Quest Journals
Journal of Research in Applied Mathematics
Volume 11 ~ Issue 12 (December 2025) pp: 51-62
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

On Purely Infinite $C*$-algebras Associated to Hausdorff - ´Etale Groupoids

Awad Mohamed[1], Abdellteaf Yahia[2], Shawgy Hussein[3]
[1.] *Ministry of Education Safiya girls Secondry School, Department of Mathematics.*
[2.] *Berhard Rolwers,Rolwers Zerspanungstechnik Schwichtelersts 36.49377Vechta-Deutschland Germany*
[3.] *Sudan University of science and Technology,college of science , Department of Mathematics Sudan*

Abstract
For G be a Hausdorff -´etale groupoid that is minimal and topologically principal. We show that the sequence $C_r^*(G)$ is purely infinite simple if and only if all the nonzero positive elements of $C_0\left(G^{(0)}\right)$ are infinite in the sequence of $C_r^*(G)$. If G is a Hausdorff-ample groupoid, then we also show that the sequence of $C_r^*(G)$ is purely infinite simple if and only if every nonzero projection in $C_0\left(G^{(0)}\right)$ is infinite in the sequence $C_r^*(G)$. We then also show how this result applies to k-graph C^*-algebras. Finally, we investigate strongly purely infinite groupoid C^*-algebras.

Key words. Groupoid; groupoid C^*-algebra ; purely infinite C^*-algebra; strongly purely infinite; topologically principal; k-graph.

*Received 05 Dec., 2025; **R**evised 13 Dec., 2025; **A**ccepted 15 Dec., 2025 © The author(s) 2025.*
P**ublished with open access at **www.questjournas.org

I. INTRODUCTION

Purely infinite simple C^*-algebras were introduced by [7] where he showed that the K_0 group of such algebras can be computed within the algebra itself without resorting to the usual direct limit construction. The K-theory groups of C^*-algebras have long been known to be computable invariants and Cuntz's result shows that this computation is easier when the C^*-algebra is purely infinite simple. Elliott initiated a program to find a suitably large class of C^*-algebras on which the K-theory groups provide a complete isomorphism invariant (see[8]).This program has achieved remarkable success, in a theorem of [11,20] which states that every Kirchberg algebra satisfying the Universal Coefficient Theorem (UCT) is classified by the isomorphism classes of its ordered K-theory groups .A Kirchberg algebra is a separable, nuclear,purely infinite simple C^*-algebra .The best of classification by the Kirchberg-Phillips theorem has lead many to study when various constructions of C^*-algebras yield purely infinite simple algebras. Kumjian, Pask and Raeburn show that a graph C^*-algebra of a cofinal graph is purely infinite simple if and only if every vertex can be reached from a loop with an entrance [17, Theorem 3.9]. Carlsen and Thomsen show that if the C^*-algebra constructed from a locally injective surjection θon a compact metric space of finite covering dimension is simple, then it is purely infinite simple if and only if θ is not a homeomorphism [5]. [24] show that if a countable exact group H acts by an essentially free action on the Cantor set $X$and the type semigroup of clopen subsets of $X$is almost imperforated, then $C_0(X) \rtimes_r H$ is purely infinite if and only if every clopen set E in X is paradoxical .The constructions in each of the above examples are special cases of groupoid C^*-algebras.

We investigate purely infinite C^*-algebras associated to Hausdorff - ´etale groupoids .We restrict our attention to simple groupoid C^*-algebras. Characterising simplicity of groupoid C^*-algebras is known and we use the following theorem from [4]:

Theorem1.1(Brown-Clark-Farthing-Sims).Let G be a second-countable, locally compact , Hausdorff -´etale groupoid .Then $C^*(G)$ is simple if and only if all of the following conditions are satisfied .

(i) $C^*(G) = C_r^*(G)$;
(ii) G is topologically principal ;
(iii) G is minimal .

However, necessary and sufficient conditions on the groupoid for the associated algebra to be purely infinite simple are not known. Anantharaman-Delaroche showed that 'locally contracting' is a sufficient condition on the groupoid in [1] but whether locally contracting is necessary remains an open question . Part of the difficulty in characterizing those groupoids that give rise to purely infinite simple C^*-algebras is relating arbitrary projections in the groupoid C^*-algebra to the groupoid itself . We show a necessary and sufficient conditions for ensuring pure infiniteness of groupoid C^*-algebras .We show that for a Hausdorff -´etale, topologically principal, and minimal groupoid G the C^*-algebra of the sequence of $C_r^*(G)$ is purely infinite simple if and only if all the non-zero positive elements of $C_0\left(G^{(0)}\right)$ are infinite in the sequence $C_r^*(G)$ .We specialize to Hausdorff-ample groupoids . This is an important class of examples because every Kirchberg algebra in UCT is Morita equivalent to an algebra associated to a Hausdorff-ample groupoid (see [26]). We show in Theorem 4.1 for a Hausdorff-ample groupoid G, that is also topologically principal and minimal, the C^*-algebra as the sequence $C_r^*(G)$ is purely infinite if and only if every nonzero projection in $C_0\left(G^{(0)}\right)$ is infinite in the sequence $C_r^*(G)$ .Theorem 4.1 is a generalization of [10] about partial actions. We demonstrate how Theorem 3.3 applies to k-graph C^*-algebras .We turn our attention to the non-simple case. In [13], the other introduce three separate notions of purely infinite C^*-algebras: weakly purely infinite,purely infinite and strongly purely infinite. Of these notions, the last one appears to be the most useful in the classification theory of non-simple C^*-algebras. Indeed [12] showed that two separable, nuclear, strongly purely infinite C^*-algebras with the same primitive ideal space X are isomorphic if and only if they are KK_X- equivalent .We provide a characterization of when groupoid C^*-algebras are strongly purely infinite in Proposition 6.3.

2.1. Groupoids . A groupoid G is a small category in which every morphism is invertible . The set of objects in $G$is identified with the set of identity morphisms and both are denoted by $G^{(0)}$. We call $G^{(0)}$ the unit space of G . Each morphism γ_j in the category has a range and source denoted $r(\gamma_j)$ and $s(\gamma_j)$ respectively and thus r and s define maps $G \to G^{(0)}$. A topological groupoid is a groupoid with a topology in which composition is continuous and inversion is a homeomorphism . An open bisection in a topological groupoid G is an open set Bsuch that both r and s restricted to B are homeomorphisms; in particular these restrictions are injective .An ´etale groupoid is a topological groupoid where s is a local homeomorphism . If a groupoid G is Hausdorff -´etale, then the unit space $G^{(0)}$ is open and closed in G. If G is a locally compact , Hausdorff groupoid, then G is ´etale if and only if there is a basis for the topology on G consisting of open bisections with compact closure. A topological groupoid is called ample if it has a basis of compact open bisections. If G is locally compact, Hausdorff -´etale groupoid, then we note that G is ample if and only if $G^{(0)}$ is totally disconnected (see [13]). For a subsets $L, K \subseteq G$, denote $LK = \{\gamma_j : \gamma_j = \xi_j \zeta_j \text{ with } \xi_j \in L, \zeta_j \in K, s(\xi_j) = r(\zeta_j)\}$. With a slight abuse of notation , for $u_j \in G^{(0)}$, we write $u_j G$ and Gu_j for $\{u_j\}G$ and $G\{u_j\}$ respectively and denote by $u_j G u_j$ the set

$$\{\gamma_j \in G : r(\gamma_j) = s(\gamma_j) = u_j\}.$$

A topological groupoid G is topologically principal if the set $\{u_j \in G^{(0)} : u_j G u_j = \{u_j\}\}$ is dense in $G^{(0)}$, and minimal if $G.u_j := \{r(\gamma_j): s(\gamma_j) = u_j\}$ is dense in $G^{(0)}$ for all $u_j \in G^{(0)}$. Recall, for a second countable, locally compact, Hausdorff-´etale groupoid G the algebra $C^*(G)$is simple if and only if G is minimal, topologically principal, and $C^*(G) = C_r^*(G)$.

2.2. Groupoid C^*-algebras. Let G be locally compact , Hausdorff -´etale groupoid and let $C_{c_r}(G)$ denote the set of compactly supported continuous functions from G to C . Since every element γ_j of Ghas a neighbourhood B such that $r|_{Br}$ is injective , the set $r^{-1}(u_j)$ is discrete for every $u_j \in G^{(0)}$. Thus if $f_j \in C_{c_r}(G)$ then $\operatorname{supp}(f_j) \cap r^{-1}(u_j)$is finite for all $u_j \in G^{(0)}$. Hence we are able to define a convolution and involution on $C_{c_r}(G)$ such that for $f_j, g_j \in C_{c_r}(G)$,

$$(f_j * g_j)(\gamma_j) := \sum_j \sum_{r(\eta_j)=r(\gamma_j)} f_j(\eta_j) g_j(\eta_j^{-1}\gamma_j) \text{and } f_j^*(\gamma_j) := \overline{f_j(\gamma_j^{-1})}$$

Under these operations, $C_{c_r}(G)$ is a $*$-algebra . Next define for$f_j \in C_{c_r}(G)$,

$$\|f_j\|_I := \sup_{u_j \in G^{(0)}} \sum_j \left\{ \left(\sum_{\gamma_j \in Gu_j} |f_j(\gamma_j)| \sum_{\gamma_j \in Gu_j} |f_j(\gamma_j)| \right) \right\} \text{and}$$

$$\|f_j\|_I := \sup\{\|\pi(f_j)\| : \pi \text{ is a } \|.\|_I - \text{decreasing representation}\} .$$

Then $C_r^*(G)$ is the completion of $C_{c_r}(G)$ in the $\|.\|$-norm .

Given a unit $u_j \in G^{(0)}$, the regular representation π_{u_j} of $C_{c_r}(G)$ on $\ell^2(Gu_j)$ associated to u_j is characterized by

$$\pi_{u_j}(f_j)\delta_{\gamma_j} = \sum_j \sum_{s(\eta_j)=r(\gamma_j)} (f_j)\delta_{\eta_j\gamma_j} .$$

The reduced C^*-norm on $C_{c_r}(G)$ is $\|f_j\|_r = \sup\left\{\left\|\pi_{u_j}(f_j)\right\| : u_j \in G^{(0)}\right\}$ and $C_r^*(G)$ is the completion of $C_{c_r}(G)$ in the $\|.\|$-norm .We focused on the reduced C^*-algebra and situations where the reduced and full algebras coincide . Also, we assuming G is second-countable, implying $C_r^*(G)$ is separable [21] . Below we recall a few standard results. (see [28] .

Lemma2.1 (cf.[21]. Let G be a locally compact, Hausdorff - ´etale groupoid . Then

(i) The extension map from $C_{c_r}(G^{(0)})$ into $C_{c_r}(G)$ (where a function is defined to be zero on $G - G^{(0)}$) extends to an embedding of $C_{c_r}(G^{(0)})$ into $C_r^*(G)$.

(ii) The restriction map $E_0 : C_{c_r}(G) \to C_{c_r}(G^{(0)})$ extends to a conditional expectation $E : C_r^*(G) \to C_{c_r}(G^{(0)})$.

(iii) The map E from item (ii) is faithful.That is , $E(b_r^* b_r) = 0$ mplies $b_r = 0$ for $b_r \in C_r^*(G)$.

(iv) For every closed invariant set $D \subseteq G^{(0)}$ we have the following commuting diagram :

$$\begin{array}{ccccc} 0 \to C_r^*(G|_U) & \xrightarrow{l_r} & C_r^*(G) & \xrightarrow{\rho_r} & C_r^*(G|_D) \to 0 \\ E_U \downarrow & & E \downarrow & & E_D \downarrow \\ 0 \to C_0(U) & \xrightarrow{l_0} & C_0(G^{(0)}) & \xrightarrow{\rho_0} & C_0(D) \to 0 \end{array}$$

Where $U = G^{(0)} - D, l_r$ and ρ_r are determined on continuous functions by extension and restriction respectively. Moreover, image$(l_r) \subseteq \ker \rho_r$.

(v) The subalgebra $C_{c_r}(G^{(0)})$ contains an approximate unit for $C_r^*(G)$.

Proof: (i) Since G is Hausdorff -´etale ,$G^{(0)}$is open and closed in G. Thus, the map$C_{c_r}(G^{(0)})$ into$C_{c_r}(G)$ is well defined . For $f_j, g_j \in C_{c_r}(G^{(0)})$, a quick computation gives

$$(f_j * g_j)(\gamma_j) = \sum_j \begin{cases} f_j(\gamma_j) g_j(\gamma_j) , & \text{if } \gamma_j \in G^{(0)}; \\ 0 , & \text{otherwise,} \end{cases}$$

so the map from $C_{c_r}(G^{(0)})$ into $C_{c_r}(G)$ is a $*$-homomorphism .We claim the map is isometric, that is, we claim the reduced norm agrees with the infinity norm for functions in$C_{c_r}(G^{(0)})$. By evaluating at point masses in $\ell^2(Gu_j)$, one can show that $\|f_j\|_\infty \leq \|f_j\|_r$ for $f_j \in C_{c_r}(G)$ .The reverse inequality can be verified for $f_j \in C_{c_r}(G^{(0)})$ and the claim follows. Thus the$*$-homomorphism from $C_{c_r}(G^{(0)})$ into $C_{c_r}(G) \subseteq C_r^*(G)$ extends by continuity to an isometric(hence injective)$*$-homomorphism from $C_{c_r}(G^{(0)})$ into $C_r^*(G)$.

(ii) Once again using that G is Hausdorff - `etale , we have that $G^{(0)}$ is open and closed in G and hence E_0 is well defined . One may easily verify that E_0 is (a) positive(b) linear (c) idempotent, and (d) of norm one. Therefore E_0 extends by continuity to a map $E : C_r^*(G) \to C_{c_r}(G^{(0)})$ with the same properties (a)–(d). By (iii) we conclude that E is a conditional expectation .

(iii) Let $b_r \in C_r^*(G)$ such that $E(b_r^* b_r) = 0$. We need to show that $b_r = 0$. Let

$$V_{\gamma_j} : \mathbb{C} \to \ell^2\left(G_{s(\gamma_j)}\right) \text{ be given by } c_r \longmapsto c_r \delta_{\gamma_j} .$$

Then $V_{\gamma_j}^* \omega = \omega(\gamma_j)$. Since

$$\|b_r\|_r = \sup_{u_j \in G^{(0)}} \left\|\pi_{u_j}(b_r)\right\| \quad \text{and}$$

$$\left\|\pi_{u_j}(b_r)\delta_{\gamma_j}\right\|^2 = \langle \pi_{u_j}(b_r)\delta_{\gamma_j}, \pi_{u_j}(b_r)\delta_{\gamma_j}\rangle = \langle \pi_{u_j}(b_r^* b_r)\delta_{\gamma_j}, \delta_{\gamma_j}\rangle = V_{\gamma_j}^* \pi_{u_j}(b_r^* b_r) V_{\gamma_j} 1 ,$$

it suffices to show that $V_{\gamma_j}^* \pi_{u_j}(b_r^* b_r) V_{\gamma_j} = 0$ for all $u_j \in G^{(0)}$ and $\gamma_j \in G$. For $f_j \in C_{c_r}(G)$, $u_j \in G^{(0)}$, and $c_r \in \mathbb{C}$, we have

$$V_{\gamma_j}^* \pi_{u_j}(f_j) V_{u_j} c_r = V_{\gamma_j}^* \pi_{u_j}(f_j) c_r \delta_{u_j} = \sum_j V_{\gamma_j}^* \left(\sum_{s(\eta_j)=u_j} f_j(\eta_j) c_r \delta_{u_j} \right) = f_j(u_j) c_r$$
$$= E(f_j)(u_j) c_r . \qquad (2.1)$$

Thus by the continuity of E, for all $a_r \in C_r^*(G)$, $E(a_r)(u_j) = V_{\gamma_j}^* \pi_{u_j}(a_r) V_{u_j}$ as operators on $\mathbb{C}$. For every open bisection B and $\gamma_j \in B$,pick a function $\phi_{\gamma_j,B} \in C_{c_r}(G)$ such that $\phi_{\gamma_j,B}(\gamma_j) = 1$, $\operatorname{supp}\left(\phi_{\gamma_j,B}\right) \subseteq B$, and $0 \leq \phi_{\gamma_j,B} \leq 1$. Now if $f_j \in C_{c_r}(G)$ and B is an open bisection with $\gamma_j \in B$,then

$$\left(E\left(\phi_{\gamma_j,B}^* f_j \phi_{\gamma_j,B}\right)\right)(u_j) = \sum_j \sum_{r(\xi_j)=r(\zeta_j)=u_j} \phi_{\gamma_j,B}(\xi_j^{-1}) f_j(\xi_j^{-1}\zeta_j) \phi_{\gamma_j,B}(\zeta_j^{-1}) ,$$

which is zero unless $\xi_j, \zeta_j \in B^{-1}$. Since $r(\xi_j) = r(\zeta_j) = u_j$,we have that $\xi_j = \zeta_j$ is the unique element of $u_j B^{-1}$. So

$$\left(E\left(\phi_{\gamma_j,B}^* f_j \phi_{\gamma_j,B}\right)\right)(u_j) = \phi_{\gamma_j,B}(\xi_j^{-1}) f_j\left(s(\zeta_j)\right) \phi_{\gamma_j,B}(\xi_j^{-1}) \leq E(f_j)\left(s(\zeta_j)\right)$$
$$\leq \|E(f_j)\|_\infty . \qquad (2.2)$$

Now if $a_r \in C_r^*(G)$ then $\phi_{\gamma_j,B}^* a_r^* a_r \phi_{\gamma_j,B}$ is positive so $E\left(\phi_{\gamma_j,B}^* a_r^* a_r \phi_{\gamma_j,B}\right) \geq 0$. Therefore by the continuity of Ewe can apply (2.2) to obtain

$$0 \leq E\left(\phi_{\gamma_j,B}^* b_r^* b_r \phi_{\gamma_j,B}\right) \leq \|E(b_r^* b_r)\|_\infty = 0 .$$

Thus $E\left(\phi_{\gamma_j,B}^* b_r^* b_r \phi_{\gamma_j,B}^*\right) = 0$ for all open bisections B and $\gamma_j \in B$. For $\gamma_j \in G$ pick an open bisection B such that $\gamma_j \in B$. Notice for $c_r \in \mathbb{C}$

$$\pi_{s(\gamma_j)}\left(\phi_{\gamma_j,B}^*\right) V_{s(\gamma_j)} c_r = \pi_{s(\gamma_j)}\left(\phi_{\gamma_j,B}^*\right) c_r \delta_{s(\gamma_j)}$$
$$= \sum_j \sum_{s(\eta_j)=s(\gamma_j)} \phi_{\gamma_j,B}(\eta_j) c_r \delta_{\eta_j} = c_r \delta_{\gamma_j} = V_{\gamma_j} c_r$$

Thus $\pi_{s(\gamma_j)}\left(\phi_{\gamma_j,B}\right) V_{s(\gamma_j)} = V_{\gamma_j}$ as operators . Now by equation (2.1) and the above observation we get for all $\gamma_j \in G$ that

$$V_{\gamma_j}^* \pi_{u_j}(b_r^* b_r) V_{\gamma_j} = V_{s(\gamma_j)}^* \pi_{s(\gamma_j)}\left(\phi_{\gamma_j,B}^* b_r^* b_r \phi_{\gamma_j,B}\right) V_{s(\gamma_j)}^* = E\left(\phi_{\gamma_j,B}^* b_r^* b_r \phi_{\gamma_j,B}\right) = 0$$

as desired . Therefore $b_r = 0$ and hence $E$is faithful .

(iv) The diagram commutes when restricting to continuous functions with compact support . Commutatively then passes to the respective completions by continuity. Since we know $\rho_r\left(l_r(f_j)\right) = 0$ for all $f_j \in C_{c_r}(G|_U)$ we obtain image $(l_r) \subseteq \ker \rho_r$ by continuity.

(v) Let $\mathbb{C}$ be the set of compact sets in $G^{(0)}$ordered by inclusion . For each $G \in C$ pick a function e_C in $C_{c_r}\left(G^{(0)}\right)$ such that $0 \leq e_C \leq 1$ and $e_C|_{c_r} \equiv 1$. Fix $f_j \in C_{c_r}(G)$ vanishing outside a compact set $K \subseteq G$. For C such that $s(K) \subset C, f_j * e_C = f_j$. It follows that (e_C) is an approximate unit for $C_r^*(G)$.

2.3. Purely infinite simple C^*-algebras .Given a C^*-algebra A we denote its positive elements by A^+. If $B$is a subalgebra of A then $B^+ \subset A^+$. In particular, if $C_0(X)$ is an abelian subalgebra of A and $f_j \in C_0(X)$ such that $f_j(x_n) \geq 0$ for all $x_n \in X$, then $f_j \in A^+$. For positive elements $a_r \in M_{n_0}(A)$ and $b_r \in M_{m_0}(A)$, a_r is Cuntz below b_r, denoted $a_r \prec b_r$, if there exists a sequence of elements x_{k_0} in $M_{m_0,n_0}(A)$ such that $x_{k_0}^* b_{rk} \to a_r$ in norm . Notice that $\precsim$ is transitive : if $a_r \precsim b_r$ and $b_r \precsim c_r$ there exist sequences of element x_{n_0} and y_{n_0} such that $x_{k_0}^* b_r x_{k_0} \to a_r$ and $y_{n_0}^* c_r y_{n_0} \to b_r$ in norm, so $x_{n_0}^* y_{n_0}^* c_r y_{n_0} x_{n_0} \to a_r$ in norm , that is $a_r \precsim c_r$. We say A is purely infinite if there are no characters on A and for all $a_r, b_r \in A^+$, $a_r \precsim b_r$ if and only if $a_r \in \overline{A b_r A}$ [14]. A non-zero positive element $a_r \in A$ is properly infinite if $a_r \oplus a_r \precsim a_r$. By [14] A is purely infinite if and only if every non-zero positive element in A is properly infinite. A projection p in a C^*-algebra A is infinite if it is Murray-von Neumann equivalent to a proper subprojection of itself, i.e., if there exists a partial isometrics such that $s^*s = p$ but $ss^* \lneqq p$. By [14] a C^*-algebra A is purely infinite if every non-zero hereditary C^*-subalgebra in every quotient of A contains an infinite projection . For simple C^*-algebras the converse is also true, thus a simple C^*-algebra is purely infinite precisely when every hereditary subalgebra contains an infinite projection .We consider, locally compact, Hausdorff - ´etale groupoids .We will show that we can determine when $C_r^*(G)$ is purely infinite simple by restricting our attention to elements of $C_0\left(G^{(0)}\right)$. Before we do that, we need the following technical lemmas .(see [28]).

Lemma3.1. Let G be a locally compact , Hausdorff - ´etale groupoid and $E: C_r^*(G) \to C_0\left(G^{(0)}\right)$ be the faithful conditional expectation extending restriction . Suppose that G is topologically principal. For every $\epsilon > 0$ and $c_r \in C_r^*(G)^+$, there exists$f_j \in C_0\left(G^{(0)}\right)^+$ such that :

(i) $\|f_j\| = 1$;

(ii) $\|f_j c_r f_j - f_j E(c_r) f_j\| < \epsilon$;

(iii) $\|f_j E(c_r) f_j\| > \|E(c_r)\| - \epsilon$.

Proof: Let $\epsilon > 0$. For $c_r = 0$ the result is trivial so let $c_r \in C_r^*(G)^+$ such that $c_r \neq 0$. Define

$$a_r := \frac{c_r}{\|E(c_r)\|} \quad .$$

To find an appropriate f_j, we use the construction in the proof of [14] ; we include the details below . Find $b_r \in C_{c_r}(G) \cap C_r^*(G)^+$ so that $\|a_r - b_r\| < \frac{\epsilon}{2\|E(c_r)\|}$. Then

$$\|E(b_r)\| > 1 - \frac{\epsilon}{2\|E(c_r)\|}$$

because E is linear and$\|E(a_r)\| = 1$. Now , let $K := \mathrm{supp}(b_r - E(b_r))$, which is a compact subset of $G \setminus G^{(0)}$. Let

$$U := \left\{u_j \in G^{(0)} \mid E(b_r)(u_j) > 1 - \frac{\epsilon}{2\|E(c_r)\|}\right\}.$$

Since G is topologically principal,[14,Lemma(2.3)] implies that there exists a nonempty open set $V \subseteq U_j$ such that $VKV = \emptyset$. Using regularity, fix a nonempty open set W such that $\overline{W} \subseteq V$ .Using normality, select a positive (nonzero) real-valued function$f_j \in C_{c_r}\left(G^{(0)}\right)$ such that $f_j|_{\overline{W}} = 1$, $\mathrm{supp}(f_j) \subseteq V$, and $0 \leq f_j(x_n) \leq 1$ for all $x_n \in G^{(0)}$ .Therefore, f_j is positive in $C_r^*(G)$ and satisfies item (i) . To see that item (ii) holds , a direct computation gives

$$f_j b_r f_j = f_j E(b_r) f_j . \qquad (3.1)$$

Since $\|a_r - b_r\| < \frac{\epsilon}{2\|E(c_r)\|}$, $\|f_j\| = 1$ and E is norm decreasing we have

$$\|f_j E(a_r) f_j - f_j E(b_r) f_j\| < \frac{\epsilon}{2\|E(c_r)\|} . \qquad (3.2)$$

Combining equations (3.1) and (3.2) we get

$$\|f_j a_r f_j - f_j E(a_r) f_j\| = \|f_j a_r f_j - f_j b_r f_j + f_j b_r f_j - f_j E(b_r) + f_j E(b_r) f_j - f_j E(a_r) f_j\| < \frac{\epsilon}{\|E(c_r)\|} .$$

Thus multiplying by$\|E(c_r)\|$ gives $\|f_j c_r f_j - f_j E(c_r) f_j\| < \epsilon$ as needed in (ii).To see item (iii) notice that since supp$f_j \subseteq U$ we have

$$f_j E(b_r) f_j \geq \left(1 - \frac{\epsilon}{2\|E(c_r)\|}\right) f_j^2 .$$

Since $\|f_j\| = 1$,from the above equation and equation (2) we get

$$\|f_j E(a_r) f_j\| > \|f_j E(b_r) f_j\| - \frac{\epsilon}{2\|E(c_r)\|} \geq 1 - \frac{\epsilon}{2\|E(c_r)\|} - \frac{\epsilon}{2\|E(c_r)\|} = 1 - \frac{\epsilon}{\|E(c_r)\|} .$$

Multiplying by $\|E(c)\|$we obtain $\|f_j E(c_r) f_j\| > \|E(c_r)\| - \epsilon$ as needed .

Lemma3.2. (see [28]). Let G be a locally compact, Hausdorff -´etale groupoid and $E : C_r^*(G) \to C_0(G^{(0)})$ be the faithful conditional expectation extending restriction .
Suppose that G is topologically principal . For every non-zero $a_r \in C_r^*(G)^+$, there exists non-zero $h_r \in C_0(G^{(0)})^+$ such that $h_r \precsim a_r$.

Proof: Let $a_r \in C_r^*(G)^+$ such that $a_r \neq 0$. Since E is faithful , $E(a_r)$ is non-zero .
Applying Lemma 3.1 to $c_r := \frac{a_r}{\|E(c_r)\|}$ and $\epsilon = 1/4$ gives us an $f_j \in C_0(G^{(0)})$ such that items (i), (ii) and (iii) of Lemma 3.2 hold . In particular $\|f_j E(c_r) f_j\| > \frac{3}{4}$. Following [14], for each $d_r \in C_0(G^{(0)})^+$we define the element

$$(d_r - 1/2)_+ := \phi_{1/2}(d_r) \in C_0(G^{(0)})^+$$

Where $\phi_{1/2}(t) = \max\{t - 1/2\}$ for $t \in \mathbb{R}^+$. Notice that

$$\|\phi_{1/2}(d_r)\| = \max\{\|d_r\| - 1/2 , 0\} ,$$

for each $d_r \in C_0(G^{(0)})^+$. Now let $h_r := (f_j E(c_r) f_j - 1/2)_+ \in C_0(G^{(0)})^+$. Using item (ii) of Lemma 3.1 and [13], we can find $g_j \in C_r^*(G)$ so that $h_r = g_j^* f_j c_r f_j g_j$. Therefore $h_r \precsim a_r$. Finally,$h_r \neq 0$ since

$$\|f_j\| = \|(f_j E(c_r) f_j - 1/2)_+\| \geq \|f_j E(c_r) f_j\| - 1/2 \geq 1/4 > 0 .$$

Theorem 3.3. (see [28]). Let G be a locally compact, Hausdorff - ´etale groupoid. Suppose that G .is minimal and topologically principal . Then $C_r^*(G)$ is purely infinite if and only if every non-zero positive element of $C_0(G^{(0)})$ is infinite in $C_r^*(G)$.

Proof: The forward implication is trivial . To see the reverse , let $a_r \in C_r^*(G)^+$ such that $a_r \neq 0$. Using Lemma 3.2 we can find a non-zero

$$h_r \in C_0(G^{(0)})^+$$

such that $h_r \precsim a_r$. By assumption , we know h_r is infinite. Since $C_r^*(G)$ is simple by [21], h_r is properly infinite by [14]. Thus a_r is properly infinite by [14].hence $C_r^*(G)$ is purely infinite .Recall that a Kirchberg algebra is a separable , nuclear , purely infinite simple C^*-algebra .We combine Theorem 3.3 with results from [2,4,21] to obtain the following characteri-zation of groupoid Kirchberg algebras .

Corollary3.4.(see [28]). Let G be a second-countable, locally compact, Hausdorff -´etale groupoid .Then $C^*(G)$ is a Kirchberg algebra if and only if G is minimal , topologically principal, measure-wise amenable and every non-zero positive element of $C_0(G^{(0)})$ is infinite in $C^*(G)$.

Proof: Suppose $C^*(G)$ is a Kirchberg algebra .Then $C^*(G)$ is simple by definition and so $C^*(G) = C_r^*(G), G$ is minimal and G topologically principal [4]. Since $C^*(G)$ is nuclear, $C_r^*(G)$ is also nuclear hence G is measure-wise amenable by [2]. Finally, we apply Theorem 3.3 to see that every non-zero positive element of $C_0(G^{(0)})$ is infinite in $C^*(G)$. Conversely, suppose G is minimal, topologically principal, measure-wise amenable and that every non-zero positive element of $C_0(G^{(0)})$ is infinite in $C^*(G)$. Then $C_r^*(G) = C^*(G)$ is nuclear by [2], simple by [24], separable because G is second countable [21] and purely infinite by Theorem 3.3 .We will restrict our attention to ample groupoids . Although this might seem a very restrictive class of groupoids , it actually includes a lot of important examples .Again, every Kirchberg algebra in UCT is Morita equivalent to a C^*-algebra associated to a Hausdorff - ample groupoid (see[26]). The ample case is far more manageable than the general case. In particular there is a large number of projections in the associated algebra. Let G be a locally compact, Hausdorff -´etale groupoid . If G is ample, then the complex Steinberg algebra associated to G is

$$A(G) := \text{span}\{\chi_B : B \text{ is a compact open bisection}\} \subseteq C_{c_r}(G)$$

where χ_B denotes the characteristic function of B, is dense in $C_r^*(G)$see [6](see also [27]). A quick computation shows that $\chi_B * \chi_D = \chi_{BD}$ and $\chi_B^* = \chi_{B-1}$, so that if $B \subseteq G^{(0)}$is compact open, then χ_B is a projection .

Theorem4.1(see [28]). Let G be a second countable, Hausdorff-ample groupoid. Suppose that G is topologically principal, minimal and that B is a basis of $G^{(0)}$ consisting of compact open sets .Then $C_r^*(G)$ is purely infinite if and only if every non-zero projection p in $C_0(G^{(0)})$ with $\text{supp}(p) \in B$ is infinite in $C_r^*(G)$.

Proof: The forward implication is trivial . To see the reverse, suppose every non-zero projection p of $C_0(G^{(0)})$ with $\text{supp}(p) = U$ for some $U \in B$ is infinite in $C_r^*(G)$. By Theorem 3.3 it suffices to show that every positive element in $C_0(G^{(0)})^+$is infinite . Let $a_r \in C_r^*(G^{(0)})^+$be a nonzero element . We show that a_r is properly infinite . We claim there is a non-zero projection $p \in C_0(G^{(0)})^+$ with $\text{supp}(p) \subseteq U$ for some $U \in B$ such that $p \precsim a_r$. To see this, first note that characteristic functions of the form χ_V are projections in $C_0(G^{(0)})$ for every compact open set $V \subseteq G^{(0)}$. Since B is a basis of compact open sets , there exists a compact open set $U_0 \in B$and a non-zero $s \in \mathbb{R}^+$ such that $\chi_{v_0}(x_n) \leq s a_r(x_n)$ for every $x_n \in G^{(0)}$. Then $p := \chi_{v_0} \leq s a_r$. Applying [14] we get that $p \precsim s a_r$ and so $p \precsim a_r$ as claimed . Since p is infinite by assumption and $C_r^*(G)$ is simple,p is properly infinite by [14].

Hence a_r is properly infinite by [14]. In the next corollary, we show how we can use the minimality of G to strengthen our result .

Corollary 4.2 . (see [28]). Let G be a second countable, Hausdorff- ample groupoid .
Suppose that $G$is topologically principal and minimal . Then $C_r^*(G)$ is purely infinite if and only if there exists a point $x_n \in G^{(0)}$ and a neighborhood basis D at x_n consisting of compact open sets so that every non-zero projection q in $C_0(G^{(0)})$ with $\text{supp}(q) \in D$ is infinite in $C_r^*(G)$.

Proof: Again, the forward direction is trivial. For the reverse implication, suppose there exist a point $x_n \in G^{(0)}$ and neighborhood basis $D$of x_n consisting of compact open sets such that that every non-zero projection q in $C_0(G^{(0)})$ with $\text{supp}(q) \in D$ is infinite in $C_r^*(G)$. Let B be a basis of $G^{(0)}$ consisting of compact open sets and suppose $p := \chi_v$ is a non-zero projection in $C_0(G^{(0)})$ with $U \in B$. By Theorem 4.1, it suffices to show that p is infinite . Since G is minimal and ample , there exists a compact open bisection B such that $x_n \in s(B)$ and $r(B) \cap U \neq \emptyset$. By shrinking B,we may assume that $r(B) \subseteq U$. Since $s(B)$is a compact open neighborhood of x_n, there exists a $V \in D$ such that $V \subseteq s(B)$. By shrinking Bagain ,we may assume that $s(B) = V$. Thus. $\chi_V = \chi_B^* \chi_{r(B)} \chi_B$. That is $\chi_V \precsim \chi_{r(B)}$. Hence , $\chi_{r(B)}$ is properly infinite by [15]. Finally, since $\chi_U = \chi_{r(B)} + \chi_{U-r(B)}, \chi_U$ is infinite .We apply Theorem 4.1 to C^*-algebras associated to k-graphs .We assume the reader is familiar with the basic definitions and constructions of k-graphs and their C^*-algebras found in [16], but we recall a few facts here. Let A be a k-graph.Then the associated C^*-algebra $C^*(A)$is the universal C^*-algebra generated by a Cuntz-Krieger A-family $\{s_{\lambda_j} : \lambda_j \in A\}$.To keep things clean , we will restrict our attention to row-finite k-graphs with no sources but similar results hold in the more general setting . We think our results will be useful in this setting because necessary and sufficient conditions on A for $C^*(A)$ to be purely infinite simple are not known. Following [16] we recall how $C^*(A)$ can be realised as the C^*-algebra of a second countable, Hausdorff - ample groupoid G_A as follows . Let A^∞ denote the infinite path space of A and $A^\infty(v)$ be the set of infinite paths with range v. Define

$$G_A := \{(x_n, n_0, y_n) \in A^\infty \times \mathbb{N}^k \times A^\infty : \sigma^l(x_n) = \sigma^{m_0}(y_n), n_0 = l - m_0\}$$

where σ is the shift map . We view (x_n, n_0, y_n) as a morphism with source y_n and range x_n. Composition is given by $(x_n, n_0, y_n)(y_n, m_0, w_n) = (x_n, n_0 + m_0, w_n)$. The unit space $G_A^{(0)}$is identified A^∞ . For $\lambda_j, \mu_j \in A$ with $s(\lambda_j) = s(\mu_j)$ we define

$$Z(\lambda_j, \mu_j) := \{(\lambda_j z_n, d(\lambda_j) - d(\mu_j), \mu_j z_n) : z_n \in A^\infty(s(\lambda_j))\}.$$

The (countable) collection of all such $Z(\lambda_j, \mu_j)$ generate a topology under which G_A is a second countable , Hausdorff - ample groupoid by [16]. Further, the relative topology on the unit space A^∞ has a basis of compact cylinder sets

$$Z(\lambda_j) := \{\lambda_j x_n \in A^\infty : x_n \in A^\infty(s(\lambda_j))\}$$

by identifying $Z(\lambda_j, \lambda_j)$ and $Z(\lambda_j)$ from [16]. Note that G_A is amenable by [16] and hence $C_r^*(G_A) = C^*(G_A)$. It was shown in [16] that

$$C^*(A) \cong C^*(G_A) .$$

More specifically, by [16,Corollary 5.3], there is a (unique) isomorphism $\phi : C^*(A) \to C^*(G_A)$ such that $\phi(s_{\lambda_j}) = \chi_{z_n(\lambda_j, s(\lambda_j))}$. Note that

$$\phi\left(s_{\mu_j}s_{\mu_j}^*\right) = \chi_{Z_n(\mu_j,s(\mu_j))}\chi_{Z_n(\mu_j,s(\mu_j))}^* = \chi_{Z_n(\mu_j,s(\mu_j))}\chi_{Z_n(\mu_j,s(\mu_j),\mu_j)} = \chi_{Z_n(\mu_j,\mu_j)} = \chi_{Z_n(\mu_j)} .$$

With all of this theory in place, along with the simplicity results of [23] and [4], the following is an immediate corollary of Theorem 4.1 and Corollary 4.2 .

Corollary 5.1(see [28]).Suppose A is a row-finite k-graph with no sources such that A is aperiodic and co final in the sense of [23]. Then

(i) For $\mu_j \in A$, $s_{\mu_j}s_{\mu_j}^*$ is infinite if and only if $s_{s(\mu_j)}$ is .

(ii) $C^*(A)$ is purely infinite simple if and only if s_v is infinite for every $v \in A^0$.

(iii) $C^*(A)$ is purely infinite simple if and only if there exists $x_n \in A^\infty$ such that s_v is infinite for every vertex v on x_n .

Proof: For (i), we use a trick used in [25]. Recall that infiniteness is preserved under von Neumann equivalence , hence $s_{\mu_j}s_{\mu_j}^*$ is infinite if and only if $s_{\mu_j}^*\ s_{\mu_j} = s_{s(\mu_j)}$ is infinite .For (ii), we apply Theorem 4.1 to the second countable, Hausdorff - ample groupoid G_A, first we check the remaining hypotheses of Theorem 4.1 . Since A is cofinal and aperiodic, $C^*(G_A) \cong C^*(A)$ is simple by [23].Thus $C^*(G_A) = C_r^*(G_A)$ is simple and hence G_A is topologically principal and minimal by [4].We have that the collection of cylinder sets of the form $Z(\mu_j)$ form a basis B of consisting of compact open sets . Now we apply Theorem 4.1 to see that $C^*(G_A)$ is purely infinite if and only if each $\chi_{Z_n(\mu_j)}$ is infinite . Let

$$\phi : C^*(A) \to C^*(G_A)$$

be the isomorphism characterized by$s_{\mu_j} \mapsto \chi_{Z_n(\mu_j)}$. Since ϕ is an isomorphism, this gives $\chi_{Z_n(\mu_j)}$ is infinite if and only if $\phi^{-1}\left(\chi_{Z_n(\mu_j)}\right) = s_{\mu_j}s_{\mu_j}^*$ is infinite. Finally,$s_{\mu_j}s_{\mu_j}^*$ if and only if $s_{s(\mu_j)}$ is infinite by (i). For (iii) , given an infinite path x_n, the collection of compact open sets of the form $Z(x_n(0,n_0))$ for $n_0 \in \mathbb{N}^k$ form a neighbourhood base at x_n. Now proceed as in the proof of (ii) replacing Theorem 4.1 with Corollary 4.2 and μ_j with $x_n(0,n_0)$.

Let A be a C^*-algebra .A pair of positive elements $(a_1,a_2) \in A \times A$ has the matrix diagonalization property in A in the sense of [15] if for every positive matrix $\begin{pmatrix} a_1 & b_{12} \\ b_{21} & a_2 \end{pmatrix}$ with $b_{ij} \in A$ and every $\epsilon_1,\epsilon_2,\delta > 0$ there exists $d_1,d_2 \in A$ with

$$\|d_i^*a_id_i - a_i\| < \epsilon_i \text{and} \|d_i^*b_{ij}d_j\| < \delta .$$

A subset $\mathcal{F}_r$of A^+ is a filling family for A, in the sense of [15], if for every hereditary C^*-subalgebra $H$of A and every primitive ideal $I$of A with $H \not\subseteq I$there exist $f_j \in \mathcal{F}_r$and $z_n \in A$ with $z_n^*z_n \in H$ and $z_nz_n^* = f_j \notin I$. By Proposition 3.13 and Lemma 3.12 [15], if A^+ contains a filling family $\mathcal{F}_r$ that is closed under ϙ-cut-downs and every pair of elements$(a_1,a_2) \in \mathcal{F}_r \times \mathcal{F}_r$ has the matrix diagonalization property, then A is strongly purely infinite .

We provide a characterization of when the reduced groupoid C^*-algebra is strongly purely infinite (Proposition (6.3)). In our proof of Proposition 6.3 we will use results from [4] to describe ideals of reduced groupoid C^*-algebras . First we need the following lemma. Recall theta subset $D \subseteq G^{(0)}$ is said to be invariant if

$$G.D := \{r(\gamma_j) : s(\gamma_j) \in D\} \subseteq D .$$

Lemma 6.1 (see [28]). Let G be a second countable, locally compact, Hausdorff - ´etale groupoid such that $C^*(G) = C_r^*(G)$ .Then the following properties are equivalent :

(i) For every closed invariant set $D \subseteq G^{(0)}$

$$C^*(G|_D) = C_r^*(G|_D) .$$

(ii) For every closed invariant set $D \subseteq G^{(0)}$ the sequence

$$0 \to C_r^*\left(G|_{G^{(0)}-D}\right) \xrightarrow{\iota_r} C_r^*(G) \xrightarrow{\rho_r} C_r^*(G|_D) \to 0$$

is exact where ι_r and ρ_r are determined on continuous functions by extension and restriction respectively. Remark 6.2 in [22,Remark 4.10],Renault mentions that if $G|_D$ is amenable for every closed invariant set $D \subseteq G^{(0)}$, then item (ii) of Lemma 6.1 follows . Thus Lemma 6.1 is a strengthening of Renault's comment .

Proof: Fix a closed invariant set $D \subseteq G^{(0)}$ and let $U = G^{(0)} - D$. Consider the following diagram :

$$\begin{array}{ccccc} 0 \to C^*(G|_U) & \xrightarrow{\iota_r} & C^*(G) & \xrightarrow{\rho_r} & C^*(G|_D) \to 0 \\ \pi_U \downarrow & & \pi \downarrow & & \pi_D \downarrow \\ 0 \to C_r^*(G|_U) & \xrightarrow{\iota_0} & C_r^*(G) & \xrightarrow{\rho_0} & C_0(D) \to 0 \end{array} \quad (6.1)$$

where π_U, π and π_D are the respective quotient maps, and ι, ι_r and ρ, ρ_r extend extension and restriction respectively. Since all of the maps involved are continuous , the diagram commutes .We also have that the top row of (6.1) is exact by [18,Lemma 1.10].(ii) ⇒(i) : We show the subjective map π_D is injective. Fix any $a_r \in C_r^*(G|_D)$ with $\pi_D(a_r) = 0$. Find $b_r \in C^*(G)$ with $\rho(b_r) = a_r$. From

$\pi_D(\rho(b_r)) = \rho_r(\pi(b_r)) = 0$, exactness of (6.1), subjectivity of π_U, and $\iota_r \circ \pi_U = \pi \circ \iota_r$ we obtain

$$\pi(b_r) \in \ker \rho_r = \iota_r\left(C_r^*(G|_U)\right) = \iota_r \circ \pi_U\left(C^*(G|_U)\right) \pi \circ \iota_r \left(C^*(G|_U)\right).$$

Find $c_r \in C^*(G|_U)$ with $\pi(b_r) = \pi \circ \iota_r (c_r)$. As π is an isomorphism by assumption we obtain that $b_r = \iota_r(c_r)$. Hence $a_r = \rho_r(b_r) = \rho_r \circ \iota_r(c_r) = 0$. and

$C^*(G|_D) = C_r^*(G|_D)$.(i)⇒(ii) :

By assumption the maps π and π_D are isomorphisms . Using the commutative diagram (6.1) and the exactness of the top line of that diagram, the exactness of the bottom line follows from a easy diagram chase. Let G be a second countable, locally compact, Hausdorff -´etale groupoid and D be a closed invariant set of $G^{(0)}$ and define $U = G^{(0)} - D$. Recall from Lemma 2.1 (4) we have the communing diagram :

$$\begin{array}{ccccc} 0 \to C_r^*(G|_U) & \xrightarrow{\iota_r} & C_r^*(G) & \xrightarrow{\rho_r} & C_r^*(G|_D) \to 0 \\ E_U \downarrow & & E \downarrow & & E_D \downarrow \\ 0 \to C_0(U) & \xrightarrow{\iota_0} & C_0\left(G^{(0)}\right) & \xrightarrow{\rho_0} & C_0(D) \to 0 \end{array} \quad (6.2)$$

Notice that the bottom row in (6.2) is exact . We will use this diagram several times . We also use the notation Ideal [S] for the ideal in $C_r^*(G)$ generated by $S \subseteq C_r^*(G)$.

Proposition 6.3(see[28]).Let G be a second countable, locally compact, Hausdorff and ´etale groupoid such that $C^*(G) = C_r^*(G)$ .Then the following properties are equivalent :

(i) The C^*-algebra $C_r^*(G)$ is strongly purely infinite, and for every ideal I in $C_r^*(G)$,

$$I = \text{Idea}\left[I \cap C_0\left(G^{(0)}\right)\right].$$

(ii) For every closed invariant set $D \subseteq G^{(0)}$, $G|_D$ is topologically principal; the sequence

$$0 \to C_r^*(G|_U) \xrightarrow{\iota_r} C_r^*(G) \xrightarrow{\rho_r} C_r^*(G|_D) \to 0 \quad (6.3)$$

is exact where $U = G^{(0)} - D$, ι_r and ρ_r are determined on continuous functions by extension and restriction respectively; and for every pair of elements a_r, b_r in $C_0\left(G^{(0)}\right)^+$ the 2-tuple (a_r, b_r) has the matrix diagonalization property in $C_r^*(G)$.

Proof: (i) ⇒(ii) : Fix a closed invariant set $D \subseteq G^{(0)}$ and $U = G^{(0)} - D$. For this D and U we have a commuting diagram as in (6.2). Define $I := \ker \rho_r \subseteq C_r^*(G)$. Using the diagram, $\rho_0\big(E(I)\big) = E_D\big(\rho_r(I)\big) = 0$, implying that $E(I) \subseteq \iota_0\big(C_0(U)\big)$. Since $E(b_r) = b_r$ for $b_r \in C_0\big(G^{(0)}\big)$, $I \cap C_0\big(G^{(0)}\big) \subseteq E(I)$. Using assumption (i) we have $I = \text{Ideal}\big[I \cap C_0\big(G^{(0)}\big)\big]$. Hence

$$\ker \rho_r = I = \text{Ideal}\big[I \cap C_0\big(G^{(0)}\big)\big] \subseteq \text{Ideal}[E(I)] \subseteq \text{Ideal}\big[\iota_0\big(C_0(U)\big)\big] \subseteq \iota_r\big(C_r^*(G|_U)\big) ;$$

that is $\ker \rho_r \subseteq$ image (ι_r) .Thus (6.3) is exact .We know that each $G|_D$ is topologically principal by [4] provided that $C^*(G|_D) = C_r^*(G|_D)$ .The latter follows from Lemma 6.1 since (6.3) is exact .Since $C_r^*(G)$is strongly purely infinite, Lemma 5.8 in [13] implies that every pair (a_r, b_r) of positive elements in $C_0\big(G^{(0)}\big)$ has the matrix diagonalization property in $C_r^*(G)$. (ii)⇒(i) : Since we assumed that $G|_D$ is topologically principal for all closed invariant $D \subseteq G^{(0)}$, by the proof of Corollary 5.9 in [4] , we know $I = \text{Ideal}\big[I \cap C_0\big(G^{(0)}\big)\big]$ for every ideal I in $C_r^*(G) = C^*(G)$ provided that $C^*(G|_D) = C_r^*(G|_D)$for every closed invariant set $D \subseteq G^{(0)}$. But this follows from Lemma 6.1 since $C^*(G) = C_r^*(G)$ and (6.3) is exact, which are assumed in (ii). Hence (ii) implies $I = \text{Ideal}\big[I \cap C_0\big(G^{(0)}\big)\big]$.

We prove $C_r^*(G)$ is strongly purely infinite . Define $\mathcal{F}_r := C_0\big(G^{(0)}\big)^+ \subseteq C_r^*(G)$.

By functional calculus we know

$$f_j(a_r) \in \mathcal{F}_r , \qquad \text{for } f_j \in C_0(\mathbb{R})^+ , \qquad a_r \in \mathcal{F}_r .$$

In particular $\mathcal{F}_r$ is closed under ε-cut-downs, i.e., for each $a_r \in \mathcal{F}_r$, and $\varepsilon \in (0, \|a_r\|)$ we have $(a_r - \varepsilon)_+ \in \mathcal{F}_r$. By (ii) each pair (a_r, b_r) with $a_r, b_r \in \mathcal{F}_r$ has the matrix diagonalization property (of [15]) . Now by Lemma 3.12 of [15] we know that $\mathcal{F}_r$ has the matrix diagonalization property of [15]. If follows from Proposition 3.13 of [15] that $C_r^*(G)$ is strongly purely infinite provided that $\mathcal{F}_r$ is a filling family for $C_r^*(G)$, which we now show. Fix any hereditary C^*-subalgebra H of $C_r^*(G)$ and any ideal $I$of $C_r^*(G)$ with $H \nsubseteq I$. We know $I = \text{Ideal}\big[I \cap C_0\big(G^{(0)}\big)\big]$, hence $I = \iota_r\big(C_r^*(G|_U)\big)$for some open invariant set $U \subseteq G^{(0)}$. Let $D = G^{(0)} - U$ and consider the corresponding commuting diagram (6.2). Select $d_r \in H^+, d_r \notin I$. Define $c_r := \rho_r(d_r)$. As $d_r \notin I = \ker \rho_r$ by exactness in (ii), we know $\rho_r(d_r) \neq 0$. Since E_D is faithful and c_r positive ,

$$\epsilon := \frac{1}{4}\|E_D(c_r)\| > 0 .$$

By (ii) the groupoid $G|_D$ is topologically principal, hence Lemma 3.1 gives $f_j \in C_0(D)^+$ such that

$$h_r := a_r^* f_j c_r f_j a_r = \big(f_j E_D(c_r) f_j - \epsilon\big)_+ \in C_0(D)^+ .$$

Notice that

$$\|h_r\| \geq \|f_j E_D(c_r) f_j\| - \epsilon > \|E_D(c_r)\| - 2\epsilon > 0 .$$

Using that ρ_r restricts to the map $C_0\big(G^{(0)}\big) \to C_0(D)$, select $b_r \in C_0\big(G^{(0)}\big)^+$such that $\rho_r(b_r) = h_r$. Also as ρ_r is surjective find $w_n \in C_r^*(G)$ such that $\rho_r(w_n) = f_j a_r$. Since $\rho_r(b_r - w_n^* d_r w_n) = h_r - a_r^* f_j c_r f_j a_r = 0$ we have $b_r = w_n^* d_r w_n + v$ for some $v \in I$. Let $\{e_{\lambda_j}\}$ denote an approximate unit of $I = (G|_U)$ with $e_{\lambda_j} \in C_0(U)$(see Lemma 2.1). Let 1 be the unit of the unitization of $C_r^*(G)$. Then $\big(1 - e_{\lambda_j}\big) u_j \big(1 - e_{\lambda_j}\big) \to 0$. For suitable λ_0 and $e := 1 - e\lambda_0$ we get $\|e w_n^* d_r w_n e - e b_r e\| = \|eve\| < \epsilon$. Use Lemma 22 of [13] to find a contraction $u_j \in C_r^*(G)$ such that

$$g_j := u_j^* e w_n^* d_r w_n e u_j = (e b_r e - \epsilon) \in C_0\big(G^{(0)}\big)^+ = \mathcal{F}_r .$$

Since $b_r e_{\lambda_0} + e\lambda_0 b_r - e\lambda_0 b_r e_{\lambda_0} \in C_0(U) \subseteq \ker \rho_r$ we obtain that $\rho_r(e b_r e) = \rho_r(b_r) = h_r$.

Moreover by functional calculus we know$(h_r - \epsilon)_+ = \big(f_j E_D(c_r) f_j - 2\epsilon\big)_+$ .We conclude

$$\|\rho_r(g_j)\| = \|(h_r - \epsilon)_+\| = \left\|\big(f_j E_D(c_r) f_j - 2\epsilon\big)_+\right\| \geq \|f_j E_D(c_r) f_j\| - 2\epsilon > \|E_D(c_r)\| - 3\epsilon > 0 ,$$

ensuring $g_j \notin I$ .Finally with $z_n := u_j^* e w_n^* d_r^{1/2} \in C_r^*(G)$ we obtain $g_j = z_n z_n^*$ and $z_n^* z_n \in H$. By definition $\mathcal{F}_r$is a filling family for $C_r^*(G)$ completing the proof .

References

[1] C. Anantharaman-Delaroche, Purely infinite C^*-algebras arising from dynamical system, Bull. Soc.Math. France 125 (1997), no. 2, 199-225.
[2] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, Monographies de L'Enseignement Math´ematique, 36, L'Enseignement Math´ematique, Geneva, 2000.
[3] B. Blackadar, Operator algebras, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Noncommutative Geometry, III.
[4] J.H. Brown, L.O. Clark, C. Farthing and A. Sims, Simplicity of algebras associated to ´etale groupoids, Semigroup forum, to appear.
[5] T. Carlsen and K. Thomsen, The structure of the C^*-algebra of a locally injective surjection, Ergodic Theory Dynam. Systems 32 (2012), no. 4, 1226–1248.
[6] L.O. Clark, C. Farthing, A. Sims andM. Tomforde, A groupoid generalization of Leavitt path algebras,Semigroup forum, to appear.
[7] J. Cuntz, K-theory for certain C^*-algebras, Ann. of Math. (2) 113 (1981), no. 1, 181-197.
[8] G. Elliott, On the classification of C^*-algebras of real rank zero, J. Reine Angew. Math. 443 (1993),179-219.
[9] R. Exel, Reconstructing a totally disconnected groupoid from its ample semigroup, Proc. Amer. Math. Soc. 138 (2010), 2991–3001.
[10] T. Giordano and A. Sierakowski, Purely infinite partial crossed products, submitted. arXiv:1303.4483v2
[11] E. Kirchberg, The classification of purely infinite C^*-algebras using Kasparov's theory, preprint, 1994.
[12] E. Kirchberg, Das nicht-kommutative Michael-Auswahlprinzip und die Klassifikation nicht-einfacher Algebren, p. 92–141 in C^*-Algebras: Proceedings of the SFB-Workshop on C^*-algebras, M¨unster, Germany, March 8-12, 1999/J. Cuntz, S. Echterhoff (ed.), Springer, Berlin etc. (2000), 272 pp.
[13] E. Kirchberg and M. Rørdam, Infinite non-simple C^*-algebras: absorbing the Cuntz algebras O1,Adv. Math. 167 (2002), 195–264.
[14] E. Kirchberg and M. Rørdam, Non-simple purely infinite C^*-algebras, Amer. J. Math. 122 (2000),637–666.
[15] E. Kirchberg and A. Sierakowski, Strong pure infiniteness of crossed products, J. Funct. Anal. (2014).
[16] A. Kumjian and D. Pask, Higher rank graph C^*-algebras, New York J. Math. 6 (2000), 1–20.
[17] A. Kumjian, D. Pask, and I. Raeburn, Cuntz-Krieger algebras of directed graphs, Pacific J. Math. 184 (1998), no. 1, 161–174.
[18] P.S. Muhly, J. Renault and D.P. Williams, Continuous trace groupoid C^*-algebras, III, Trans. Amer. Math. Soc. 348 (1996), 3621–3641.
14 J. BROWN, L. O. CLARK, AND ADAM SIERAKOWSKI
[19] A. Paterson, Groupoids, Inverse Semigroups, and their Operator Algebras, Progress in Mathematics,170, Birkhauser, Boston, 1998.
[20] N. C. Phillips, A classification theorem for nuclear purely infinite simple C^*-algebras, Doc. Math. 5 (2000), 49-114.
[21] J. Renault, A groupoid approach to C^*-algebras. Lecture Notes in Mathematics, 793. Springer,Berlin, 1980.
[22] J. Renault, The ideal structure of groupoid crossed product C^*-algebras, J. Operator Theory 25 (1991), 3–36.
[23] D.I. Robertson and A. Sims, Simplicity of C^*-algebras associated to higher-rank graphs, Bull. Lond. Math. Soc. 39 (2007), 337–344.
[24] M. Rørdam and A. Sierakowski, Purely infinite C^*-algebras arising from crossed products, Ergodic Theory Dynam. Systems 32 (2012), no. 1, 273-293.
[25] A. Sims, Gauge-invariant ideals in the C^*-algebras of finitely aligned higher-rank graphs, Canad. J.Math. 58 (2006), 1268-1290.
[26] J. Spielberg, Graph-based models for Kirchberg algebras, J. Operator Theory 57 (2007), 347–374.
[27] B. Steinberg, A groupoid approach to inverse semigroup algebras, Adv. Math. 223 (2010), 689-727.
[28] Jonathan brown, lisa orloff clark , and adam sierakowski Purely Infinite C^*-algebras Associated to Hausdorff -´Etale Groupoids. March 21, 2014.