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Research Paper

Highlightness of Unifom Rigidity and Multi-Sensitivity on Uniform Spaces

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Abstract
In the same clear waywe follow [47] to introduce highlghits concepts of weak uniformity, uniform rigidity and multi-sensitivity for uniform spaces and obtain some equivalent characterizations of uniform rigidity. We show that a dynamical system (X, f) defined on a Hausdorff uniform space is uniformly rigid if and only if (X, f^n) is uniformly rigid for certainn $\in \mathbb{N}$ if and only if its hyperspatial dynamical system is uniformly or weakly rigid. Besides,we show that every non-minimal point transitive dynamical system defined on a Hausdorff uniform space with dense Banach almost periodic points is sensitive and obtain the equivalence of the multi-sensitivity between original dynamical system and its hyperspatial for Hausdorff uniform spaces.
Keywords: *Uniform rigidity, Weak rigidity, Uniform space, Hyperspace, Almost equicontinuity, Sensitivity.*

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I. Introduction

A dynamical system is a pair (X, f), where X is a nontrivial Hausdorff uniform space with no isolated points and $f : X \to X$ is a uniformly continuous map.Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. The complexity of a dynamical system has been a central problem of study since the term of chaos was introduced by [23]and known as Li–Yorke chaos today. Recently, [14] pointed out that there exists two feasible ways to generalize many dynamical properties to Hausdorff spaces. One is finite open covers and the other is uniformities. Weconsider the latter one to extend the notion of rigidity to uniform spaces. The notion of uniform rigidity was introduced by [13] being a topological analogue of rigidity in ergodic theory [11]. Meanwhile, [13] also proved that every uniformly rigid system has zero topological entropy. Then, [12] obtained that a topologically transitive dynamical system is uniformly rigid if and only if it is a factor of an almost equicontinuous dynamical system. It should be noted that all above results were obtained for dynamical systems defined on compact metric spaces.

Another interesting question for a dynamical system is when orbits from nearby points start to deviate after finite steps. This is also one of the most important features depicting the complexity of a dynamical system. This notion, the 'butterfly effect', has been widely studied and is termed as sensitivity, introduced by [5] and [8]. [6] showed that a topologically transitive system defined on an infinite metric space with the dense periodic points is sensitive (also see [12],[31]), which was extended to a uniform space for group action by [7]. [4] extended the main results of [18] to the uniform spaces and proved that a Devaney chaotic continuous action of an Abelian group on a second countable Baire Hausdorff uniform space is Li–Yorke chaotic. Then, [40] proved that a point transitive dynamical system is either sensitive or almost equicontinuous. Then, [38] extended the notions of mean sensitivity and Banach mean sensitivity to uniform spaces and proved that a point-transitive dynamical system on a Hausdorff uniform space is either almost (Banach) mean equicontinuous or (Banach) mean sensitive. Recently, [2] generalized concepts of entropy points, expansivity and shadowing property for dynamical systems on uniform spaces and obtained a relation betweentopological shadowing property and positive uniform entropy. For more results on sensitivity, see[14,20,21,24,28,29,32,34–37,39,43,44,46]. [30] obtained that a dynamical system on a totally bounded uniform space which is topologically shadowing, mixing, and topologically expansive has the topological specification property. Recently, [41] gave a systematic study on the shadowing properties with average error in tracing such as (asymptotic) average shadowing, d-shadowing, d-shadowing and almost specification and obtained a few equivalent characterizations of the average shadowing property which implies d-shadowing and d-shadowing. Then, [1] proved that a dynamical system with ergodic shadowing is topologically chain transitive. [14] obtained some equivalent characterizations and iteration invariance of various definitions of shadowing in the compact uniformity sense generalizing the compact metric sense.

We firstly study the rigidity for Hausdorff uniform spaces and obtain some equivalent characterizations of uniform and weak rigidity. In particular, it is proved that both uniform and weak rigidity are preserved under iterations and that uniform and weak rigidity are coincident for hyperspatial dynamical systems. Besides, some results on sensitivity and multi-sensitivity for Hausdorff uniform spaces are obtained.

II. Basic definitions and preliminaries

2.1. Uniform Space

The concept of uniform space was introduced by [33]. We give only a brief description of the uniform space. Formore see[9,Chapter8]. Let Xbe a nonempty set and $A_{r-1} \subset X \times X$. Set

$A_{r-1}^{-1} = \{(y_{r-1}, x_{r-1}) \mid (x_{r-1}, y_{r-1}) \in A_{r-1}\}$which is called the inverse of A_{r-1}.A_{r-1} is symmetric if $A_r = A_{r-1}^{-1}$. Clearly, $A_r \cap A_{r-1}^{-1}$is symmetric for any $A_{r-1} \subset X \times X$. For any $A_r \circ A_{r+1} \subset X \times X$, the composite $A_r \circ A_{r+1}$ of A_r and A_{r+1} is defined by $A_r \circ A_{r+1} = \{(x_{r-1}, y_{r-1}) \in X \times X \mid \exists\, z_{r-1} \in X$ such that $(x_{r-1}, z_{r-1}) \in A_r$and $(z_{r-1}, y_{r-1}) \in A_{r+1}\}$.

For any$A_{r-1} \subset X \times X$and anyn $\in \mathbb{N}$, denote $A_{r-1}^{n} = \underbrace{A_{r-1} \circ \ldots \circ A_{r-1}}_{n \text{ times}} \mid \{z_{r-1}\}$. Let

$\mathcal{D}_X = V_{r-1} \subset X \times X \mid \Delta \subset V_{r-1}andV_{r-1} = V_{r-1}^{-1}$.

Let $x_{r-1} \in X$and $V_{r-1} \in \mathcal{U}$. The set $B(x_{r-1}, V_{r-1}) = \{y_{r-1} \in X \mid (x_{r-1}, y_{r-1}) \in V_{r-1}\}$ is called the V_{r-1}-ball about x_{r-1}. For a set$A_{r-1} \subset X$and $V_{r-1} \in \mathcal{D}_X$, by the V_{r-1}-ball about A_{r-1} we mean the set $B(A_{r-1}, V_{r-1}) = Sx_{r-1} \in XB(x_{r-1}, V_{r-1})$.

Definition 1. A nonempty collection$\mathcal{U}$of subsets of $X \times X$ is called a uniform structure on X if the following conditions are satisfied:

(1) $\mathcal{U} \subset \mathcal{D}_X$;
(2) If $E_r \in \mathcal{U}$ and $E_r \subset E_{r+1} \in \mathcal{D}_X$, then$E_{r+1} \in \mathcal{U}$;
(3) For any $E_r, E_{r+1} \in \mathcal{U}, E_r \cap E_{r+1} \in \mathcal{U}$;
(4) For any$E_{r-1} \in \mathcal{U}$, there exists $G_{r-1} \in \mathcal{U}$ such that $G_{r-1} \circ G_{r-1} \subset E_{r-1}$.

The elements of $\mathcal{U}$are called entourages. WhenX has a uniform structure$\mathcal{U}$, then $(X, \mathcal{U})$, or simply X is called a uniform space.

For a uniform space $(X, \mathcal{U})$,there exists a uniform topology which will be denoted by $|\mathcal{U}|$ on X characterized by a neighborhood base at every point$x_{r-1} \in X$ consisting of the sets $B(x_{r-1}, V_{r-1})$, where V_{r-1}runs through all entourages of$(X, \mathcal{U})$. It is known that a uniform space $(X, \mathcal{U})$ is Hausdorff if and only if $\mathcal{U}$ is separated, i.e., $\cap\, \mathcal{U} = \Delta$ (see [19,Proposition8.10]). It can be verified that for any $E_{r-1} \in \mathcal{U}$, there exists an entourage $G_{r-1} \in \mathcal{U}$ such that$G_{r-1} \circ G_{r-1} \subset E_{r-1}$ and $\overline{G_{r-1}} \subset E_{r-1}$.

For$(X, \mathcal{U})$ and $(Y, \mathcal{V})$be two uniform spaces. A map $f : X \to Y$ is uniformly continuous if $(f \times f)^{-1}(\mathcal{V}) \subset \mathcal{U}$.

2.2. Rigidity, Almost Equicontinuity and Sensitivity

For(X, f) be a dynamical system defined on a uniform space $(X, \mathcal{U})$. For anyn $\in$ N, write $(X^n, f^{(n)})$ as the n-fold product system $(\underbrace{X \times \ldots \times X}_{n}, \underbrace{f \times \ldots \times f}_{n})$. A point $x_{r-1} \in X$is called a recurrent point of(f) if for every neighborhood $U_{r-1}$of x_{r-1}, there exists $n \in N$ such that $f^n(x_{r-1}) \in U_{r-1}$. Denote by Rec(f) the set of all recurrent points of f. According to [13], a dynamical system (X, f)defined on a compact metric space X is

(1) n-rigid if $\mathrm{Rec}(f^{(n)}) = X^n$;
(2) weakly rigid if (X, f) is n-rigid for any $n \in N$;
(3) uniformly rigid if there exists an increasing sequence $\{n_k\}_{k=1}^{\infty} \subset N$ such that f^{n_k} converges uniformly to id_X.

Clearly, a dynamical system (X, f) is uniformly rigid if and only if for any $\varepsilon > 0$, there exists $n \in N$ such that for any $x_{r-1} \in X, d(x_{r-1}, f^n(x_{r-1})) < \varepsilon$. This equivalent characterization of the uniform rigidity allows us to define uniform rigidity for uniform spaces. It is clear that the uniform rigidity implies the weak rigidity. Let $\alpha \in R\backslash Q$ and $X = R^2/Z^2$.Define $f : X \to X$ by $f(x_{r-1}, y_{r-1}) = (x_{r-1} + \alpha, x_{r-1} + y_{r-1})$. According to [13], (X, f) is minimal and weakly rigid, but not uniformly rigid. This shows that the uniform rigidity is strictly stronger than the weak rigidity

Definition 2. ([13]) A dynamical system (X, f) defined on a uniform space $(X, \mathcal{U})$ is

(1) n-rigid if $\mathrm{Rec}(f^{(n)}) = X^n$;
(2) weakly rigid if(X, f) is n-rigid for any $n \in \mathbb{N}$

(3) uniformly rigid if for any entourage $E_{r-1} \in \mathcal{U}$, there exists $n \in \mathbb{N}$ such that for any $x_{r-1} \in X$, $(x_{r-1}, f^n(x_{r-1})) \in E_{r-1}$, i.e., $\{(x_{r-1}, f^n(x_{r-1})) \mid x_{r-1} \in X\} \subset E_{r-1}$.

Definition 3. ([29]) Let (X, f) be a dynamical system defined on a uniform space $(X, \mathcal{U})$ and $x_{r-1} \in X$.

(1) $x_{r-1}$is an equicontinuous point if for any entourage $E_{r-1} \in \mathcal{U}$, there exists an entourage $D_{r-1} \in \mathcal{U}$ such that for any $y_{r-1} \in B(x_{r-1}, D_{r-1})$ and any $n \in \mathbb{Z}^+$, $\left(f^n(x_{r-1}), f^n(y_{r-1})\right) \in E_{r-1}$;
(2) (X, f) is almost equicontinuous if it has an equicontinuous point;
(3) (X, f) is equicontinuous if every point of X is an equicontinuous point;
(4) (X, f) is sensitive if there exists an entourage$E_{r-1} \in \mathcal{U}$(a sensitive entourage) such that for any $x_{r-1} \in X$ and any entourage $D_{r-1} \in \mathcal{U}$, there exist $y_{r-1} \in B(x_{r-1}, D_{r-1})$ and $n \in \mathbb{Z}^+$such that $(f^n(x_{r-1}), f^n(y_{r-1})) \notin E_{r-1}$.

2.3. Furstenberg Families and $\mathcal{F}$-Recurrence(Transitivity)

For$\mathcal{P}$ be the collection of all subsets of$\mathbb{Z}^+$. A collection $\mathcal{F} \subset \mathcal{P}$ is called a Furstenberg family if it is hereditary upwards, i.e., $F_r \subset F_{r+1}$ and$F_r \in \mathcal{F}$imply $F_{r+1} \in \mathcal{F}$. A family $\mathcal{F}$ is proper if it is a proper subset of $\mathcal{P}$, i.e., neither empty nor the whole$\mathcal{P}$. It is easy to see that $\mathcal{F}$ is proper if and only if$\mathbb{Z}^+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. All the families considered below are assumed to be proper. For a Furstenberg family $\mathcal{F}$, denote $\Delta(\mathcal{F}) = \{A_{r-1} \in \mathcal{P} \mid A_{r-1} \supset F_{r-1} - F_{r-1}$for some$F_{r-1} \in \mathcal{F}\}$, where $F_{r-1} - F_{r-1} = \{i - j \in \mathbb{Z}^+ \mid i, j \in F_{r-1}\}$. For$A_{r-1} \subset \mathbb{Z}^+$, define

$$\overline{d}(A_{r-1}) = \limsup_{n\to+\infty} \frac{1}{n} |A_{r-1} \cap [0, n-1]| \text{ and} \underline{d}(A_{r-1}) = \liminf_{n\to+\infty} \frac{1}{n} |A_{r-1} \cap [0, n-1]|.$$

Then, $\overline{d}(A_{r-1})$ and $\underline{d}(A_{r-1})$ are the upper and the lower density of A_{r-1}, respectively.
Similarly, define the upper and the lower Banach density of A_{r-1} as

$$BD^*_{r-1}(A_{r-1}) = \limsup_{|I|\to+\infty} \frac{|A_{r-1} \cap I|}{|I|} \text{ and } \quad B(D_{r-1})_*(A_{r-1}) = \liminf_{|I|\to+\infty} \frac{|A_{r-1} \cap I|}{|I|},$$

where I is over all non-empty finite intervals of $\mathbb{Z}^+$.
A subset F_{r-1} of $\mathbb{Z}_+$ is

(1) syndetic if it has a bounded gap, i.e., if there is $N \in \mathbb{N}$such that $\{i, i+1, \dots, i+N\} \cap F_{r-1} \neq \emptyset$ for every $i \in \mathbb{Z}_+$;
(2) thick if it contains arbitrarily long runs of positive integers, i.e., for every $n \in \mathbb{N}$ there exists some$a_n \in \mathbb{Z}^+$ such that$\{a_n, a_n+1, \dots, a_n+n\} \subset F_{r-1}$;
(3) an IP-set if there exists a subset $\{p_i : i \in \mathbb{N}\}$ such that$F_{r-1} \supset \{p_{i_1} + \cdots + p_{i_k} \mid k \in \mathbb{N}, i_1 < \cdots < i_k\}$.

The set of all thick subsets of $\mathbb{Z}^+$, all syndetic subsets of$\mathbb{Z}^+$, all IP-subsets of $\mathbb{Z}^+$, all subsets of $\mathbb{Z}^+$ with positive upper density, all subsets of $\mathbb{Z}^+$ with upper density equal to 1, and all subsets of $\mathbb{Z}^+$ with positive upper Banach density, are denoted by $\mathcal{F}_t, \mathcal{F}_s, \mathcal{F}_{ip}, \mathcal{F}_{pud}, \mathcal{F}_{ud1}$, and $\mathcal{F}_{pubd}$, respectively. Clearly, all of them are Furstenberg families.

For any subsets U_{r-1}, V_{r-1} of X, define the set of transfer times by$N_f(U_{r-1}, V_{r-1}) = \{n \in \mathbb{Z}^+ \mid f^n(U_{r-1}) \cap V_{r-1} \neq \emptyset\}$. Similarly, for any $x_{r-1} \in X$, let $N_f(x_{r-1}, V_{r-1}) = \{n \in \mathbb{Z}^+ \mid f^n(x_{r-1}) \in V_{r-1}\}$. When the map f is clear from the context, we simply write $N(U_{r-1}, V_{r-1})$ and $N(x_{r-1}, V_{r-1})$.

The orbit of a point $x_{r-1} \in X$ is the set $orb(x_{r-1}, f) := \{f^n(x_{r-1}) \mid n \in \mathbb{Z}^+\}$. A_{r-1} point $x_{r-1} \in X$ is transitive if $\overline{orb(x_{r-1}, f)} = X$. Denote by Trans$(f)$ the set of all transitive points of f. A dynamical system (X, f) is point transitive if there exists a transitive point. A_{r-1}dynamical system (X, f) is topologically transitive if for any pair of nonempty open subsets U_{r-1}, V_{r-1} of $X, N_f(U_{r-1}, V_{r-1}) \neq \emptyset$. A dynamical system (X, f) is minimal if every orbit under f is dense in X. It is easy to see that (X, f) is a minimal system if and only if X has no proper, nonempty, closed invariant subset. Let $\mathcal{F}$ be a Furstenberg family. A_{r-1} point$x_{r-1} \in X$is

(1) a periodic point if $f^n(x_{r-1}) = x_{r-1}$ for some $n \in \mathbb{N}$;
(2) a minimal point if $(\overline{orb(x_{r-1}, f)}, f)$ is a minimal system;
(3) a$\mathcal{F}$-recurrent point if for every neighborhood U_{r-1} of $x_{r-1}, N(x_{r-1}, U_{r-1}) \in \mathcal{F}$;
(4) an almost periodic point if for every neighborhood U_{r-1} of $x_{r-1}, N(x_{r-1}, U_{r-1}) \in \mathcal{F}_s$;
(5) aBanach almost periodic point if for every neighborhood U_{r-1} of $x_{r-1}, N(x_{r-1}, U_{r-1}) \in \mathcal{F}_{pubd}$.

The sets of all periodic, minimal, almost periodic, and Banach almost periodic points of (X, f) are denoted by $Per(f), M(f), A_{r-1}(f)$, and $BA_{r-1}(f)$, respectively. [15] proved that for a dynamical system (X, f) defined on a locally compact metric space $X, M(f) = A_{r-1}(f)$.

For a Furstenberg family $\mathcal{F}, a$ dynamical system is called $\mathcal{F}$-transitive if $N_f(U_{r-1}, V_{r-1}) \in \mathcal{F}$ for every pair of nonempty open subsets U_{r-1}, V_{r-1} of X. The $\mathcal{F}_{pud}$-transitivity and $\mathcal{F}_{ud1}$-transitivity are called topological ergodicity and strongly topological ergodicity respectively in [17,22]. Clearly, $\mathcal{F}_s$-transitivity is stronger that $\mathcal{F}_{pud}$ -transitivity.

III. Rigidity On Uniform Spaces

3.1. Uniform Rigidity on Uniform Spaces

Firstly, we obtain the iteration invariance of the uniform rigidity, which clearly holds for dynamical systems defined on compact metric spaces (see [3, Theorem 3.4]). However, this invariance for uniform spaces is more difficult.

Theorem 4 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$. Then, the following statements are equivalent:
(1) (X, f) is uniformly rigid;
(2) (X, f_n) is uniformly rigid for any $n \in \mathbb{N}$;
(3) (X, f_n) is uniformly rigid for some $n \in \mathbb{N}$.

Proof. (2) ⇒ (3) ⇒ (1)hold trivially.
(1) ⇒ (2). Assume that (X, f) is uniformly rigid and fix any$n > 2$. For any $D_{r-1} \in \mathcal{U}$, let $N_{D_{r-1}} = mink \in \mathbb{Z}^+ | \{(x_{r-1}, f^k(x_{r-1}))| \, x_{r-1} \in X\} \subset D_{r-1}$ (as (X, f) is uniformly rigid, $N_{D_{r-1}}$exists). It can be verified that for any $D_r, D_{r+1} \in \mathcal{U}$ with $D_r \subset D_{r+1}, N_{D_r} \geq N_{D_{r+1}}$.

Consider the following two cases:
Case 1.$\{N_{D_{r-1}} \, |D_{r-1} \in \mathcal{U}\}$is finite.
Take $N = max \, \{N_{D_{r-1}} \, |D_{r-1} \in \mathcal{U}\}$ and fix an entourage $E_{r-1} \in \mathcal{U}$ such that $N_{E_{r-1}} = N$. Then, for any entourage $D_{r-1} \in \mathcal{U}, (x_{r-1}, f^N(x_{r-1})) \in D_{r-1} \cap E_{r-1}$ for all $x_{r-1} \in X$.This implies that $\{(x_{r-1}, f^N(x_{r-1}))|x_{r-1} \in X\} \subset E_{r-1} \cap (\cap \mathcal{U}) = E_{r-1} \cap \Delta = \Delta$, i.e., $f^N \equiv id_X$. Therefore, $(X, f_n)$is uniformly rigid.

Case 2.$\{N_{D_{r-1}}|D_{r-1} \in \mathcal{U}\}$ is infinite.
Case 2-1. If there exist an entourage $F_{r-1} \in \mathcal{U}$ and a subset $A_{r-1} \subset \{0, 1, \dots, n-1\}$ with $|A_{r-1}| = n-1$ such that for any $q \in A_{r-1}, \{p \in \mathbb{Z}^+|\{(x_{r-1}, f^{pn+q}(x_{r-1}))|x_{r-1} \in X\} \subset F_{r-1}\}$ is finite, then there exists an entourage $F_r \in \mathcal{U}$ such that $\{j \in N|\{(x_{r-1}, f^j\,(x_{r-1}))|x_{r-1} \in X\} \subset F_r\} \subset \{pn + q^{'} | \, p \in \mathbb{Z}^+, q^{'} \in \{0, 1, \dots, n\, - 1\} \, A_{r-1}\}$. For any entourage $E_{r-1} \in \mathcal{U}$, take an entourage $E_r \in \mathcal{U}$ such that$(E_{r-1})_n^1 \subset E_{r-1} \cap F_r$. Then, there exists an increasing sequence$\{pk\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\{npk + q' | k \in \mathbb{N}\} = \{j \in N|\{(x_{r-1}, f^j\,(x_{r-1}))|x_{r-1} \in X\} \subset E_r\}$. Let $E_{r+1} = E_r \cap (f \times f)^{-(np_1+q^{'})}(E_r)$ and $k_1 = 1$. Then, there exists$k_2 \geq 2$ such that $\{(x_{r-1}, f^{npk_2+q^{'}}\,(x_{r-1}))|x_{r-1} \in X\} \subset E_{r+1}$. By induction, there exist entourages$E_{r+2}, \dots, E_{r+n-1} \in \mathcal{U}$ and $k_2 < k_3 < \cdots < k_n$such that$(E_{r-1})_{j+1} = (E_{r-1})_j \cap (f \times f)^{\,-(npk_j\,+q^{'})}\,((E_{r-1})_j)$ $(2 \leq j \leq n-1)$and$\{(x_{r-1}, f^{npk_j\,+q^{'}}\,(x_{r-1}))|x_{r-1} \in X\} \subset (E_{r-1})_j$ $(3 \leq j \leq n)$. For any $x_{r-1} \in X, (x_{r-1}, f^{\,npk_1+q^{'}}\,(x_{r-1})) \in E_r, (f^{npk_1+q^{'}}\,(x_{r-1}), f^{\sum_{j=1}^{2}(npk_j\,+q^{'})}(x_{r-1})) = (f \times f)^{npk_1+q^{'}}\,(x_{r-1}, f^{npk_2+q^{'}}\,(x_{r-1})) \in E_r, \dots$, and $(f^{\sum_{j=1}^{n-1}(npk_j\,+q^{'})}(x_{r-1}), f^{\sum_{j=1}^{n}(npk_j\,+q^{'})}(x_{r-1})) \in E_r$. This implies that for any$x_{r-1} \in X$,

$$\left(x_{r-1}, (f^n)^{\left(q^{'} + \sum_{j=1}^{n} pk_j\right)}(x_{r-1})\right) = \left(x_{r-1}, f^{\sum_{j=1}^{n}\left(npk_j\,+q^{'}\right)}P(x_{r-1})\right) \in E_r^n \subset E_{r-1}.$$

Case 2-2. If for any entourage $F_{r-1} \in \mathcal{U}$ and for any subset $A_{r-1} \subset \{0, 1, \dots, n-1\}$ with $|A_{r-1}| = n-1$, there exists some$q \in A_{r-1}$such that $\wp(F_{r-1}, q, n) := \{p \in \mathbb{Z}^+|\{(x_{r-1}, f^{pn+q}(x_{r-1}))|x_{r-1} \in X\} \subset F_{r-1}\}$ is infinite, then we claim that there exists a subset$B \subset \{0, 1, \dots, n-1\}$ with$|B| \geq 2$such that for any entourage$F_{r-1} \in \mathcal{U}$and any $q \in B, \wp(F_{r-1}, q, n)$ is infinite.
(1) $n = 3$. If for any $q \in \{0, 1, 2\}$ and any entourage $F_{r-1} \in \mathcal{U}, \wp(F_{r-1}, q, 3)$ is infinite, this claim holds trivially. If there exist some $q \in \{0, 1, 2\}$ and an entourage $F_{r-1} \in \mathcal{U}$ such that $\wp(F_{r-1}, q, 3)$ is finite, then for any $q^{'} \in \{0, 1, 2\} \backslash \{q\}$ and any entourage $F_r \in \mathcal{U}, \wp(F_r, q', 3)$ is infinite. In fact, suppose that $\wp(F_r, q', 3)$ is finite, then both $\wp(F_r \cap F, q, 3)$ and $\wp(F_1 \cap F, q', 3)$ are finite. This is a contradiction. Therefore, there exists a subset $B \subset \{0, 1, 2\}$ with $|B| \geq 2$ such that for any entourage$F_{r-1} \in \mathcal{U}$ and any $q \in B, \wp(F_{r-1}, q, 3)$ is infinite.
(2) $n = 4$. If for any $q \in \{0, 1, 2, 3\}$ and any entourage$F_{r-1} \in \mathcal{U}, \wp(F_{r-1}, q, 4)$ is infinite, this claim holds trivially. If there exist some $q \in \{0, 1, 2, 3\}$ and an entourage $F_{r-1} \in \mathcal{U}$ such that $\wp(F_{r-1}, q, 4)$ is finite, then the hypothesis implies that for any $A_{r-1} \subset \{0, 1, 2, 3\} \backslash \{q\}$with $|A_{r-1}| = 2$ and any entourage $F_r \in \mathcal{U}$, there

exists $q^{'} \in A_{r-1}$ such that $\wp(F_r, q', 4)$ is infinite. Similarly to the proof of $n = 3$, it can be verified that there exists a subset $B \subset \{0,1,2,3\} \setminus \{q\}$ with $|B| \geq 2$ such that for any entourage $F_{r-1} \in \mathcal{U}$ and any $q \in B, \wp(F_{r-1}, q, 4)$ is infinite. By analogy, it can be verified that this claim holds.

Applying the claim implies that there exists some $q \in \{0,1,\ldots,n-1\}$ such that for any entourage $F_{r-1} \in \mathcal{U}, \wp(F_{r-1}, q, n)$ is infinite. For any fixed entourage $E_{r-1} \in \mathcal{U}$, choose an entourage $E_r \in \mathcal{U}$ such that $E_r^n \subset E_{r-1}$. Then, there exists$p_1 \in \mathbb{N}$ such that $\{(x_{r-1}, f^{np_1+q}(x_{r-1})) | x_{r-1} \in X\} \subset E_r$. Let$E_{r+1} = E_r \cap (f \times f)^{-(np_1+q)}(E_r)$. As $\wp(E_{r+1}, q, n)$ is infinite, there exists $p_2 \in \mathbb{N}$such that$\{(x_{r-1}, f^{np_2+q}(x_{r-1})) | x_{r-1} \in X\} \subset E_{r+1}$. By induction, there exist entourages $E_{r+2}, \ldots, E_{r+n-1} \in \mathcal{U}$ and $p_3 < \cdots < p_n$such that $(E_{r-1})_{j+1} = (E_{r-1})_j \cap (f \times f)^{-(np_j+q)}((E_{r-1})_j)$ $(2 \leq j \leq n-1)$ and $\{(x_{r-1}, f^{np_j+q}(x_{r-1})) | x_{r-1} \in X\} \subset (E_{r-1})_j$ $(3 \leq j \leq n)$. Similarly to the proof of Case 2-1, it can be verified that for any $x_{r-1} \in X$,

$$(x_{r-1}, (f^n)^{(q+\sum_{j=1}^{n} p_j)}(x_{r-1})) = (x_{r-1}, f^{\sum_{j=1}^{n}(np_j+q)}(x_{r-1})) \in E_r^n \subset E_{r-1}.$$

Since 2|4 and f^4 is uniform rigid, it follows that f^2 is also uniform rigid. Summing up Case 1, Case 2-1 and Case 2-2, it follows that (X, f^n) is uniformly rigid for any $n \in \mathbb{N}$.
The proof is completed.

Remark 5. For the claim in Case 2-2, the reviewer provides the following simpler proof: Suppose that thereexists no such subset as made in the claim. Then, for $B \subset \{0,1,\ldots,n-1\}$ with $|B| \geq 2$, there exist $F_{r-1} \in \mathcal{U}$and $q \in B$ such that $\wp(F_{r-1}, q, n)$ is finite. Taking $Q = \{q \in \{0,1,\ldots,n-1\} \,|\, \wp((F_{r-1})_q, q, n)$ is finite for some$(F_{r-1})_q \in \mathcal{U}\}$, it follows that $n - 1 \leq |Q| < n$ ($|Q| \neq n$, since then (X, f) is not uniformly rigid, contrary to thestarting assumption). For any $q \in Q$, pick an $(F_{r-1})_q \in \mathcal{U}$ such that $\wp((F_{r-1})_q, q, n)$ is finite and take

$$F_{r-1} = \bigcap_{q \in Q} (F_{r-1})_q.$$

Then we are in Case 2-1 with F_{r-1} and taking $A_{r-1} \subset Q$, a contradiction

Lemma 6. ([40, Theorem 6]) Let (X, f) be a dynamical system defined on a Hausdorff uniform space X. If (X, f) is topologically transitive and almost equicontinuous, then $E_{r-1}q(f) = Trans(f)$. In particular, (X, f) is point transitive.
We have the following theorem.

Theorem 7 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$. If (X, f) is topologically transitive and almost equicontinuous, then (X, f) is uniformly rigid.
Proof. Applying Lemma 6 implies that there exists a transitive point x which is equicontinuous. For any entourage $E_{r-1} \in \mathcal{U}$, there exists an entourage $E_r \in \mathcal{U}$ such that $\overline{E_r} \subset E_{r-1}$. Then, there exists an entourage $D_{r-1} \in \mathcal{U}$ such that for any $y_{r-1} \in B(x_{r-1}, D_{r-1})$ and any $n \in \mathbb{Z}^+, (f^n(x_{r-1}), f^n(y_{r-1})) \in E_r$. As $\overline{\text{orb}(x_{r-1}, f)} = X$, it follows that there exists $k \in \mathbb{Z}^+$ such that $f^k(x_{r-1}) \in B(x_{r-1}, D_{r-1})$. This implies that for any $n \in \mathbb{Z}^+$,
$(f^n(x_{r-1}), f^{n+k}(x_{r-1})) \in E_r$, i.e.,$\{(f^n(x_{r-1}), f^n(x_{r-1})) | \, n \in \mathbb{Z}^+\} \subset (id_X \times f^k)^{-1}(\overline{E_r})$.
Therefore,

$$\Delta = \overline{\{(f^n(x_{r-1}), f^n(x_{r-1})) | n \in \mathbb{Z}^+\}} \subset (id_X \times f^k)^{-1}(\overline{E_r}) \subset (id_X \times f^k)^{-1}(E_{r-1}),$$

implying that for any $y_{r-1} \in X, (y_{r-1}, f^k(y_{r-1})) \in E_{r-1}$, i.e., (X, f) is uniformly rigid.

3.2. Weak Rigidity on Uniform Spaces

For$x_{r-1} \in Rec(f)$and any $D_{r-1} \in \mathcal{U}$, write $N_{D_{r-1}}(x_{r-1}) = min\{n \in N | f^n(x_{r-1}) \in B(x_{r-1}, D_{r-1})\}$.
Proposition 8 [47]. Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$. Then, for any $n \in \mathbb{N}, Rec(f) = Rec(f^n)$.
Proof. Similarly to the proof of Theorem 4, it can be verified that this holds.
Proposition 9 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$ and $x_{r-1} \in Rec(f)$. Then, $x_{r-1} \in Per(f)$ if and only if $\{N_{D_{r-1}}(x_{r-1}) | D_{r-1} \in \mathcal{U}\}$ is finite.
Proof. From the definition of periodic points, the necessity holds trivially. It suffices to verify the sufficiency. As $\{N_{D_{r-1}}(x_{r-1}) | D_{r-1} \in \mathcal{U}\}$ is finite, let $N = max\{N_{D_{r-1}}(x_{r-1}) | D_{r-1} \in \mathcal{U}\}$. Then, there exists an entourage $E_{r-1} \in \mathcal{U}$ such that $N_{E_{r-1}}(x_{r-1}) = N$. This implies that for any entourage $D_{r-1} \in \mathcal{U}, N_{D_{r-1} \cap E_{r-1}}(x_{r-1}) = N$, i.e., $(x_{r-1}, f^N(x_{r-1})) \in \bigcap_{D_{r-1} \in \mathcal{U}} (D_{r-1} \cap E_{r-1}) = E_{r-1} \cap (\bigcap \mathcal{U}) = \Delta$. Therefore, $f^N(x_{r-1}) = x_{r-1}$.
The following theorem extends [10, Theorem 2.17] which states that every recurrent point of a dynamical system defined on a metric space is$\mathcal{F}_{ip}$-recurrent to a Hausdorff uniform space.

Theorem 10 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$and $x_{r-1} \in Rec(f)$. Then, for any $D_{r-1} \in \mathcal{U}, N(x_{r-1}, B(x_{r-1}, D_{r-1})) \in \mathcal{F}_{ip}$.

Proof. If $x_{r-1} \in Per(f)$,this holds trivially. If $x_{r-1} \in Rec(f) \setminus Per(f)$, applying Proposition (4.2.8) yields that $\{N_{D_{r-1}}(x_{r-1}) | D_{r-1} \in \mathcal{U}\}$ is infinite, implying that $N(x_{r-1}, B(x_{r-1}, D_{r-1}))$ is infinite for any entourage $D_{r-1} \in \mathcal{U}$. For any fixed entourage $D_{r-1} \in \mathcal{U}$, denote $F_{r-1} = N(x_{r-1}, Int(B(x_{r-1}, D_{r-1})))$.Choose $p_1 \in F_{r-1}$. Then, $f^{p_1}(x_{r-1}) \in Int(B(x_{r-1}, D_{r-1}))$. Let $U_r = Int(B(x_{r-1}, D_{r-1})) \cap f^{-p_1}(Int(B(x_{r-1}, D_{r-1})))$. Clearly, U_r is an open neighborhood of $x_{r-1}$and for any $y_{r-1} \in U_r, f^{p_1}(y_{r-1}) \in Int(B(x_{r-1}, D_{r-1}))$. Noting that $N(x_{r-1}, U_r)$ is infinite, choose $p_2 \in (p_1, +\infty) \cap N(x_{r-1}, U_r)$. Clearly, $f^{p_2}(x_{r-1}) \in U_r$. Then, $f^{p_1+p_2}(x_{r-1}) = f^{p_1}(f^{p_2}(x_{r-1})) \in Int(B(x_{r-1}, D_{r-1}))$. In particular,$p_1, p_2, p_2 + p_1 \in F_{r-1}$. Continue this inductively. Assume that $p_1, p_2, \dots, p_n$ have been obtained satisfying that $f^m(x_{r-1}) \in Int(B(x_{r-1}, D_{r-1}))$ holds for all $m \in P_n := \{p_{i_1} + \cdots + p_{i_k} | 1 \le i_1 < \cdots < i_k \le n\}$ and $p_k > \sum_{i=1}^{k-1} p_i$ for all $2 \le k \le n$. Let $(U_{r+n-1}) = Int(B(x_{r-1}, D_{r-1})) \cap (\bigcap_{m \in P_n} f^{-m} (Int(B(x_{r-1}, D_{r-1}))))$. Choose $p_{n+1} \in (p_n, +\infty) \cap N(x_{r-1}, (U_{r+n-1}))$. It can be verified that $p_{n+1}, p_{n+1} + m \in F_{r-1}$ for any $m \in P_n$. Clearly,$\{p_{i_1} + \cdots + p_{i_k} | k \in \mathbb{N}, i_1 < \cdots < i_k\} \subset N(x_{r-1}, B(x_{r-1}, D_{r-1}))$. Therefore, $N(x_{r-1}, B(x_{r-1}, D_{r-1}))$ is an IP_{-set}.

Theorem 11 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$. Then, the following statements are equivalent:
(1) (X, f) is weakly rigid;
(2) (X, f^n) is weakly rigid for any $n \in \mathbb{N}$;
(3) $(X, f^n)$is weakly rigid for some $n \in \mathbb{N}$.
Proof. Applying Proposition 9, this holds trivially.

IV. Rigidity On Hyperspaces

For a uniform space$(X, \mathcal{U})$, set $C(X) = \{A_{r-1} \subset X | A_{r-1} is\ compact\ and\ nonempty\}$. Let $\mathcal{B}_{\mathcal{U}}$ be the family of all sets

$$C^{V_{r-1}} = \{(A_{r-1}, A'_{r-1}) \in C(X) \times C(X) | A_{r-1} \subset B(A'_{r-1}, V_{r-1})\ and A'_{r-1} \subset B(A_{r-1}, V_{r-1})\}, V_{r-1} \in \mathcal{U}.$$

The uniformity on the set $C(X)$ generated by the base$\mathcal{B}_{\mathcal{U}}$ is denoted by $C^{\mathcal{U}}$.
The topology on $C(X)$ generated by the family of all the sets of the form

$$\mathcal{V}(U_r, U_{r+1}, \dots, U_{r+n-1}) = \left\{A_{r-1} \in C(X) A_{r-1} \subset \bigcup_{i=1}^{n} (U_{r-1})_i\ and A_{r-1} \cap (U_{r-1})_i \neq \emptyset\ for\ i = r, r+1, \dots, r+n-1\right\},$$

where $U_r, U_{r+1}, \dots, U_{r+n-1}$is a finite sequence of open subsets of $(X, |\mathcal{U}|)$, is called the Vietoris topology on $C(X)$ and $C(X)$ with the Vietoris topology is called the hyperspace of X. [26] shows that the topology $|C^{\mathcal{U}}|$ generated by $C^{\mathcal{U}}$ coincides with the Vietoris topology on $C(X)$.
Let (X, f)be a dynamical system. Define $C^f : (C(X), |C^{\mathcal{U}}|) \longrightarrow (C(X), |C^{\mathcal{U}}|)$ by

$$C^f(A_{r-1}) = f(A_{r-1}), \qquad \forall A_{r-1} \in C(X).$$

Clearly, C^f is well defined and uniformly continuous. Therefore,$(C(X), C^f)$is a dynamical system. For any $n \in \mathbb{N}$, define $C_n(X) = \{A_{r-1} \in C(X) | |A_{r-1}| \le n\}$ and $C_\infty(X) = \bigcup_{n \in \mathbb{N}} C_n(X)$. The following result is easy to check.
Lemma 12. Let $(X, \mathcal{U})$ be a Hausdorff uniform space. Then, $C_\infty(X)$ is dense in $(C(X), |C^{\mathcal{U}}|)$.
The next theorem shows that both weak and uniform rigidity of $(C(X), C^f)$ are equivalent to the uniform rigidity of (X, f).

Lemma 13 (see [47]). Let $(X, \mathcal{U})$ be a Hausdorff uniform space, $A_{r-1} \in \mathcal{U}$ and $z_{r-1} \in X$. Then,
(1) for any $n \in \mathbb{N}, B(z_{r-1}, A_{r-1}^n) \subset Int(B(z_{r-1}, A_{r-1}^{n+1}))$;
(2) $\overline{B(z_{r-1}, A_{r-1})} \subset B(z_{r-1}, \overline{A_{r-1}})$.

Proof. (1) It has been proved in [14, Lemma 2.5].
(2) It suffices to verify that$B(z_{r-1}, \overline{A_{r-1}})$ is a closed subset. For any $y_{r-1} \in X \setminus B(z_{r-1}, \overline{A_{r-1}})$, there exist open sunsets U_{r-1}, V_{r-1} of Xsuch that$(z_{r-1}, y_{r-1}) \in U_{r-1} \times V_{r-1} \subset X \times X \setminus \overline{A_{r-1}}$. This implies that $y_{r-1} \in V_{r-1} \subset X \setminus B(z_{r-1}, \overline{A_{r-1}})$, i.e., $X \setminus B(z_{r-1}, \overline{A_{r-1}})$ is an open subset of X. Therefore, $B(z_{r-1}, \overline{A_{r-1}})$ is a closed subset of X.

Theorem 14 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$. Consider the following statements:
(1) (X, f) is uniformly rigid;
(2) $(C(X), C^f)$ is uniformly rigid;
(3) $(C(X), C^f)$ is weakly rigid.

Then, (1) and (2) are equivalent. If additionally$X$ is compact, then (1)–(3) are all equivalent.
Proof.(1) ⇒ (2). For any $W \in C^{\mathcal{U}}$, there exists $V_{r-1} \in \mathcal{U}$ such that $C^{V_{r-1}} \subset W$. The uniform rigidity of (X, f) implies that there exists $n \in \mathbb{N}$ such that for any $x_{r-1} \in X, (x_{r-1}, f^n(x_{r-1})) \in V_{r-1}$ and $(f^n(x_{r-1}), x_{r-1}) \in V_{r-1}$. This implies that for any $A_{r-1} \in C(X), A_{r-1} \subset B(f^n(A_{r-1}), V_{r-1})$ and $f^n(A_{r-1}) \subset B(A_{r-1}, V_{r-1})$,i.e., $(A_{r-1}, f^n(A_{r-1})) \in C^{V_{r-1}} \subset W$. Therefore, $\{(A_{r-1}, f^n(A_{r-1})) | A_{r-1} \in C(X)\} \subset W$,implying that $(C(X), C^f)$ is uniformly rigid.
(2) ⇒ (1). This holds trivially.
When X is compact, it suffices to check (3) ⇒ (1). Suppose that $(C(X), C^f)$ is weakly rigid and fix any entourage$E_{r-1} \in \mathcal{U}$. Choose an entourage $E_{r-1} \in \mathcal{U}$ such that $\overline{E_r^3} \subset E_{r-1}$. The compactness of X implies that there exists$x_r, \dots, x_{r+n-1} \in X$ such that $\bigcup_{i=1}^{n} B((x_{r-1})_i, E_r) = X$. As $(C(X), C^f)$ weakly rigid, it follows that$(B(x_r, E_r), \dots, B(x_{r+n-1}, E_r))$ is a recurrence of $(C(X)^n, (C^f)(n))$. Applying Lemma 13 implies that for any $1 \leq i \leq n, \overline{B((x_{r-1})_i, E_r)} \subset B((x_{r-1})_i, \overline{E_r}) \subset Int(B((x_{r-1})_i, \overline{E_r^2}))$. Then, there exists $k \in \mathbb{N}$such that

$$((C^f)^{(n)}))^k \overline{(B(x_r, E_r)}, \dots, \overline{B((x_{r-1})_n, E_r)}) \in \mathcal{V}(Int(B(x_r, \overline{E_r^2}))) \times \dots \times \mathcal{V}(Int(B(x_{r+n-1}, \overline{E_r^2}))),$$

i.e., for any $1 \leq i \leq n, f^k(\overline{B((x_{r-1})_i, E_r)}) \subset Int(B((x_{r-1})_i, \overline{E_r^2}))$. For any $x_{r-1} \in X$, there exists $1 \leq i \leq n$ such that $x_{r-1} \in B((x_{r-1})_i, E_r)$. Therefore, $f^k(x_{r-1}) \in Int(B((x_{r-1})_i, \overline{E_r^2}))$, i.e., $((x_{r-1})_i, f^k(x_{r-1})) \in \overline{E_r^2}$. This, together with $(x_{r-1}, (x_{r-1})_i) \in E_r$, implies that $(x_{r-1}, f^k(x_{r-1})) \in \overline{E_r^3} \subset E_{r-1}$. Hence, (X, f) is uniformly rigid.

V. A Remark On Sensitivity

The authors in[6, Proposition 2.2]] obtained the following result for compact metric spaces, whose proof is straightforward.
Lemma 15. Let X be a topological space and $f : X \longrightarrow X$ be continuous. If $Trans(f) \neq \emptyset$, then for any $x_{r-1} \in Trans(f)$ and any neighborhood $U_{r-1}$of $x_{r-1}, N(U_{r-1}, U_{r-1}) = N(x_{r-1}, U_{r-1}) - N(x_{r-1}, U_{r-1})$.
For minimal and almost periodic points, we have the following result. Note that the proof is similar to [15, Lemma 3, Lemma 4]], but for completeness, a proof is provided here.
Lemma 16 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space $(X, \mathcal{U})$. Then, $A_{r-1}(f) \subset M(f)$.
Proof. Take any fixed $x_{r-1} \in A_{r-1}(f)$ and suppose that x_{r-1} is not minimal, i.e., $x_{r-1} \notin M(f)$. Then, there exists some $z_{r-1} \in \overline{orb(x_{r-1}, f)}$such that $\overline{orb(z_{r-1}, f)} \not\subseteq \overline{orb(x_{r-1}, f)}$. As $(X, \mathcal{U})$ is completely regular, there exist open subsets $U_{r-1}, V_{r-1}$of X such that $x_{r-1} \in U_{r-1}, \overline{orb(z_{r-1}, f)} \subset V_{r-1}$, and $U_{r-1} \cap V_{r-1} = \emptyset$. For any $n \in \mathbb{N}$, let $V_{r+n-1} = \bigcap_{i=0}^{n} f^{-i}(V_{r-1})$. Then, there exists $k \in \mathbb{N}$ such that $f^k(x_{r-1}) \in V_{r+n-1}$as $V_{r+n-1}$is a neighborhood ofz_{r-1}. This implies that for any $0 \leq i \leq n, f^{k+i}(x_{r-1}) \in V_{r+n-1} \subset V_{r-1}$, i.e., $\{i \in \mathbb{Z}^+ | f^i(x_{r-1}) \in V_{r-1}\} \in \mathcal{F}_t$, which is a contradiction as $\{i \in \mathbb{Z}^+ | f^i(x_{r-1}) \in U_{r-1}\} \in \mathcal{F}_s$ and $U_{r-1} \cap V_{r-1} = \emptyset$.

Theorem 17 (see [47]). Let (X, f) be a dynamical system defined on a Hausdorff uniform space$(X, \mathcal{U})$ and $\mathcal{F}$be a Furstenberg family. If$(X, f)$is $\mathcal{F}$-transitive and almost equicontinuous, then for any $x_{r-1} \in Trans(f)$ and any neighborhood U_{r-1} of $x_{r-1}, N(x_{r-1}, U_{r-1}) \in \Delta(\mathcal{F})$.
Proof. Clearly, $Trans(f) \neq \emptyset$ (see Lemma 6). For any fixed$x_{r-1} \in Trans(f)$ and any neighborhood $U_{r-1}$of x_{r-1}, it follows from Lemma 6 that $x_{r-1}$is an equicontinuous point. Choose an entourage $E_{r-1} \in \mathcal{U}$such that $B(x_{r-1}, E_{r-1} \circ E_{r-1}) \subset U_{r-1}$. Then, there exists an entourage $E_r \subset E_{r-1}$ such that for any $y_{r-1} \in B(x_{r-1}, E_r)$ and any $n \in \mathbb{Z}^+, (f^n(x_{r-1}), f^n(y_{r-1})) \in E_{r-1}$. For any $n \in \mathbb{N}(B(x_{r-1}, E_r), B(x_{r-1}, E_r))$, there exists $y_{r-1} \in B(x_{r-1}, E_r)$ such that $f^n(y_{r-1}) \in B(x_{r-1}, E_r)$. Thus, $(x_{r-1}, f^n(x_{r-1})) \in E_r \circ E_{r-1} \subset E_{r-1} \circ E_{r-1}$, implying that

$$N(B(x_{r-1}, E_r), B(x_{r-1}, E_r)) \subset N(x_{r-1}, B(x_{r-1}, E_{r-1} \circ E_{r-1})) \subset N(x_{r-1}, U_{r-1}) \in \mathcal{F}.$$

This, together with Lemma 15, implies that for any neighborhood $U_{r-1}$of x_{r-1}, $N(U_{r-1}, U_{r-1}) = N(x_{r-1}, U_{r-1}) - N(x_{r-1}, U_{r-1}) \in \Delta(\mathcal{F})$.
For any fixed neighborhood U_{r-1} of x_{r-1}, take two entourages $E_r \subset E_{r+1}$ such that:(a) $B(x_{r-1}, E_{r+1} \circ E_{r+1}) \subset U_{r-1}$; (b) for any $y_{r-1} \in B(x_{r-1}, E_r)$ and any $n \in \mathbb{Z}^+, (f^n(x_{r-1}), f^n(y_{r-1})) \in E_{r+1}$. The above proof implies

that $N\big(B(x_{r-1},E_r),B(x_{r-1},E_r)\big)\in\Delta(\mathcal{F})$. For any $n\in N\big(B(x_{r-1},E_r),B(x_{r-1},E_r)\big)$, there exists some $y_{r-1}\in B(x_{r-1},E_r)$such that $f^n(y_{r-1})\in B(x_{r-1},E_r)$, i.e.,$(f^n(y_{r-1}),x_{r-1})\in E_r\subset E_{r+1}$. This, together with $\big(f^n(x_{r-1}),f(y_{r-1})\big)\in E_{r+1}$, implies that $(f^n(x_{r-1}),x_{r-1})\in E_{r+1}\circ E_{r+1}$.
Therefore,$f^n(x_{r-1})\in B(x_{r-1},E_{r+1}\circ E_{r+1})\subset U_{r-1}$, implying that

$$N(B(x_{r-1},E_r),B(x_{r-1},E_r))\subset N(x_{r-1},U_{r-1})\in\Delta(\mathcal{F}).$$

Lemma 18. ([10, Proposition 3.19]) $\Delta(\mathcal{F}_{pubd})\subset\mathcal{F}_s$.
Lemma 19.([40, Theorem 7]) Let (X,f) be a dynamical system. If $(X,f)$is point transitive, then exactly one of the following holds:
(1) $Eq(f)\neq\emptyset.(X,f)$ is almost equicontinuous and $Eq(f)=Trans(f)$;
(2) $Eq(f)=\emptyset.(X,f)$ is sensitive.
In particular, if (X,f) is minimal, then it is either equicontinuous or sensitive.

Corollary 20 [47]. Let (X,f)be a dynamical system defined on a Hausdorff uniform space $(X,\mathcal{U})$. If $(X,f)$is $(F_{r-1})_{pubd}$ -transitive and almost equicontinuous, then (X,f) is minimal and equicontinuous.
Proof. For any $x_{r-1}\in Trans(f)$, applying Theorem 17 and Lemma 18 yields that $x_{r-1}\in A_{r-1}(f)$. This, together with Lemma 17, implies that $(X,f)$is minimal. The proof is completed by applyingLemma 19.
Corollary 21 [47]. Let (X,f) be a dynamical system defined on an infinite Hausdorff uniform space $(X,\mathcal{U})$. If (X,f) is point transitive and has dense periodic points, then (X,f) is sensitive.
Proof. Suppose that (X,f) is not sensitive, it follows from Lemma 19 that it is almost equicontinuous. As (X,f) has dense periodic points, it is easy to see that it is $\mathcal{F}_s$-transitive. Applying Corollary 20 yields that (X,f) is minimal, implying that $X=orb(p,f)$ for any periodic point $p\in X$, which is a contradiction as X is infinite.
Corollary 22 [47]. Let (X,f)be a non-minimal dynamical system defined on a Hausdorff uniform space $(X,\mathcal{U})$. If (X,f) is point transitive and has dense Banach almost periodic points, then (X,f) is sensitive.
Proof. Suppose that (X,f) is not sensitive, it follows from Lemma 19 that it is almost equicontinuous. As (X,f) has dense Banach almost periodic points, it is easy to see that it is $\mathcal{F}_{pubd}$ -transitive. Applying Corollary 20 yields that(X,f) is minimal, which is a contradiction.
A point transitive dynamical system $(X,\mathcal{U})$ is called an E-system if for every $x_{r-1}\in Trans(f)$ and every neighborhood U_{r-1} of x_{r-1}, $N(x_{r-1},U_{r-1})\in\mathcal{F}_{pubd}$. It has been proved that this is equivalent to the original definition of E_{r-1}-system provided that the base space is a compact metric space (see [45, Theorem 2.6.2]).

Remark 23.
(1) Applying Corollary 21 implies that [14, Theorem 3.3] holds trivially.
(2) Clearly, every E-system is Fpubd-transitive. This, together with Corollary 20, implies that every almostequicontinuous E-system is minimal and equicontinuous (see [12, Theorem 1.3]).

[16] proved that a dynamical system defined on a compact metric space is sensitive provided that itshyperspatial dynamical system is sensitive. Now, we extend this result to the Hausdorff uniform spaces. Itis worthwhile to note that [25] constructed an example to show that the converse may not hold.

Theorem 24 (see [47]). Let (X,f) be a dynamical system defined on a Hausdorff uniform space $(X,\mathcal{U})$. If $(C(X),C^f)$ is sensitive, then (X,f) is sensitive.
Proof. Assume $(C(X),C^f)$ is sensitive with a sensitive entourage $C^{V_{r-1}}$ for some$V_{r-1}\in\mathcal{U}$. For any $x_{r-1}\in X$and any entourage $D_{r-1}\in\mathcal{U}$, the sensitivity of C^f implies that there exist$A_{r-1}\in B(\{x_{r-1}\},C^{D_{r-1}})$ and$n\in\mathbb{N}$ such that $((C^f)^n(\{x_{r-1}\}),(C^f)^n(A_{r-1}))=(f^n(\{x_{r-1}\}),f^n(A_{r-1}))\notin C^{V_{r-1}}$. Thus, $f^n(\{x_{r-1}\})\nsubseteq B(f^n(A_{r-1}),V_{r-1})$ or $f^n(A_{r-1})\nsubseteq B(f^n(\{x_{r-1}\}),V_{r-1})$, implying that there exists $y_{r-1}\in A_{r-1}$ such that $(f^n(x_{r-1}),f^n(y_{r-1}))\notin V_{r-1}$. Combining this with $y_{r-1}\in A_{r-1}\subset B(x_{r-1},D_{r-1})$, it follows that$(X,f)$ is sensitive with a sensitive entourage$V_{r-1}\in\mathcal{U}$.
The multi-sensitivity was introduced by [27]. Recently, [42] proved that a product system is multi-sensitive if and only if there exists a factor system is multi-sensitive. A dynamical system(X,f) defined on a Hausdorff uniform space $(X,\mathcal{U})$is multi-sensitive if there exists an entourage $E_{r-1}\in\mathcal{U}$ (a multi-sensitive entourage) such that for any $k\in\mathbb{N}$ and nonempty open subsets $U_r,U_{r+1},\ldots,(U_{r+k-1})$ of X, there exists $n\in\mathbb{Z}^+$such that for any $1\leq i\leq k,(f\times f)^n((U_{r+i-1})\ \times(U_{r+i-1})\ )\cap(X\times X\backslash E_{r-1})\neq\emptyset$. It can be verified that a dynamical system (X,f) is multi-sensitive if and only if there exists an entourage $E_{r-1}\in\mathcal{U}$ such that for any$k\in\mathbb{N}$, any x_r, $x_{r+1},\ldots,(x_{r-1})_k\in X$, and any neighborhood (U_{r+i-1}) of (x_{r+i-1}) , there exist$n\in\mathbb{N}$ and $(y_{r+i-1})\in(U_{r+i-1})$ such that $(f^n((x_{r+i-1})\ ),f^n((y_{r+i-1})\ ))\notin E_{r-1}$ for all $1\leq i\leq k$.
Theorem 25 (see [47]). Let (X,f) be a dynamical system defined on a Hausdorff uniform space $(X,\mathcal{U})$. Then, (X,f) is multi-sensitive if and only if $(C(X),C^f)$ is multi-sensitive.
Proof.$(\Leftarrow)$. Assume that $(C(X),C^f)$ is multi-sensitive with a multi-sensitive entourage $C^{V_{r-1}}$ for some entourage$V_{r-1}\in\mathcal{U}$. For any $k\in\mathbb{N}$ and nonempty open subsets $U_r,U_{r+1},\ldots,(U_{r+k-1})$ of X, it is clear that

$\mathcal{V}(U_r), \mathcal{V}(U_{r+1}), \ldots, \mathcal{V}((U_{r+k-1})\ )$ are nonempty open subsets of$C(X)$. Then, there exists $n \in \mathbb{N}$ such that for any $1 \leq i \leq k, (C^f \times C^f)^n(\mathcal{V}((U_{r+i-1})\ ) \times \mathcal{V}((U_{r+i-1})\ )) \cap C(X) \times C(X) \backslash C^{V_{r-1}} \neq \emptyset$, i.e., there exist $(A_{r-1})_i, B_i \in \mathcal{V}((U_{r+i-1})\ )$ such that$(f^n((A_{r+i-1})\ ), f^n(B_i)) \in C^{V_{r-1}}$, implying that $f^n((A_{r+i-1})\ ) \nsubseteq B(f^n(B_i), V_{r-1})$ or $f^n(B_i) \nsubseteq B(f^n((A_{r+i-1})\ ), V_{r-1})$.Therefore, there exist $(x_{r+i-1})\ \in (A_{r+i-1})$ and $(y_{r+i-1})\ \in B_i$ such that $(f^n((x_{r+i-1})\ ), f^n((y_{r+i-1})\ )) \notin V_{r-1}$. This, together with $(x_{r+i-1})\ , (y_{r+i-1})\ \in (U_{r+i-1})$, implies that $(f \times f)^n((U_{r+i-1})\ \times (U_{r+i-1})\ ) \cap (X \times X \backslash V_{r-1}) \neq \emptyset$. Hence, (X, f) is multi-sensitive.

$(\Rightarrow)$. Assume that $(X, f)$is multi-sensitive with a multi-sensitive entourage $E_{r-1} \in \mathcal{U}$. Choose an entourage $\widehat{E_{r-1}} \in \mathcal{U}$ such that $\widehat{E_{r-1}^4} \subset E_{r-1}$. Applying Lemma 12, it suffices to check that$(C_\infty(X), C^f)$ is multi-sensitive. For any $k \in \mathbb{N}$, any $(A_{r-1})_j = \left\{(x_{r-1})_i^{(j)}\right\}_{i=1}^{n_j} \in C_\infty(X)$, and any neighborhood $\left(U_{r+j-1}\right)$ of $\left(A_{r+j-1}\right)$ $(1 \leq j \leq k$, there exist entourages $Er, Er+1, \ldots, Er+k-1 \in \mathcal{U}$ such that for any $1 \leq j \leq k$,

$$\left(A_{r+j-1}\right)\ \in \mathcal{V}\left(Int\left(B\left(x_r^{(j)}, \left(E_{r+j-1}\right)\ \right)\right), \ldots, Int\left(B\left((x_{r-1})_{n_j}^{(j)}, \left(E_{r+j-1}\right)\ \right)\right)\right) \subset \left(U_{r+j-1}\right)\ .$$

The multi-sensitivity of (X, f) implies that there exist $n \in \mathbb{N}$ and$(y_{r-1})_i^{(j)} \in Int(B((x_{r+i-1})^{(j)}, \left(E_{r+j-1}\right)\ ))$ such that for any $1 \leq j \leq k$and any $1 \leq i \leq n_j, (f^n((x_{r-1})_i^{(j)}),\ f^n((y_{r-1})_i^{(j)})) \notin \hat{E}_{r-1}^2$. For any $1 \leq j \leq k$ and any $1 \leq i \leq n_j$, take $(z_{r-1})_i^{(j)}$ as follows:

(1) $z_r^{(j)} = y_r^{(j)}$;

(2) If $(f^n(x_r^{(j)}), f^n((x_{r-1})_i^{(j)})) \notin \widehat{E_{r-1}}$, then $(z_{r-1})_i^{(j)} = (x_{r-1})_i^{(j)}$;

(3) If $(f^n((x_{r-1})_r^{(j)}), f^n\ ((y_{r+-1})_i^{(j)})) \notin \widehat{E_{r-1}}$, then $(z_{r-1})_i^{(j)} = (y_{r-1})_i^{(j)}$.

Clearly, $(z_{r+i-1})^{(j)}$ exists. Let $Z_j = \left\{(z_{r+i-1})^{(j)}\right\}_{i=1}^{n_j}$. It can be verified that $Z_j \in \left(U_{r+j-1}\right)$ and $f^n(x_r^{(j)}) \notin B(f^n(Z_j), \widehat{E_{r-1}})$, implying that$\left((C^f)^n((A_{r-1})_j\ ), (C^f)^n(Z_j)\right) = (f^n((A_{r-1})_j), f^n(Z_j\ )) \notin C^{\widehat{E_{r-1}}}$. Therefore, $(C(X), C^f)$is sensitive.

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