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Research Paper

Complex Stochastic Dynamics and Information Measures in Quantum-Inspired Models

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ABSTRACT : *This paper presents a comprehensive framework for probability theory in the complex domain, extending classical measure-theoretic concepts to accommodate complex-valued random variables, probability amplitudes, and non-Hermitian dynamics. Motivated by foundational questions in quantum mechanics, signal analysis, and stochastic processes, both theoretical advancements and numerical simulations are reviewed to illustrate the behavior and utility of complex-valued probabilistic models. Key contributions include the formalization of complex measures, analytic continuation of characteristic functions, entropy estimation for complex density matrices, and spectral analysis of non-Hermitian operators, such as those in the Ginibre ensemble. Simulations of complex Brownian motion and Ornstein–Uhlenbeck processes are performed, along with the computation of von Neumann entropy and fidelity to quantify uncertainty and coherence in quantum-like systems. These results underscore the necessity of complex probability theory in modeling interference, non-equilibrium behavior, and information flow in modern physical and mathematical systems. The paper concludes by outlining future directions in the geometry, computation, and foundational axioms of complex-valued stochastic systems.*

KEYWORDS: *Complex probability, quantum mechanics, non-Hermitian dynamics, complex Brownian motion, entropy estimation, Ginibre ensemble, Ornstein-Uhlenbeck processes, Von Neumann entropy, interference, stochastic systems.*

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I. INTRODUCTION

Probability theory has long served as the mathematical foundation for analyzing randomness, uncertainty, and statistical inference. Traditionally, it is built upon real-valued measures, random variables, and expectations, governed by Kolmogorov's axiomatic framework [16][18]. However, the need to model systems whose behavior depends on complex quantities, such as phase, interference, and analytic continuation, has stimulated interest in developing probability theory within the complex domain.

In contrast to conventional probability measures that take non-negative real values summing to one, complex probability constructs allow for complex-valued measures, amplitudes, or expectation functionals. These do not always obey standard probabilistic interpretations, yet they maintain internal consistency under appropriate mathematical formalism. Notably, in quantum mechanics, probabilities are derived from the modulus squared of complex amplitudes, while the amplitudes themselves evolve unitarily in complex Hilbert spaces [6],[23]. This intrinsic reliance on complex values motivates the study of more general frameworks, such as complex-valued stochastic processes and analytic probability distributions.

From a mathematical standpoint, complex probability entails extending the integration theory to accommodate complex measures, redefining characteristic functions as complex-analytic functions, and generalizing random variables to complex-valued measurable functions. These tools provide a natural setting for analyzing random holomorphic fields, Gaussian analytic functions, and non-Hermitian operators whose spectra lie in the complex plane [11],[14]. Moreover, concepts such as complex martingales, complex Brownian motion,

and complex Lévy processes are under active development, with applications ranging from quantum optics to statistical learning in the frequency domain [13].

The aim of this article is to introduce the theory of probability in complex spaces. We first present the historical motivations and foundational developments in this area. Then, we construct the formal framework of complex probability measures, complex-valued random variables, and integration over complex manifolds. Subsequent sections address analytic continuation, complex characteristic functions, and stochastic processes with complex state spaces. Finally, we discuss physical and mathematical applications, highlighting ongoing challenges and open research directions.

1.1 Historical background

The origins of probability theory in the complex domain can be traced to two seemingly distinct lines of development: the emergence of quantum mechanics in the early 20th century, and the mathematical investigation of complex integration and analytic functions.

The formalism of quantum mechanics, as established by Heisenberg, Schrödinger, and Dirac, inherently relies on complex vector spaces and operators. Dirac's bracket notation formalized the use of complex Hilbert spaces where probability amplitudes, rather than probabilities themselves, are the primary objects [6]. The Born rule, which computes the probability density as $|\psi(x)|^2$, presumes an underlying complex structure not found in classical statistics.

Simultaneously, mathematicians began studying random variables and measures on complex spaces. The theory of complex measures, introduced as early as the 1930s, generalizes signed measures by allowing values in $\mathbb{C}$ instead of $\mathbb{R}$, and found application in harmonic analysis and spectral theory [29]. Despite initial skepticism regarding their interpretability, complex measures provided the analytic backbone for various probabilistic constructs.

In the 1960s, complex-valued random processes gained attention in statistical signal processing, particularly in the context of analytic signals and envelope detection. By the 1990s and 2000s, the rise of non-Hermitian quantum mechanics, random matrix theory, and holomorphic probability theory further expanded the role of complex-valued probabilities [1],[16].

A notable development is the reinterpretation of the Feynman path integral, which is central to quantum field theory, as not being a sum of real-valued probabilities, but over complex-valued weights $e^{iS/\hbar}$, where S is the classical action [9]. Although such quantities defy standard probabilistic interpretation, they are indispensable in accurately predicting experimental outcomes.

In recent decades, researchers such as Khrennikov [16],[30], and others have attempted to formalize the concept of complex probabilities in a mathematically consistent and physically meaningful way. These efforts have culminated in the emergence of an interdisciplinary field at the intersection of functional analysis, stochastic processes, and mathematical physics.

II. DEFINITIONS AND MATHEMATICAL FRAMEWORK

In this section, we provide formal definitions and foundational concepts necessary to develop probability theory within the complex domain. We start by extending classical notions, such as measurable spaces, random variables and expectation, to their complex counterparts. This section is self-contained and presumes familiarity with standard measure-theoretic probability and functional análisis.

2.1 Complex-valued probability measures

Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω is a sample space and $\mathcal{F}$ is a σ-algebra of subsets of Ω. A complex measure $\mu: \mathcal{F} \to \mathbb{C}$ is a countably additive function such that for any disjoint sequence $\{A_n\} \subset \mathcal{F}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1}$$

where the series converges in the complex norm. This generalizes the classical definition of a probability measure, which restricts μ to be non-negative and satisfy $\mu(\Omega) = 1$.

In the complex case, the total variation of a complex measure μ is defined as

$$|\mu|(A) := sup \sum_i |\mu(A_i)| \tag{2}$$

where the supremum is over all finite partitions $\{A_i\}$ of A. This makes $|\mu|$ a positive measure and allows us to define integration of complex-valued functions with respect to μ via the Radon–Nikodym derivative relative to $|\mu|$.

Although $\mu(\Omega)$ may not equal 1, many applications normalize complex measures by requiring $|\mu(\Omega)| = 1$, or interpret μ as a complex amplitude whose squared modulus provides a conventional probability distribution.

2.2 Complex-valued random variables

A complex-valued random variable is a measurable function $X: \Omega \to \mathbb{C}$. As in the real case, the distribution of X is defined by the pushforward measure μ_X on $\mathbb{C}$, given by

$$\mu_X(B) := \mu\big(X^{-1}(B)\big), B \in \mathcal{B}(\mathbb{C}) \tag{3}$$

where $\mathcal{B}(\mathbb{C})$ denotes the Borel σ-algebra on $\mathbb{C}$. Expectations and moments are defined analogously:

$$\mathbb{E}[X] := \int_\Omega X(\omega) d\mu(\omega), \mathbb{E}[|X|^2] = \int_\Omega |X(\omega)|^2 d|\mu|(\omega) \tag{4}$$

2.3 Complex characteristic functions

In applications, particularly in quantum mechanics and signal processing, complex random variables often represent analytic signals or quantum amplitudes. Their statistical properties are characterized not only by their modulus but also by their phase, making tools like the complex correlation function and complex characteristic function essential.

2.4 Complex characteristic functions

Given a complex-valued random variable X, the characteristic function is defined as

$$\phi_X(t) := \mathbb{E}\big[e^{itX}\big] = \int_\Omega e^{itX(\omega)} d\mu(\omega), \qquad t \in \mathbb{R} \tag{5}$$

where $i = \sqrt{-1}$. For real-valued random variables, ϕ_X is uniformly continuous and positive-definite. For complex X, ϕ_X becomes an analytic function in a neighborhood of zero, and its analyticity properties can be studied using tools from complex analysis [20].

In quantum mechanics, the Fourier transform of a wave function ψ corresponds to the characteristic function of the associated position or momentum observable. Thus, understanding the behavior of ϕ_X in the complex plane is key to analyzing localization, coherence, and interference.

2.5 Complex Hilbert spaces and inner product spaces

In many applications, particularly quantum mechanics, random variables are modeled not as functions on a sample space but as elements of a complex Hilbert space $\mathcal{H}$. In this framework, expectation is replaced by an inner product:

$$\mathbb{E}[X] \sim \langle \psi | \hat{X} \psi \rangle \tag{6}$$

where $\psi \in \mathcal{H}$ is a normalized state vector and $\hat{X}$ is a linear operator acting on $\mathcal{H}$. This formulation underlies quantum probability theory [24], in which random variables correspond to observables and expectation values depending on the underlying quantum state.

This non-commutative probabilistic setting naturally extends to complex-valued measures on operator algebras and leads to phenomena such as non-locality**,** superposition, and quantum entanglement, which are inherently complex in nature.

III. APPLICATIONS IN PHYSICS AND MATHEMATICS

The formalism of complex probability theory is not merely of abstract interest; it has concrete applications in several domains of physics and mathematics. This section outlines key areas where complex-valued measures and random processes play a fundamental role, with a particular focus on quantum mechanics**,** complex stochastic processes, and path integral formulations. These applications highlight the necessity of extending classical probabilistic tools to accommodate complex-valued structures.

3.1 Quantum probability and measurement theory

Quantum mechanics is the most prominent example of complex probability in physics. Here, probability amplitudes (complex-valued wave functions) encode all measurable information about a physical system. The probability density is derived from the modulus squared of the wave function:

$$\rho(x) = |\psi(x)|^2 \tag{7}$$

However, the complex phase of $\psi(x)$ influences phenomena such as interference**,** entanglement, and coherence, which are not captured by classical probability theory [23]. In the quantum probability framework, random variables correspond to self-adjoint operators on a complex Hilbert space, and expectations take the form:

$$\mathbb{E}[X] = \langle \psi | \hat{X} \psi \rangle \tag{8}$$

where ψ is a normalized state vector. Complex probability measures can also be associated with non-Hermitian

observables, as in PT-symmetric quantum mechanics [1], where the eigenvalue spectrum remains real despite the underlying non-Hermiticity.

The probabilistic interpretation of such theories relies on a complex inner product structure and non-classical Born rules, requiring the reinterpretation of measurement and normalization conditions. These developments further emphasize the need for a consistent theory of complex-valued probability and expectation.

3.2 Complex stochastic processes

Complex-valued stochastic processes naturally arise in various fields, including analytic signals in signal processing, where the Hilbert transform constructs a complex-valued representation of real-valued time series [10]; complex Brownian motion (or planar Brownian motion), defined as $B_t = X_t + iY_t$, with X_t and Y_t being independent real-valued Brownian motions; and holomorphic Gaussian processes, which model random analytic functions with applications in complex geometry, such as the distribution of zeros of random polynomials and entire functions [14].

In mathematical finance, complex stochastic differential equations (SDEs) have been proposed to model oscillatory or wave-like behaviors in asset dynamics, especially in high-frequency trading. More formally, let Z_t be a complex Itô process defined by the SDE:

$$dZ_t = \mu(Z_t, t)dt + \sigma(Z_t, t)dW_t \tag{9}$$

where $\mu: \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$, $\sigma: \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$, and W_t is a complex-valued Wiener process.

These processes also arise in non-equilibrium thermodynamics, where complex noise models can represent non-conservative forces or coupled quantum-classical interactions.

3.3 Path integrals and complex weight measures

One of the most profound uses of complex-valued probability is found in the Feynman path integral formulation of quantum mechanics. Unlike the classical probabilistic path space measure (e.g., Wiener measure for Brownian motion), the Feynman measure assigns to each path $\gamma(t)$ a complex weight of the form: $A[\gamma] = e^{iS[\gamma]/\hbar}$ where $S[\gamma]$ is the classical action functional and $\hbar$ is the reduced Planck constant [9]. The sum-over-paths or path integral becomes:

$$\langle x_f, t_f | x_i, t_i \rangle = \int_{\gamma(t_i)=x_i}^{\gamma(t_f)=x_f} \mathcal{D}[\gamma]\, e^{iS[\gamma]/\hbar} \tag{10}$$

Despite its success, the mathematical status of the path integral is subtle, since $e^{iS/\hbar}$ does not define a true probability measure. Attempts to rigorously define the path integral have employed Wick rotation (transforming time to imaginary values), yielding real-valued measures related to the Euclidean path integral:

$$\int \mathcal{D}[\gamma] e^{-S_E[\gamma]/\hbar} \tag{11}$$

Nevertheless, the original complex-valued weight remains central to the physical interpretation of quantum transitions, making the case for complex probability theory as a foundational component of modern physics.

3.4 Random matrix theory and complex spectral distributions

Random matrix theory (RMT) provides another natural setting for complex probability. In the Ginibre ensemble, matrices with independent, identically distributed complex Gaussian entries have eigenvalues distributed in the complex plane [11]. The joint probability density of eigenvalues $\{z_i\}$ is given by:

$$P(z_1, \dots, z_n) \alpha \prod_{i<j} |z_i - z_j|^2 \cdot \prod_{k=1}^{n} e^{-|z_k|^2} \tag{12}$$

This non-Hermitian ensemble lacks real symmetry and requires probabilistic reasoning in $\mathbb{C}^n$. Applications of complex RMT span from open quantum systems to non-equilibrium dynamics and neural networks.

IV. NON-HERMITIAN OPERATORS, ANALYTIC CONTINUATION, AND COMPLEX ENTROPY

Recent theoretical developments have deepened our understanding of complex probability by linking it to advanced mathematical structures such as non-Hermitian operators, complex analytic functions, and generalized entropy measures. These concepts broaden the landscape of probabilistic modeling, enabling applications beyond traditional domains and motivating new foundational questions.

4.1 non-Hermitian operators and complex spectra

In classical quantum mechanics, observables are represented by Hermitian (self-adjoint) operators whose spectra are real and whose eigenfunctions form a complete orthonormal basis. Nevertheless, research into non-Hermitian operators, especially those that are PT-symmetric (i.e., they remain the same under parity and time-reversal transformations), has demonstrated that real eigenvalues can be found in complex Hamiltonians [1]. These operators challenge the classical axioms of quantum probability since they do not generate unitary time evolution in the standard Hilbert space norm.

Let H be a non-Hermitian operator with complex eigenvalues $\{E_n\} \subset \mathbb{C}$, and corresponding biorthogonal eigenvectors $\{|\psi_n\rangle, |\phi_n\rangle\}$ satisfying:

$$H|\psi_n\rangle = E_n|\psi_n\rangle, H^\dagger|\phi_n\rangle = \overline{E_n}|\phi_n\rangle \tag{13}$$

In this framework, the expectation value of an observable $\hat{O}$ is defined as:

$$\langle \hat{O} \rangle = \frac{\langle \phi_n | \hat{O} | \psi_n \rangle}{\langle \phi_n | \psi_n \rangle} \tag{14}$$

a formula that requires complex inner products and careful normalization [22]. H is not Hermitian, and therefore, the eigenvalues E_n are not guaranteed to be real.

A biorthogonal system $\{|\psi_n\rangle\}$ and $\{|\phi_n\rangle\}$ is introduced, where H acts on $|\psi_n\rangle$, and its adjoint $H^\dagger$ acts on $|\phi_n\rangle$. The expectation value is not the usual $\langle \phi_n | \hat{O} | \psi_n \rangle$, but instead involves the biorthogonal vectors, which is common in non-Hermitian operator theories (such as in PT-symmetric quantum mechanics).

These constructions illustrate how complex-valued probabilities naturally emerge in extended quantum theories, and how biorthogonal bases redefine the geometry of measurement.

4.2 Analytic continuation and complex characteristic functions

Analytic continuation plays a fundamental role in both mathematical analysis and probability theory. For a real-valued random variable X, the moment-generating function $M_x(t) = \mathbb{E}[e^{tX}]$ is defined on a real interval but often extends analytically to a neighborhood of the origin in $\mathbb{C}$. In complex probability, this continuation is more than a technical tool, it provides access to entire functions**,** pole structures, and singularities that reveal global probabilistic behavior.

For example, the characteristic function $\phi_X(t) = \mathbb{E}[e^{itX}]$ can be extended to a complex Fourier transform of the underlying probability measure, enabling reconstruction via the inversion theorem:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_X(t)dt \tag{15}$$

In the complex domain, such transforms are used to define complex measures on analytic manifolds and to model random fields over domains in $\mathbb{C}^n$. The Laplace transform of a complex-valued distribution, similarly extended, plays a key role in quantum field theory and fractional calculus [31].

Analytic continuation is also used to regularize divergent series and integrals in path integrals and zeta-function regularization, where physical observables are analytically extended into the complex plane to extract meaningful finite results.

4.3 Complex entropy and generalized information measures

The concept of entropy serves as a bridge between probability and information theory. Classical Shannon entropy for a discrete probability distribution $\{p_k\}$ is defined as:

$$H = -\sum_k p_k \log(p_k) \tag{16}$$

where p_k represents the probability associated with the k-th event in the distribution. When $\{p_k\} \subset \mathbb{C}$, this expression becomes ill-defined due to the complex logarithm's multivalued nature. Nonetheless, generalized entropy measures have been proposed to accommodate complex-valued distributions. These include Complex Rényi entropies, defined as:

$$H_\alpha^{\mathbb{C}} = \frac{1}{1-\alpha} log\left(\sum_k |p_k|^\alpha\right), \alpha \in \mathbb{R}_+ \tag{17}$$

which preserve scale-invariance and remain well-defined for complex-valued amplitudes p_k. This entropy measure generalizes the Shannon entropy and is defined for $\alpha > 0$, capturing the scale-invariant properties of complex probability amplitudes p_k. The use of $|p_k|^\alpha$ ensures that the entropy remains well-defined for complex amplitudes and is sensitive to the magnitudes of the complex numbers involved. Phase-sensitive entropies, such as:

$$H_{phase} = -\sum_k |p_k|^2 \arg(p_k) \tag{18}$$

which explicitly encodes the role of phase in probabilistic systems, where $\arg(p_k)$ is the phase (or argument) of the complex number p_k. This entropy measures the information content derived from the distribution of the phases of the complex amplitudes, offering a complementary perspective to magnitude-based measures like the Rényi entropy.

These two types of entropies, magnitude-based (Complex Rényi) and phase-sensitive (phase entropy), provide a complete description of the information carried by a complex-valued probability distribution. Together, they are useful tools in complex probability theory and can be applied to fields like quantum information and statistical mechanics, where systems naturally exhibit complex-valued amplitudes.

4.4 Research Examples

We show how this entropy measures can be applied in some specific contexts. We use as an example aspect that relates to quantum mechanics**,** communication theory, and statistical mechanics to show how Complex Rényi entropies and phase-sensitive entropies are used**.**

4.4.1 Complex Rényi entropy in quantum states

In quantum mechanics, the density matrix ρ describes the state of a quantum system. If we consider a pure quantum state represented by a wavefunction ψ, we can express this state in terms of a probability distribution for the measurement outcomes on a chosen basis. The Complex Rényi entropy can then be used to quantify the mixedness of the quantum state, especially when we deal with quantum systems in a superposition of states.

For example, in a system with two energy levels, $\{|0\rangle,|1\rangle\}$ the probability distribution for measuring the system in these states might be $p_0 = |\langle 0|\psi\rangle|^2$ and $p_1 = |\langle 1|\psi\rangle|^2$, where $\psi = \alpha|0\rangle + \beta|1\rangle$ with complex coefficients α and β.

Using the Complex Rényi entropy $H_\alpha^{\mathbb{C}}$, we could quantify the information content of the system's state as:

$$H_\alpha^{\mathbb{C}} = \frac{1}{1-\alpha} log(p_0^\alpha + p_1^\alpha) \tag{19}$$

where α controls the sensitivity of the entropy to different probabilities. As α approaches 1, this entropy tends to the Shannon entropy, which is a measure of uncertainty in the state of the system. For $\alpha > 1$, it emphasizes the dominance of certain outcomes (e.g., a specific state becoming more probable).

This entropy measure helps in quantum information theory where it is crucial to quantify quantum uncertainty and mixedness of states, especially in quantum communication or quantum cryptography protocols [4].

4.4.2 Phase-sensitive entropy in quantum coherent states

In quantum optics, phase-sensitive entropies are important when describing coherent states, which are often used to model the behavior of light. A coherent state $|\alpha\rangle$ has a well-defined phase and is typically represented as a superposition of Fock (photon number) states.

For a coherent state $|\alpha\rangle$, the probability of distribution of the number of photons n in the state is given by a Poisson distribution:

$$p_{n=}\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \tag{20}$$

The phase-sensitive entropy can be used to quantify the uncertainty in the phase of the quantum state. For example, in a coherent state $|\alpha\rangle$, the phase of the field (described by arg (α)) can be extracted using the entropy:

$$H_{phase} = -\sum_k |p_k|^2 \arg(p_k) \tag{21}$$

where p_k represents the probability of measuring a photon in state $|k\rangle$. This entropy measures the phase uncertainty of the system, which is important when analyzing quantum noise in optical systems.

For example, in quantum cryptography protocols like quantum key distribution (QKD), understanding the phase of quantum signals is essential for ensuring the security of the transmission.

4.4.3 Statistical mechanics: Entropies in thermodynamics

In statistical mechanics, phase-sensitive entropy can be used to understand the distribution of energy states or particle velocities in complex systems such as gases or liquids. For instance, in a system of interacting particles, the probability distribution p_k may correspond to the occupation probabilities of certain states in a thermodynamic system.

The phase-sensitive entropy captures the distribution of these states, specifically the phase or angular momentum associated with particle configurations. This is particularly relevant when studying systems with rotational symmetries or spin systems.

For example, in a spin system, the probability distribution of the spin states $p_k = |\langle s_k|\psi\rangle|^2$ could be complex valued if the spins are interacting in a quantum field. The phase-sensitive entropy quantifies the uncertainty about the orientation of the spins in space, which is critical for studying spin glass systems or magnetic systems at different temperatures.

In the scientific context, the Complex Rényi entropy and the phase-sensitive entropy find applications across several key fields. In quantum mechanics, these measures are used to evaluate the mixedness and uncertainty of quantum states, with the phase-sensitive entropy being particularly useful for characterizing phase uncertainty in coherent states and optical systems. In quantum optics, phase-sensitive entropy provides crucial insights into phase uncertainty in coherent light, such as that produced by lasers, which is essential for quantum communication and quantum key distribution protocols. Finally, in statistical mechanics, both entropy measures are applied to complex systems like gases or spin systems, where phase-sensitive entropy aids in analyzing rotational symmetries and the distribution of energy states.

The key roles played by entropy measures in fields ranging from quantum information theory to statistical mechanics are illustrated by these examples. The measures in question are both complex Rényi and phase sensitive.

In quantum information theory, the Von Neumann entropy: $S(\rho) = -Tr(\rho log \rho)$ remains central, even when the density matrix ρ is complex-valued or non-Hermitian (e.g., in open systems described by Lindblad equations). In such cases, complex entropy quantifies coherence and decoherence processes beyond thermal uncertainty [3].

Furthermore, in mathematical thermodynamics and complex networks, entropy measures defined on complex-valued transition matrices or probability fluxes offer novel ways to capture dynamic imbalance, feedback, and self-organization in non-equilibrium systems.

4.4.4 Gaussian Wave-Packet Solution of the Free Schrödinger Equation

In one dimension the time-dependent free particle Schrödinger equation is

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} \tag{22}$$

Because this is a linear partial differential equation, any superposition of planewave solutions is itself a solution. A plane wave of wavenumber k has the form

$$\psi_k(x,t) = e^{i(kx-\omega_k t)}, \text{ with } \omega_k = \frac{\hbar k^2}{2m} \tag{23}$$

Hence one may write a general solution by Fourier superposition

$$\psi(x,t) = \int_{-\infty}^{\infty} A(k)\ e^{i(kx-\omega_k t)}\ dk \tag{24}$$

here $A(k)$ is the spectral amplitude in momentum space [15].

To obtain a localized wave-packet, one chooses $A(k)$ to be a Gaussian centered at k_0, for example

$$A(k) = \frac{1}{(2\pi\,\sigma_k^2)^{\frac{1}{4}}}\ e^{\left[\frac{(k-k_0)^2}{4\sigma_k^2}-ikx_0\right]} \tag{25}$$

Substituting (25) into (24) gives

$$\psi(x,t) = \frac{1}{(2\pi\,\sigma_k^2)^{\frac{1}{4}}}\int_{-\infty}^{\infty} e^{\left[-\frac{(k-k_0)^2}{4\sigma_k^2}-ikx_0+ikx-i\frac{\hbar k^2 t}{2m}\right]}dk \tag{26}$$

One rewrites $(k-k_0)^2 = k^2 - 2k\,k_0 + k_0^2$ and groups like powers of k in the exponent:

$$-\left(\frac{1}{4\sigma_k^2}+i\,\frac{\hbar t}{2m}\right)k^2 + \left(\frac{k_0}{2\sigma_k^2}+i\,(x-x_0)\right)k - \frac{k_0^2}{4\sigma_k^2}.$$

Defining

$$a = \frac{1}{4\sigma_k^2}+i\,\frac{\hbar t}{2m}, b = \frac{k_0}{2\sigma_k^2}+i\,(x-x_0),$$

the integral becomes a standard Gaussian integral in k:

$$\int_{-\infty}^{\infty} e^{-ak^2+bk}\,dk = \sqrt{\frac{\pi}{a}}\ e^{\frac{b^2}{4a}},\ \Re(a) > 0 \tag{27}$$

Applying this result and reorganizing the algebra yields the closed-form solution

$$\psi(x,t) = \left(\frac{1}{2\pi\,\sigma_t^2}\right)^{\frac{1}{4}} e^{\left[-\frac{(x-x_0-v_g\,t)^2}{4\,\sigma_t^2}+i(k_0\,x-\omega_0\,t+\phi(t))\right]} \tag{28}$$

where

$$\sigma_t = \sigma_0\sqrt{1+(\frac{\hbar\,t}{2\,m\,\sigma_0^2})^2}, v_g = \frac{\hbar\,k_0}{m}, \omega_0 = \frac{\hbar\,k_0^2}{2m}$$

and $\sigma_0 = 1/(2\,\sigma_k)$. Equation (28) is the well-known analytic Gaussian wave packet that propagates with group-velocity v_g and disperses (spreads) with time [2],[21].

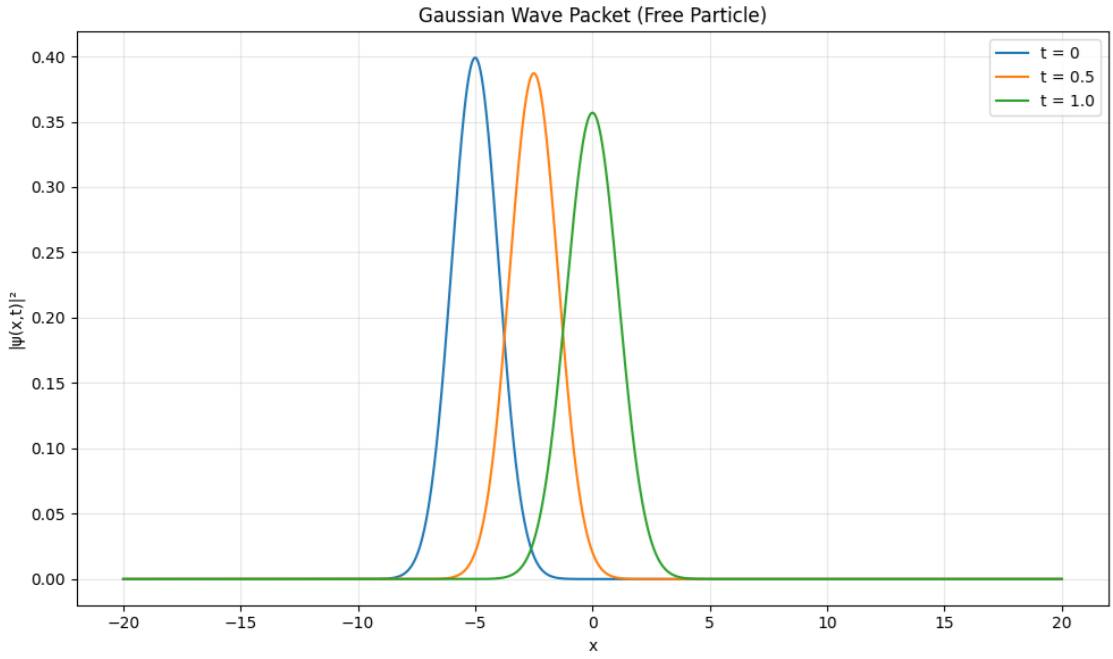

Figure 1. Time evolution of a Gaussian wave packet for a free particle. The curves show the probability density $|\,\psi(x,t)\,|^2$ at different times. The packet's center propagates with the group velocity $v_g = \hbar k_0/m$, while the wave packet spreads due to dispersion according to the time-dependent width σ_t.

An equivalent derivation uses the propagator (or kernel) representation:

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,t;x',0)\;\psi(x',0)\;dx' \tag{29}$$

with the free-particle propagator

$$K(x,t;x',0) = \sqrt{\frac{m}{2\pi\, i\, \hbar\, t}}\; e^{\left[\frac{i\, m\, (x-x')^2}{2\hbar t}\right]} \tag{30}$$

Substituting a Gaussian initial state $\psi(x',0)$ and evaluating the convolution integral again via Gaussian integrals reproduces (28) [25].

Finally, the Feynman path-integral formulation gives the same propagator by summing over all possible trajectories $x(t)$ with amplitude $e^{(iS[x(t)]/\hbar)}$ where $S[x(t)]$ is the classical action [9]. The integrand is complex-analytic and Gaussian in discretized form; the result is the same kernel (30). This reveals that the analytic (holomorphic) structure of the quantum evolution underlies the Gaussian wave-packet solution [19].

Physically, the Gaussian wave-packet has both a localized envelope (governed by the real Gaussian term) and an oscillatory carrier (complex exponential). Its normalization ensures $\int\ |\psi(x,t)|^2\ dx = 1$. The dispersion σ_t grows in time because different Fourier components of the spectral amplitude have different phase velocities ω_k/k, leading to spreading of the packet [7]. The analytic form therefore provides a complete demonstration that a superposition of plane waves (with Gaussian weight) propagates under the free Schrödinger equation into an analytic Gaussian wave packet, connecting Fourier methods, propagator approach and path-integral interpretation.

4.4.4.1 Gaussian Wave Packet in the Harmonic Oscillator

A Gaussian wave packet in a one-dimensional harmonic potential provides a paradigmatic example of quantum dynamics that closely resembles classical motion. The harmonic potential is defined as $V(x) = \frac{1}{2}m\omega^2x^2$, and the corresponding time-dependent Schrödinger equation is

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2\right]\psi(x,t) \tag{31}$$

An initial Gaussian wave packet centered at position x_0 with momentum $p_0 = \hbar k_0$ can be expressed as

$$\psi(x,0) = \frac{1}{(\pi\sigma_0^2)^{\frac{1}{4}}}\, e^{\exp\left[-\frac{(x-x_0)^2}{2\sigma_0^2} + ik_0x\right]} \tag{32}$$

where σ_0 determines the initial width, x_0 is the initial position, and k_0 defines the central momentum. If the width is chosen as $\sigma_0 = \sqrt{\hbar/(m\omega)}$, the wave packet is a coherent state, which oscillates in the harmonic potential without changing shape. In this case, the analytical solution is

$$\psi(x,t) = \frac{1}{(\pi\sigma_0^2)^{\frac{1}{4}}}\exp\left[-\frac{(x-x_c(t))^2}{2\sigma_0^2} + i\frac{p_c(t)}{\hbar}x + i\phi(t)\right] \tag{33}$$

where the center of the packet follows $x_c(t) = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t)$ and the momentum evolves as $p_c(t) = p_0 \cos(\omega t) - m\omega x_0 \sin(\omega t)$ [5],[26]. This shows that the expectation values of position and momentum, $\langle x(t) \rangle$ and $\langle p(t) \rangle$, follow classical harmonic motion, while the wave packet maintains its Gaussian shape.

For numerical simulations, the time evolution can be implemented using implicit methods such as Crank–Nicolson, which preserves the norm and stability of the solution, or split-step Fourier methods, which alternate between kinetic and potential evolution in small time steps [8]. These methods allow one to propagate the wave packet efficiently and observe its dynamics. For Gaussian wave packets that are not coherent states ($\sigma_0 \neq \sqrt{\hbar/(m\omega)}$), the packet still oscillates around the potential minimum but exhibits a breathing behavior in its width due to interference between different momentum components.

Visualizations of $|\psi(x,t)|^2$ reveal the packet's oscillation and, for coherent states, the invariance of its width. Observables such as $\langle x(t) \rangle$ and $\langle p(t) \rangle$ track classical trajectories, while $\sigma_x(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ remains constant for coherent states but oscillates for non-coherent packets. Such simulations not only provide intuition about the connection between classical and quantum dynamics but also illustrate fundamental quantum phenomena such as coherent states, dispersion, and the role of momentum distribution in temporal evolution [5], [8],[26].

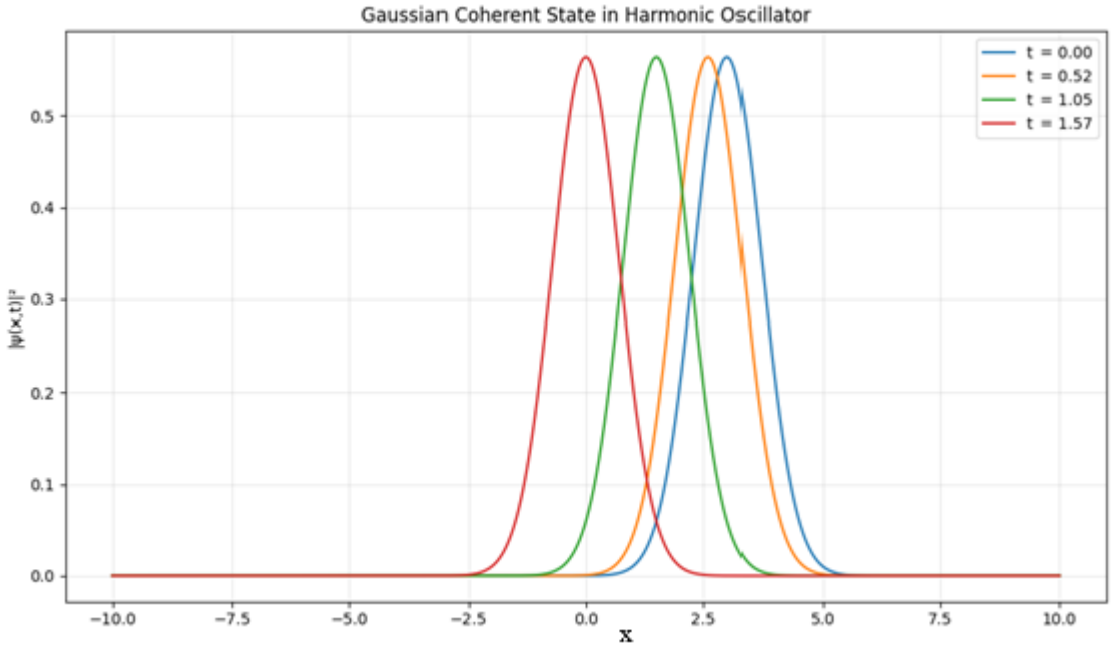

Figure 2. Dynamics of a Gaussian wave packet in a harmonic oscillator potential. The initial state is a coherent state with width $\sigma_0 = \sqrt{\hbar/(m\omega)}$, resulting in nondispersive motion. The probability densities $|\psi(x,t)|^2$ are shown at several instants. The center of the packet follows the classical trajectory $x_c(t)$, while the width remains constant, demonstrating the correspondence between classical and quantum dynamics in the harmonic oscillator [5],[26].

The time evolution of a Gaussian wave packet in a one-dimensional harmonic oscillator is simulated using the split-step Fourier method. Initially, the wave packet is defined as a coherent state, characterized by a specific

width $\sigma_0 = \sqrt{\hbar/(m\omega)}$, which ensures minimal dispersion during evolution. The harmonic potential is represented as $V(x) = \frac{1}{2}m\omega^2x^2$, and the wave function is propagated in small time steps by alternating between kinetic and potential evolution. In momentum space, the kinetic operator is applied via a Fourier transform, while the potential operator is applied directly in position space. This approach preserves the norm of the wave function and allows efficient computation of the dynamics over many time steps.

The resulting simulation provides a clear visualization of the quantum-classical correspondence. The center of the Gaussian packet oscillates sinusoidally around the potential minimum, following the trajectory predicted by classical mechanics, while the shape of the packet remains approximately Gaussian, reflecting the properties of a coherent state. Observables such as the expectation values of position and momentum, $\langle x(t) \rangle$ and $\langle p(t) \rangle$, follow the classical harmonic motion, demonstrating that coherent quantum states can exhibit dynamics that closely resemble classical particles. Additionally, the simulation illustrates the concept of wave packet dynamics in quantum systems: the spread, interference, and oscillation of the probability density $| \psi(x,t) |^2$ can be directly observed, providing an intuitive understanding of fundamental quantum phenomena in the context of the harmonic oscillator.

Overall, this code and the corresponding animation serve as a powerful pedagogical and research tool, linking analytic solutions, numerical methods, and physical interpretation. It allows students and researchers to visualize how quantum states evolve over time and how coherent states bridge the gap between classical and quantum descriptions of motion, offering insights into both the mathematical structure and physical intuition of quantum mechanics [5], [8],[26].

V. Numerical implementation and simulation

This section illustrates practical aspects of complex systems through implementation and visualization of two fundamental phenomena: (1) complex-valued stochastic processes, including complex Brownian motion and stochastic differential equations (SDEs); and (2) non-Hermitian random matrices, with an emphasis on the eigenvalue distributions of the Ginibre ensemble. These simulations offer empirical insights into the geometric and statistical properties inherent to complex-valued random systems.

5.1 Complex Brownian motion simulation

We define a complex Brownian motion $B_t = X_t + iY_t$, where X_t and Y_t are independent real-valued Brownian motions. The trajectory lies in the complex plane and exhibits isotropic diffusion. Figure 1 illustrates a simulated trajectory of a complex Brownian motion B_t, where X_t and Y_t are two independent real Brownian motions. The horizontal axis corresponds to the real part $\Re$(B(t)), while the vertical axis corresponds to the imaginary part $\Im$(B(t)). The path in blue depicts the stochastic evolution of the process in the complex plane. The green dot indicates the starting point (t = 0), and the red dot marks the final position at time t = T.

As in the real-valued case, the trajectory is continuous but nowhere differentiable, showing the irregular oscillations typical of Brownian paths. The complex extension, however, generates a bidimensional diffusion process that explores both the real and imaginary directions simultaneously. Such processes play a central role in complex stochastic analysis, conformal invariance, and quantum field theory [12],[17],[27].

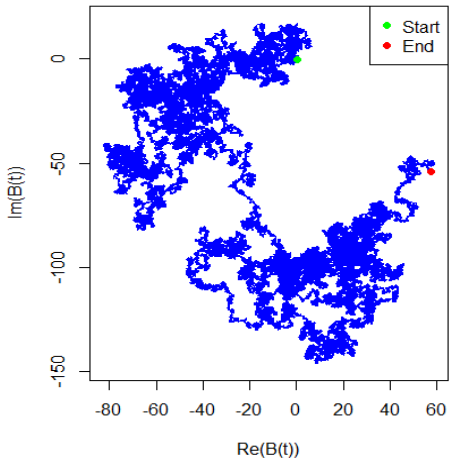

Figure 3. Simulated trajectory of a complex Brownian motion in the complex plane, with the starting position (green) and the final position (red).

This simulation shows a random walk in $\mathbb{C}$, with no preferential direction. The isotropy reflects the independence and identical variance of the real and imaginary parts.

5.2 Complex-valued SDE: Ornstein–Uhlenbeck process

We now simulate a complex Ornstein–Uhlenbeck process, governed by the SDE:

$$dZ_t = -\theta Z_t dt + \sigma dW_t$$

where $Z_t \in \mathbb{C}$, $\theta > 0$, and W_t is complex Brownian motion.

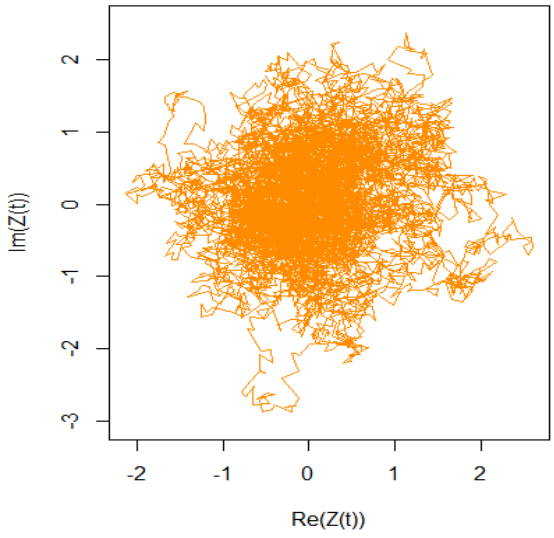

Figure 4. Trajectory of a complex Ornstein–Uhlenbeck process.

This process models a mean-reverting complex signal, such as quantum noise or a decaying analytic field. The center-attraction imposed by $-\theta Z_t$ keeps the trajectory localized.

5.3 Eigenvalue distributions of Ginibre matrices

To visualize non-Hermitian spectral statistics, we simulate the Ginibre ensemble- random matrices with i.i.d. complex Gaussian entries-and plot their eigenvalues.

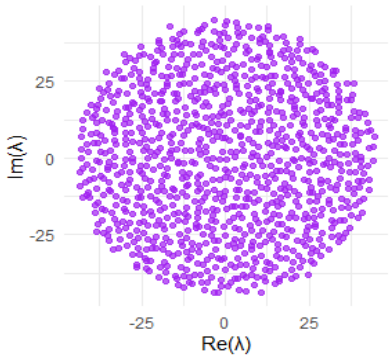

Figure 5. Eigenvalue distribution of a 1000×1000 times Ginibre matrix.

The eigenvalues lie approximately in a circular disk in $\mathbb{C}$, as predicted by the circular law [28]. This demonstrates a key difference from Hermitian ensembles, whose spectra lie on $\mathbb{R}$.

These simulations demonstrate that complex-valued stochastic processes and operators reveal dynamics that are inherently inaccessible to purely real-valued models. Potential extensions of this work include simulating PT-symmetric Hamiltonians with complex eigenvalues and verifying their biorthogonality, exploring complex Lévy flights and other non-Gaussian analytic processes, and estimating complex entropy from simulated density.

5.4 Entropy estimation and fidelity measures for complex density matrices

In complex probability and quantum mechanics, density matrices $\rho \epsilon \mathbb{C}^{nxn}$ represent the state of mixed systems. We extend our simulations by: Estimating the von Neumann entropy $S(\rho) = -Tr(\rho log \rho)$ and computing the fidelity between two density matrices $F(\sigma, \sigma) = \left[Tr(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})\right]^2$.

This allows us to quantify information content**,** coherence, and closeness between quantum-like complex states.

Visualizing the eigenvalue distributions helps understand entropy and state similarity.

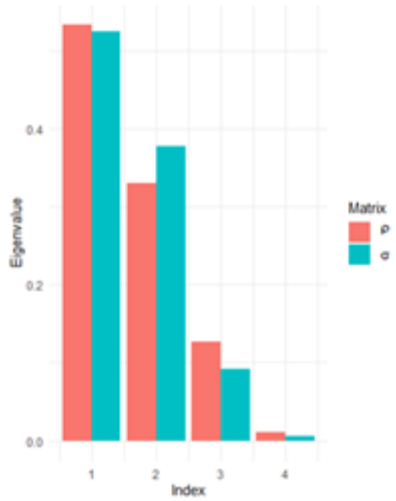

Figure 6. Eigenvalue spectra of two complex density matrices used to compute von Neumann entropy and fidelity.

The bar plot displays the four largest eigenvalues (ranked by magnitude) for each matrix, plotted against their corresponding index on the x-axis. The y-axis represents the magnitude of the eigenvalues, providing insight into the variance captured by each principal component or dimension associated with the respective matrix.

Both matrices exhibit a similar decay pattern in their eigenvalue distributions, with the majority of the variance concentrated in the first two components. At index 1, the eigenvalue associated with the ρ matrix is slightly higher than that of σ, with both values approaching 0.5. This suggests that the leading principal component of ρ captures slightly more variance than that of σ. In contrast, at index 2, σ shows a notably larger eigenvalue than ρ, indicating that the second dimension in σ explains a greater proportion of variance compared to the same component in ρ.

For indices 3 and 4, the eigenvalues drop significantly for both matrices, each falling below 0.15. At index 3, the matrix σ retains a marginally higher eigenvalue than ρ, while at index 4, the pattern reverses, with ρ having a slightly larger, albeit negligible, eigenvalue. These observations indicate that the underlying structures of both matrices are predominantly low-dimensional, with most of the informative variance captured in the first two components.

Overall, the similarity in eigenvalue decay across the two matrices suggests comparable spectral characteristics, though minor differences in the contributions of specific components may reflect subtle structural or statistical distinctions. This comparison of spectral profiles is relevant for assessing matrix similarity in applications such as covariance modeling, dimensionality reduction, or signal processing.

VI. DISCUSSION

Theoretical foundations and numerical applications of probability theory in complex spaces have been explored in this research, with perspectives from mathematical analysis, quantum theory and stochastic modelling being unified. By extending classical probabilistic concepts to accommodate complex-valued

measures**,** random variables**,** and density matrices, we have exemplified the utility and necessity of this framework across several scientific domains.

The extension of probability into the complex domain raises fundamental questions regarding interpretability, normalization, and measure-theoretic rigor. While classical probability theory is confined to real-valued, non-negative measures, complex probability relaxes these constraints, enabling richer mathematical structures such as probability amplitudes in quantum mechanics, complex-valued characteristic functions and their analytic continuations, and non-Hermitian operators with spectra in the complex plane. These generalizations, though mathematically subtle, are essential for accurate modeling interference, coherence, and dynamical transitions in quantum systems. The emergence of biorthogonal inner products, PT-symmetry, and complex-valued entropy measures further supports the view that classical probability is a limiting case of a broader, more general complex framework.

Our simulations exemplified the computational feasibility and empirical depth of complex-valued probabilistic structures. Complex Brownian motion and Ornstein–Uhlenbeck processes capture geometric behaviors in $\mathbb{C}$, modeling analytic signals and quantum fields. The Ginibre ensemble reveals how eigenvalue distributions of non-Hermitian random matrices form rotationally symmetric patterns in the complex plane, in stark contrast to the real spectra of Hermitian matrices. We quantify uncertainty and state similarity in complex-valued density matrices using von Neumann entropy estimation and fidelity calculations. These are key metrics in quantum information and non-equilibrium thermodynamics. Collectively, these simulations validate the theoretical framework and offer a versatile toolkit for future experimental and computational studies.

The theory and simulations presented here open multiple avenues for future research. Formal foundations remain an open challenge, particularly the development of a complete axiomatic system for complex probability in infinite-dimensional settings or when time is treated as a complex variable. Advances in numerical methods are essential, including efficient algorithms for simulating complex stochastic differential equations, computing matrix functions (e.g., ρ, log (ρ)), and visualizing spectral behavior for practical applications. In quantum computing, as algorithms increasingly depend on fidelity and entropy estimation under noise and decoherence, complex probability will play a pivotal role. Furthermore, exploring the information geometry of complex probability spaces may uncover new invariants, symmetries, and constraints relevant to physical and computational systems. As non-Hermitian physics and complex network models gain prominence, complex probability is poised to become not merely an abstract generalization but a fundamental mathematical framework for contemporary science.

REFERENCES

[1]. Bender, C. M., & Boettcher, S. (1998). Real spectra in non-Hermitian Hamiltonians having PT symmetry. Physical Review Letters, 80(24), 5243–5246. https://doi.org/10.1103/PhysRevLett.80.5243

[2]. Bräuer, K. (2018). 5.3 Gaussian wave packet as solution of the free Schrödinger equation [Lecture notes]. University of Tübingen. https://unituebingen.de/fileadmin/Uni_Tuebingen/Fakultaeten/MathePhysik/Institute/ITP/Braeuer/Skripte/2018_Basic_Physics_Course/Chap05_3.pdf

[3]. Breuer, H.-P., & Petruccione, F. (2002). The theory of open quantum systems. Oxford University Press.

[4]. Cervantes-López, E. (2025). Quantum entanglement in chaotic systems: The role of the quantum Baker's transformation. International Journal of Modern Engineering Research, 15(5), 73–85. https://www.researchgate.net/publication/396358458_Quantum_Entanglement_in_Chaotic_Systems_The_Role_of_the_Quantum_Baker's_Transformation

[5]. Cohen-Tannoudji, C., Diu, B., & Laloë, F. (1991). Quantum mechanics (Vol. 1). Wiley.

[6]. Dirac, P. A. M. (1981). The principles of quantum mechanics (4th ed.). Oxford University Press. (Original work published 1930).

[7]. Drexel University. (2021). Evolution of a wave packet. https://physics.drexel.edu/~steve/Courses/Comp_Phys/Physics325/wavepacket_evolution.pdf

[8]. Feit, M. D., Fleck, J. A., & Steiger, A. (1982). Solution of the Schrödinger equation by a spectral method. Journal of Computational Physics, 47(3), 412–433. https://doi.org/10.1016/0021-9991(82)90143-4

[9]. Feynman, R. P., & Hibbs, A. R. (1965). Quantum mechanics and path integrals. McGraw-Hill.

[10]. Gabor, D. (1946). Theory of communication. Journal of the Institution of Electrical Engineers - Part III: Radio and Communication Engineering, 93(26), 429–457. https://doi.org/10.1049/ji-3-2.1946.0074

[11]. Ginibre, J. (1965). Statistical ensembles of complex, quaternion, and real matrices. Journal of Mathematical Physics, 6(3), 440–449. https://doi.org/10.1063/1.1704292

[12]. Hida, T. (1980). Brownian motion. Springer.

[13]. Hida, T., & Hitsuda, M. (1993). Gaussian processes. American Mathematical Society.

[14]. Hough, J. B., Krishnapur, M., Peres, Y., & Virág, B. (2009). Zeros of Gaussian analytic functions and determinantal point processes. American Mathematical Society.

[15]. Huber, D., & Littlejohn, R. G. (1988). Generalized Gaussian wave packet dynamics, Schrödinger equation, and stationary phase approximation. The Journal of Chemical Physics, 89.
[16]. Khrennikov, A. (2009). Interpretations of probability (2nd ed.). De Gruyter.
[17]. Klebaner, F. C. (2012). Introduction to stochastic calculus with applications (2nd ed.). Imperial College.
[18]. Kolmogorov, A. N. (1950). Foundations of the theory of probability (N. Morrison, Trans.). Chelsea Publishing Company. (Original work published 1933).
[19]. Kozdron, M. (2008). Wave packet (Chapter 3). En A. Einstein's proof of the existence of Brownian motion. https://en.wikipedia-on-ipfs.org/wiki/Wave_packet
[20]. Lukacs, E. (1970). Characteristic functions (2nd ed.). Griffin.
[21]. PhysicsPages. (2021, January 7). Free particle as Gaussian wave packet. https://physicspages.com/pdf/Quantum%20mechanics/Free%20particle%20as%20Gaussian%20wave%20packet.pdf
[22]. Mostafazadeh, A. (2002). Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. Journal of Mathematical Physics, 43(1), 205–214. https://doi.org/10.1063/1.1418246
[23]. Nielsen, M. A., & Chuang, I. L. (2010). Quantum computation and quantum information (10th anniversary ed.). Cambridge University Press.
[24]. Parthasarathy, K. R. (1992). An introduction to quantum stochastic calculus. Birkhäuser.
[25]. Reed, C., & Wheeler, J. (n.d.). Gaussian wave packets: The free-particle propagator method. https://www.reed.edu/physics/faculty/wheeler/documents/Quantum%20Mechanics/Miscellaneous%20Essays/Gaussian%20Wavepackets.pdf.
[26]. Schleich, W. P. (2011). Quantum optics in phase space (2nd ed.). Wiley-VCH.
[27]. Simon, B. (2005). Trace ideals and their applications (2nd ed.). American Mathematical Society.
[28]. Tao, T., & Vu, V. (2010). Random matrices: Universality of local eigenvalue statistics. Acta Mathematica, 206(1), 127–204. https://doi.org/10.1007/s11511-010-0059-1
[29]. Yosida, K., & Hewitt, E. (1952). Finitely additive and countably additive complex measures. Transactions of the American Mathematical Society, 72(1), 46–66. https://doi.org/10.2307/1990633
[30]. Youssef, P. (2012). Complex probability theory and quantum mechanics. International Journal of Theoretical Physics, 51(6), 1905–1913. https://doi.org/10.1007/s10773-011-1054-4
[31]. Zemanian, A. H. (1987). Distribution theory and transform analysis: An introduction to generalized functions, with applications. Dover Publications.