaos and DC2 are equivalent for dynamical systems on compact metric spaces, it follows from [23, Theorem 3.1] that the conclusion of the above corollary is not possible for such dynamical systems.

Remark 40. It is possible to modify slightly the example in Theorem 37 in order to obtain T such that every non-zero vector is absolutely mean semi-irregular for both T and T^{-1} (thus, both operators are generically mean Li-Yorke chaotic), but neither T nor T^{-1} are completely absolutely mean irregular. To do this, the only change would be to set $v_{(1+\epsilon)_{1+\epsilon}} = 1, (1+\epsilon) \in \mathbb{Z}$. That is,

- (a) $n_0 = -1, m_0 = 1, n_1 = 4, m_{-1} = -4, v_{n_0} = 1, v_{m_0} = 1, v_{n_1} = 2^{-\frac{1}{3}}, v_{m_{-1}} = 1$, and
- (b) $(1+\epsilon)_{1+\epsilon} = -(1+2\epsilon)_{-(1+\epsilon)}, (1+\epsilon) \in \mathbb{Z}, v_{(1+2\epsilon)_{1+\epsilon}} = (2(1+\epsilon))^{-\frac{1}{3}}, v_{(1+\epsilon)_{1+\epsilon}} = 1, v_{(1+2\epsilon)_{-(1+\epsilon)}} = (3+2\epsilon)^{-\frac{1}{3}}, v_{(1+\epsilon)_{-(1+\epsilon)}} = 1, (1+\epsilon) \in \mathbb{N}, \frac{v_i}{v_{i-1}} = \frac{v_j}{v_{j-1}}$ if $i, j \in [(1+2\epsilon)_{1+\epsilon}, (1+\epsilon)_{1+\epsilon}]$, or if $i, j \in [(1+\epsilon)_{(1+\epsilon)}, (1+2\epsilon)_{1+\epsilon}], (1+\epsilon) \in \mathbb{Z}$, and
- (c) $(1+\epsilon)_{1+\epsilon} = (16(1+\epsilon)^3 + 1)(1+2\epsilon)_{1+\epsilon}, (1+2\epsilon)_{(2+\epsilon)} = (16(1+\epsilon)^3 + 1)(1+\epsilon)_{1+\epsilon}, (1+\epsilon) \in \mathbb{N}$.

In that case, the vectors of the unit basis are not absolutely mean irregular. Also, we observe that the hypercyclicity condition is preserved.

7 Mean Li-Yorke Chaotic Semigroups

We recall that a one-parameter family $(T_t)_{t \geq 0}$ of operators on X is called a C_0 -semigroup if $T_0 = I, T_t T_{1+\epsilon} = T_{t+(1+\epsilon)}$ ($t, 1+\epsilon \geq 0$) and $\lim_{t \rightarrow 1+\epsilon} T_t x^r = T_{1+\epsilon} x^r$ ($x^r \in X$ and $\epsilon \geq 0$). It is well-known that such a semigroup is always locally equicontinuous, in the sense that

$$\sup_{t \in [0, 1+\epsilon]} \|T_t\| < \infty \quad \text{for every } \epsilon \geq 0.$$

See [21] for a detailed study of C_0 -semigroups. In the sequel, $\mathcal{T} = (T_t)_{t \geq 0}$ will denote an arbitrary C_0 -semigroup, unless otherwise specified.

Definition 41. \mathcal{T} is said to be mean Li-Yorke chaotic if there is an uncountable subset S of X (a mean Li-Yorke set for \mathcal{T}) such that every pair (x^r, y^r) of distinct points in S is a mean Li-Yorke pair for \mathcal{T} , in the sense that

$$\liminf_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t x^r - T_t y^r\| dt = 0 \quad \text{and} \quad \limsup_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t x^r - T_t y^r\| dt > 0.$$

If S can be chosen to be dense (resp. residual) in X , then we say that T is densely (resp. generically) mean Li-Yorke chaotic.

Li-Yorke chaos and distributional chaos for C_0 -semigroups were studied in [1], [2], [3], [17], [19], [37], for instance.

Definition 42. \mathcal{T} is called absolutely Cesàro bounded if there is $\epsilon \geq 0$ such that

$$\sup_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t x^r\| dt \leq (1+\epsilon) \sum_r \|x^r\| \quad \text{for all } x^r \in X.$$

Definition 43. We say that $x^r \in X$ is an absolutely mean irregular (resp. absolutely mean semi-irregular) vector for \mathcal{T} if

$$\liminf_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t x^r\| dt = 0 \quad \text{and} \quad \limsup_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t x^r\| dt = \infty \quad (\text{resp. } > 0).$$

Definition 44. An absolutely mean irregular manifold for \mathcal{T} is a vector subspace Y of X such that every nonzero vector in Y is absolutely mean irregular for \mathcal{T} .

As we did in [18] for hypercyclicity, or in [1] for distributional chaos, we will establish equivalences between the above notions for C_0 -semigroups and the corresponding ones for the operators of the semigroup. The following simple lemma will be very useful for this purpose.

Lemma 45 [43]. For each $\epsilon > 0$, let $(1+\epsilon)_{1+\epsilon} := \sup_{t \in [0, 1+\epsilon]} \|T_t\| < \infty$. Then

$$\frac{1}{(1+\epsilon)_{1+\epsilon}} \frac{1}{N+1} \sum_{r,j=1}^N \|(T_{1+\epsilon})^j x^r\| \leq \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t\| dt \leq (1+\epsilon)_{1+\epsilon} \frac{1}{N} \sum_{r,j=0}^N \|(T_{1+\epsilon})^j x^r\|,$$

Whenever $x^r \in X$ and $N_{1+\epsilon} \leq 1+\epsilon < (N+1)_{1+\epsilon}$ with $N \geq 1$.

The next two propositions follow easily from this lemma.

Proposition 46. Given $x^r \in X$ and a C_0 -semigroup \mathcal{T} , the following are equivalent:

- (a) x^r is an absolutely mean semi-irregular (resp. irregular) vector for \mathcal{T} ;
- (b) there exists $\epsilon > 0$ such that x^r is an absolutely mean semi-irregular (resp. irregular) vector for $T_{1+\epsilon}$;
- (c) x^r is an absolutely mean semi-irregular (resp. irregular) vector for $T_{1+\epsilon}$, for all $\epsilon > 0$.

Proposition 47. Given a AC_0 -semigroup, the following are equivalent:

- (a) \mathcal{T} is absolutely Cesàro bounded (resp. (densely, generically) mean Li-Yorke chaotic, admits a dense absolutely mean irregular manifold);
- (b) there exists $\epsilon > 0$ such that $T_{1+\epsilon}$ is absolutely Cesàro bounded (resp. (densely, generically) mean Li-Yorke chaotic, admits a dense absolutely mean irregular manifold);
- (c) $T_{1+\epsilon}$ is absolutely Cesàro bounded (resp. (densely, generically) mean Li-Yorke chaotic, admits a dense absolutely mean irregular manifold), for all $\epsilon > 0$.

Remark 48. With these propositions at hand, it is easy to transport many of our previous theorems on operators to the semigroup setting. For instance, Theorems 5 (without (v)), 17 and 33 remain valid if we replace the operator T by the semigroup \mathcal{T} . In particular, absolutely Cesàro bounded C_0 -semigroups are never mean Li-Yorke chaotic.

Theorem 49 (see [43]). Suppose that the set $\{x^r \in X : \lim_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t x^r\| dt = 0\}$ is dense in X . If any of the following conditions is true:

- (a) $\limsup_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \sum_r \|T_t y_0^r\| dt > 0$ for some $y_0^r \in X$,
 - (b) X is a Banach space and $\limsup_{t \rightarrow \infty} \frac{\|T_t\|}{t} > 0$,
 - (c) X is a Hilbert space and $\limsup_{t \rightarrow \infty} \frac{\|T_t\|}{t^{\frac{1}{2}}} > 0$,
 - (d) there is some $\lambda \in \sigma_{1+\epsilon}(A_r)$ with $\operatorname{Re} \lambda \geq 0$, where A_r is the infinitesimal generator of \mathcal{T} ,
- then \mathcal{T} has a residual set of absolutely mean irregular vectors. If, in addition, X is separable, then \mathcal{T} admits a dense absolutely mean irregular manifold.

Proof. It is a consequence of Theorem 22 and Corollary 25 (note that (b) and (c) imply that $\limsup_{\epsilon \rightarrow \infty} \frac{\|T_1^{(1+2\epsilon)}\|}{(1+2\epsilon)} > 0$ and $\limsup_{\epsilon \rightarrow \infty} \frac{\|T_1^{(1+2\epsilon)}\|}{(1+2\epsilon)^{\frac{1}{2}}} > 0$, respectively).

As an immediate consequence of the previous theorem, we have the following dichotomy for translation semigroups on weighted $L^{1+\epsilon}$ spaces.

Corollary 50. Let $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ be an admissible weight function and consider the translation semigroup \mathcal{T} , given by

$$T_t(f)(x^r) = f(x^r + t), \quad t, x^r \geq 0,$$

on the space $L_v^{1+\epsilon}(\mathbb{R}_+)$ [26]. Then either

- (a) $\lim_{\epsilon \rightarrow \infty} \frac{1}{1+\epsilon} \int_0^{1+\epsilon} \|T_t f\| dt = 0$ for every $f \in L_v^{1+\epsilon}(\mathbb{R}_+)$, or
- (b) \mathcal{T} admits a dense absolutely mean irregular manifold.

As a consequence of Theorem 49(b) and [35], we obtain

Corollary 51. The C_0 -semigroup \mathcal{T} defined on $L^1(1, \infty)$ by

$$T_t f(x^r) := \left(\frac{x^r + t}{x^r} \right) f(x^r + t)$$

is mean Li-Yorke chaotic.

Question 52. Is there a C_0 -semigroup which is mean Li-Yorke chaotic but is not distributionally chaotic? Is the semigroup defined in the above corollary distributionally chaotic?

Theorem 53 (see [43]). There exists a mixing absolutely Cesàro bounded C_0 -semigroup \mathcal{T} on $L^{1+\epsilon}(1, \infty)$ for $0 \leq \epsilon \leq \infty$.

Proof. Let $0 < \epsilon < 1$ and consider the weighted translation semigroup (T_t) on $L^{1+\epsilon}(1, \infty)$ defined by

$$T_t f(x^r) := \left(\frac{x^r + t}{x^r} \right)^{\frac{1-\epsilon}{1+\epsilon}} f(x^r + t).$$

Given $f \in L^{1+\epsilon}(1, \infty)$ with $\|f\|_{1+\epsilon} = 1$, and $\epsilon \geq 0$, we have that

$$\begin{aligned}
 \int_0^{1+\epsilon} \|T_t f\|_{1+\epsilon} dt &= \int_0^{1+\epsilon} \sum_r \left(\int_1^\infty \left(\frac{x^r + t}{x^r} \right)^{1-\epsilon} |f(x^r + t)|^{1+\epsilon} dx^r \right) dt \\
 &= \int_0^{1+\epsilon} \left(\int_{1+t}^\infty \sum_r \left(\frac{x^r}{x^r - t} \right)^{1-\epsilon} |f(x^r)|^{1+\epsilon} dx^r \right) dt \\
 &= \int_1^\infty \sum_r (x^r)^{1-\epsilon} |f(x^r)|^{1+\epsilon} \int_0^{\min\{x^r-1, 1+\epsilon\}} \left(\frac{1}{x^r - t} \right)^{1-\epsilon} dt dx^r \\
 &\leq \int_1^{1+2(1+\epsilon)} \sum_r (x^r)^{1-\epsilon} |f(x^r)|^{1+\epsilon} \int_0^{x^r-1} \left(\frac{1}{x^r - t} \right)^{1-\epsilon} dt dx^r \\
 &\quad + \int_{1+2(1+\epsilon)}^\infty \sum_r |f(x^r)|^{1+\epsilon} \int_0^{1+\epsilon} \left(\frac{x^r}{x^r - t} \right)^{1-\epsilon} dt dx^r \\
 &\leq \int_1^{1+2(1+\epsilon)} \sum_r \frac{x^r - 1}{\epsilon} |f(x^r)|^{1+\epsilon} dx^r + 2(1 + \epsilon) \leq \left(2 + \frac{2}{\epsilon} \right) (1 + \epsilon).
 \end{aligned}$$

So,

$$\left(\frac{1}{1 + \epsilon} \int_0^{1+\epsilon} \|T_t f\| dt \right)^{1+\epsilon} \leq \frac{1}{1 + \epsilon} \int_0^{1+\epsilon} \|T_t f\|^{1+\epsilon} dt \leq 2 + \frac{2}{\epsilon}.$$

Thus, (T_t) is an absolutely Cesàro bounded C_0 -semigroup. Let us now see that (T_t) is mixing. If $v(x^r) := \left(\frac{1}{x^r} \right)^{1-\epsilon}$, then T_t can be rewritten as

$$T_t f(x^r) = \left(\frac{v(x^r)}{v(x^r + t)} \right)^{\frac{1}{1+\epsilon}} f(x^r + t).$$

Since v is an admissible weight function, the translation semigroup defined as

$$\tau_t f(x^r) := f(x^r + t)$$

is a C_0 -semigroup on $L_v^{1+\epsilon}(1, \infty)$. Moreover, T_t on $L_v^{1+\epsilon}(1, \infty)$ and τ_t on $L_v^{1+\epsilon}(1, \infty)$ are topologically conjugate. Since (τ_t) is mixing because $\lim_{x^r \rightarrow \infty} v(x^r) = 0$ [26], we conclude that (T_t) is also mixing.

Corollary 54. There exists a mixing not absolutely mean irregular C_0 -semigroup \mathcal{T} on $L_v^{1+\epsilon}(1, \infty)$ for $0 \leq \epsilon \leq \infty$.

Theorem 55 (see [43]). There exists a forward translation C_0 -semigroup \mathcal{T} on $L_v^{1+\epsilon}(\mathbb{R}_+)$ which is distributionally chaotic and not mean Li-Yorke chaotic.

Proof. It was proved in [13] that there is a sequence $w = (w_{(1+2\epsilon)})_{(1+2\epsilon) \in \mathbb{N}}$ of positive weights such that the unilateral weighted forward shift

$$F_w : (x_1^r, x_2^r, \dots) \in \ell^{1+\epsilon}(\mathbb{N}) \mapsto (0, w_1, w_2, w_2 x_2^r, \dots) \in \ell^{1+\epsilon}(\mathbb{N})$$

is distributionally chaotic and satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r,j=1}^N \|(F_w)^j x^r\| = \infty \quad \text{for all } x^r \in \ell^{1+\epsilon}(\mathbb{N}) \setminus \{0\}.$$

By using conjugacy, we see that there is a sequence $v' = (v_{(1+2\epsilon)})_{(1+2\epsilon) \in \mathbb{N}}$ of positive weights such that the unweighted forward shift on $\ell^{1+\epsilon}(v', \mathbb{N})$ is distributionally chaotic and not mean Li-Yorke chaotic. Now, if we consider as admissible weight function v the polygonal formed by the sequence v' , then the forward translation C_0 -semigroup on $L_v^{1+\epsilon}(\mathbb{R}_+)$ is distributionally chaotic and not mean Li-Yorke chaotic.

Remark 56. The example in Theorem 37 can be easily adapted to the semigroup setting in order to construct a completely absolutely mean irregular C_0 -semigroup T . Indeed, the translation semigroup on $L_v^{1+\epsilon}(\mathbb{R})$ does the job if we fix $v(1 + \epsilon) = v_{1+\epsilon}, (1 + \epsilon) \in \mathbb{Z}$, where $(v_{1+\epsilon})_{(1+\epsilon) \in \mathbb{Z}}$ is the sequence of weights of the example in Theorem 37, and we set $v(x^r) = v(1 + \epsilon), x^r \in](\epsilon, 1 + \epsilon], (1 + \epsilon) \in \mathbb{Z}$.

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