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Research Paper

On the way of Pair wise Orthogonal Frames Generated by Regular and Unitary Representations of LCA Groups

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Abstract
A. Gumber and N.K. Shukla [29] study orthogonality for strongly disjointness of a pair of frames over locally compact abelian (LCA) groups. Following [29] we show the investigation of the dual Gramian analysis tools of Ron and Shen through a pre-Gramian operator over the set-up of LCA groups. Then we fiberize some operators associated with Bessel families generated by unitary actions of co-compact subgroups of LCA groups. Using this fiberization, we study and characterize a pair of orthogonal frames generated by the action of a unitary representation ρ*of a co-compact subgroup* $\Gamma \subset G$*on a separable Hilbert space* $L^2(G)$*, where* G*is a second countable LCA group. We consider sequences of frames of the form* $\{\rho(\gamma^m)\psi_m : \gamma^m \in \Gamma, \psi_m \in \Psi\}$*for a countable family* Ψ *in* $L^2(G)$*. We pay special attention to this problem of translationinvariant space by assuming* ρ*as the action of* Γ *on* $L^2(G)$ *by left-translation. The representation of* Γ *acting on* $L^2(G)$ *by (left-)translation is called the (left-)regular representation of* Γ*. Further, we apply our results on co-compact Gabor systems over LCA groups. We note that the resulting characterization can be useful for constructing new frames by using various techniques by [24] and to LCA groups by [7].*
Keywords: *Translation-invariant space; Orthogonal frames; LCA group; Unitary representation;Gramian operator; Co-compact Gabor system.*

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I. Introduction

We note that the theory of frames has a very close connection with unitary group representations since various structured systems, for example, wavelet and Gabor systems can be realized as a sequence obtained by applying a family of unitary operators to a particular window function. The study of frames of locally compact abelian (LCA) groups enables us to consider key questions in frame analysis. Moreover, it provides a unified way to the analysis on the four elementary finite groups $\mathbb{R}, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_m$ and their higher dimensional variants. So in signal processing we consider products of the above four groups, and the LCA group approach applies to all groups of the form $\mathbb{R}^s \times \mathbb{Z}^p \times \mathbf{T}^q \times \mathbb{Z}_m$, we are concerned here with frames generated by unitary actions of LCA groups. Many authors have contributed in analysing various interesting properties and results of frame theory in different ways (see $[3-7,16,19,26,27]$). Among these properties, the "orthogonality or strongly disjointness" of a pair of frames studied by [1], and [12]. The property says thatthe associated mixed dual-Gramian operator is zero, which is further equivalent to the orthogonality of the ranges of the analysis operators for the two frames. The corresponding frames are called pairwise orthogonal (simply, orthogonal). Due to potential applications in multiplexing techniques and in the synthesis of frames [1,2,10,14], this natural geometric concept in the frame theory has been significantly developed (see $[10,12-15,21,28]$).

[29] study and characterize pairwise orthogonal frames generated by unitary representations of LCA groups. They investigate the dual Gramian analysis tools of [25] by introducing the notion of pre-Gramian operator associated with Bessel families generated by unitary actions of co-compact subgroups of LCA groups,and also study the fiberization of various operators corresponding to these families via the pre-Gramian. We examine the orthogonality of frames for subspaces of a separable Hilbert space $\mathcal{H}$ that are invariant under the action of unitary representations of a closed and co-compact abelian group $\Gamma \subset G$ on $\mathcal{H}$, where G is a second countable LCA group. Here, by a unitary representation of Γ, we mean a pair $(\rho, \mathcal{H})$ with ρ as a group homomorphism of Γ into $\mathbb{U}(\mathcal{H})$, that is, the group of linear unitary operators over $\mathcal{H}$. We assume that the map $\Gamma \times \mathcal{H} \to \mathcal{H}; (\gamma^m, h) \mapsto \rho(\gamma^m)h$ is continuous. Note that a subspace of $\mathcal{H}$ that is invariant under the action of unitary representation ρ of Γ on the Hilbert space $\mathcal{H}$ is called (ρ, Γ) -invariant. To be precise about our concern,

we let $\Gamma \ni \gamma^m \mapsto \rho(\gamma^m) \in \mathbb{U}(\mathcal{H})$ as a unitary representation of Γ on $\mathcal{H}$. and study an interesting research problem regarding the orthogonality of a pair of frames for (ρ, Γ) -invariant subspaces in $\mathcal{H}$. The problem can be stated as follows.

(Q1) For which families Ψ and Φ in $\mathcal{H}$ do the collections $\langle\Psi\rangle := \{\rho(\gamma^m)\psi_m : \gamma^m \in \Gamma, \psi_m \in \Psi\}$ and $\langle\Phi\rangle := \{\rho(\gamma^m)\varphi_m : \gamma^m \in \Gamma, \varphi_m \in \Phi\}$ form a pair of orthogonal frames for $\overline{\mathrm{span}\langle\Psi\rangle}^{\mathcal{H}}$, that is, for the closed linear span of $\langle\Psi\rangle$, which is indeed the smallest (ρ, Γ) -invariant subspace in $\mathcal{H}$? We mention that by considering ρ as the action of a closed, co-compact subgroup Γ on $\mathcal{H}$ by left-translation, the main focus is to explore the question (Q1) for the (ρ, Γ) -invariant subspaces of LCA groups. Note that the representation of Γ acting on $\mathcal{H}$ by (left-) translation is called the (left-) regular representation, and the invariant subspaces of this representation are called translation-invariant (TI).We note that our work extends the results from the discrete setting $G = \mathbb{Z}^n$ and from the case of uniform lattices in $L^2(\mathbb{R}^n)$, studied respectively by [22] and [21], to the set-up of (not necessarily discrete) co-compact subgroups over LCA groups.

The significance of considering co-compact subgroups in function systems is related to the necessity of overcoming the limitation on the existence of uniform lattices for an LCA group, which says there exist LCA groups that do not contain any uniform lattices. For example, the field $\mathbb{Q}_p$ of p -adic numbers, whose only discrete (and closed) subgroup is given by the neutral element which is not a C uniform lattice. On the other hand, the field $\mathbb{Q}_n$ have only one closed and co-compact subgroup

uniform lattice. On the other hand, the field $\mathbb{Q}_p$ have only one closed and co-compact subgroup Γ, namely the subgroup given by the entire group itself, $\Gamma = \mathbb{Q}_p$. Another example is the p -adic integers (a compact group) which have only trivial examples of uniform lattices but have a lot of non-trivial co-compact subgroups,see[5].

By G we denote a second countable LCA group, with the additive group composition, denoted by the symbol " $+$" and neutral element 0. Unless mentioned otherwise we assume Γ to be a closedand co-compact subgroup in G. Note that a subgroup Γ in G is called co-compact if the quotient group G/Γ is compact, whereas Γ in G is said to be a uniform lattice if in addition, Γ is discrete.

Now, we consider $\mathcal{H}$ as $L^2(G)$ which is a Hilbert space with inner product given by $\langle f_m, g_m\rangle = \int f_m(x)\overline{g_m(x)}d\mu_G(x)$, for all $f_m, g_m \in L^2(G)$, where the symbol μ_G (not identically zero) denotes a unique Haar measure (with a well-known existence) possessed by G. It should be noted that the Haar measure of a locally compact group is unique only up to a positive multiplicative constant. Similarly, the notation μ_Γ represents a Haar measure on the subgroup Γ. Now, we let the unitary representation

$$\rho: \Gamma \to \mathbb{U}(L^2(G)); \gamma^m \mapsto \rho(\gamma^m)$$

act by a left-translation, i.e., $\rho(\gamma^m)f_m = f_m(\cdot - \gamma^m)$ for all $f_m \in L^2(G)$, and define $E^\Gamma(\Psi)$ to be the family

$$\{\rho(\gamma^m)\psi_m : \gamma^m \in \Gamma, \psi_m \in \Psi\} =: E^\Gamma(\Psi) \qquad (1.1)$$

which is generated by a countable subset Ψ in $L^2(G)$. Next, we proceed to formulate the statement of the question (Q1) in terms of TI subspaces which were introduced and studied by [5].We observe the following definition (see [29]):

Definition 1.1. Suppose that $\Gamma \subset G$ is a closed, co-compact subgroup of G. Let $V \subset L^2(G)$ be a closed subspace. Then

(i) we say that V is (ρ, Γ) -invariant or more appropriately translation-invariant (TI) under Γ, in short $\Gamma - \mathrm{TI}$, if $f_m \in V$ implies $\rho(\gamma^m)f_m \in V$ for all $\gamma^m \in \Gamma$.

(ii) we call $E^\Gamma(\Psi)$, that is, the system generated by Ψ (defined in (1.1)), as a Γ - TI system with its closed linear span in $L^2(G)$ denoted by $\overline{\mathrm{span}(E^\Gamma(\Psi))}^{L^2(G)} =: S^\Gamma(\Psi)$ as a Γ - TI space which is the smallest closed subspace in $L^2(G)$ containing $E^\Gamma(\Psi)$.

(iii) for $\Psi = \{\psi_m\}$ as a singleton subset in $L^2(G)$, let the principle translation-invariant $(\Gamma - \mathrm{PTI})$ system be given by $E^\Gamma(\psi_m) := \{\rho(\gamma^m)\psi_m : \gamma^m \in \Gamma\}$. In this case, $\overline{\mathrm{span}(E^\Gamma(\psi_m))}^{L^2(G)} =: S^\Gamma(\psi_m)$ is called a principle translation-invariant (\Gamma-PTI) space generated by ψ_m.

(iv) in case Γ is a uniform lattice, the term translation-invariant is replaced by shift-invariant (SI).

Now, we state the main problem associated with (Q1) in the context of Γ -TI systems over LCA groups. For this, let $\Psi := \{(\psi_m)_p\}_{p\in\mathcal{P}}$ and $\Phi := \{(\varphi_m)_p\}_{p\in\mathcal{P}}$ be subsets in $L^2(G)$, where $\mathcal{P}$ is a countable index set. Then, by assuming $S^r(\Psi) = S^\Gamma(\Phi)$, the main problem that we will investigate is:

(Q2) To find necessary and sufficient conditions on the above discussed generators Ψ and Φ in the Fourier domain such that the Γ - TI systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ as defined in (1.1), form a pair of orthogonal frames in the Γ -TI space $S^\Gamma(\Psi)$. For answering the above question, we introduce the notion of pre-Gramian operator associated ctiva with a Γ -TI space $S^\Gamma(\Psi)$ and fiberize the analvsis. svnthesis and mixed dual-Gramian onen corresponding to the Γ -TI systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$. Note that the fiberization is helpful in analysing various frame properties of the Γ -TI systems.

We devoted to the study of the orthogonality property of a pair of frames for Γ -TI spaces over LCA groups by using the approach of a pre-Gramian operator. The major tool used here is the dual Gramian analysis of [25] over the above set-up. Note that the novelty of our results lies in understanding that we do not require Γ to be discrete. As a consequence, our results for two Γ -TI systems to be pairwise orthogonal frames are new even in the classical setting of $G = \mathbb{R}^n$, where the results of [21] are not applicable. Moreover, it also applies to. LCA groups G that do not have uniform lattices such as the p -adic field, $\mathbb{Q}_p$.

We apply our results on systems with co-compact Gabor structure to obtain a characterization so that the above-structured systems become pairwise orthogonal Γ -TI Bessel (frame) systems.

We introduce the notion of pre-Gramian operator associated to Γ -TI systems over LCA groups, and discuss the fiberization of operators associated with these systems. We devoted to the study of orthogonal Γ -TI Bessel (frame) systems by using the fiberization approach. We deal with the application of our results on co-compact Gabor frame systems over LCA groups.

We deal with the development of the necessary machinery required to answer the question (Q2) imposed on the orthogonality of two Γ -TI frame systems. Now, let

$$L^2(\Gamma, \ell^2(\mathcal{P})) := \left\{ \text{measurable } h: \Gamma \to \ell^2(\mathcal{P}) \text{ such that } \| h \|^2 := \int_\Gamma \sum_m \| h(\gamma^m) \|^2_{\ell^2(\mathcal{P})} d\mu_\Gamma(\gamma^m) < \infty \right\}$$

be the Hilbert space of $\ell^2(\mathcal{P})$ -valued square integrable functions over Γ with inner product $\langle h^1, h^2 \rangle := \int \langle h^1(\gamma^m), h^2(\gamma^m) \rangle_{\ell 2()} d\mu_\Gamma(\gamma^m)$, for all $h^i := (h^i(\gamma^m))_{\gamma^m \in \Gamma}$ in $L^2(\Gamma, \ell^2(\mathcal{P}))$ with $i = 1,2$, and $h^i(\gamma^m) := \left(h^i_p(\gamma^m)\right)_{p \in \mathcal{A}}$ in $\ell^2(\mathcal{P})$ for each γ^m. Note that $\Gamma \subset G$ is a second countable locally compact abelian (LCA) group. Thus, Γ is σ -compact, and hence σ -finite. Further, $\mathcal{P}$ is a countable index set, therefore, $L^2(\Gamma, \ell^2(\mathcal{P})) = L^2(\mathcal{P} \times \Gamma)$ since $\ell^2(\mathcal{P})$ is separable.

2.1. Γ-TI frame systems and associated operators.

We give the following definition of a Γ-TI system (see [5]) to be a frame, and study various operators associated with such systems. See[4,6,20,23] for the general definitions of frames and associated operators along with other basic concepts of frame theory.

Definition 2.1 (see [29]). The Γ -TI system $E^\Gamma(\Psi)$ is called a Γ - TI frame system for $S^\Gamma(\Psi)$, if there exist frame bounds $0 < A \le \mathcal{A} + \epsilon < \infty$ such that the following relation is satisfied:

$$\mathcal{A}||f_m||^2 \le \sum_{p \in \mathcal{P}} \int_\Gamma \sum_m \left|\langle f_m, \rho(\gamma^m)(\psi_m)_p \rangle\right|^2 d\mu_\Gamma(\gamma^m) \le (\mathcal{A} + \epsilon)||f_m||^2 \text{ , for all } f_m \in S^\Gamma(\Psi) \quad (2.1)$$

A $\Gamma -$ TI frame system $E^\Gamma(\Psi)$ is called tight Γ - TI frame system for $S^\Gamma(\Psi)$ with frame bound $\mathcal{A}$ if we can choose $\epsilon = 0$. Note that $E^\Gamma(\Psi)$ is a Γ - TI Bessel system for $S^\Gamma(\Psi)$ with $(\mathcal{A} + \epsilon)$ as its Bessel constant if the right side of inequality in (2.1) holds. Similarly, we can define all the above terms for the case ctiv of $S^\Gamma(\Psi) = L^2(G)$

Further, let $E^\Gamma(\Psi)$ be a Γ -TI Bessel system for $L^2(G)$. Then, for $h := (h(\gamma^m))_{\gamma^m \in \Gamma}$ in $L^2(\Gamma, \ell^2(\mathcal{P}))$ with $h(\gamma^m) := \left(h_p(\gamma^m)\right)_{p \in \mathcal{O}}$ in $\ell^2(\mathcal{P})$ for each γ^m, the synthesis operator Θ_{ψ_m} associated to $E^\Gamma(\Psi)$ is defined by

$$\Theta_\Psi: L^2(\Gamma, \ell^2(\mathcal{P})) \to L^2(G); \quad h \mapsto \sum_{p \in \mathcal{P}} \int_T \sum_m h_p(\gamma^m) \rho(\gamma^m)(\psi_m)_p d\mu_\Gamma(\gamma^m). \quad (2.2)$$

Note that Θ_Ψ is well-defined, linear and bounded [23, Theorem 2.6], and hence its adjoint given by

$$\Theta^*_{\psi_m}: L^2(G) \to L^2\left(\Gamma, \ell^2(\mathcal{P})\right); \quad f_m \mapsto \left(\left(\langle f_m, \rho(\gamma^m)(\psi_m)_p \rangle\right)_{p \in \mathcal{P}}\right)_{\gamma^m \in \Gamma} \quad (2.3)$$

is also linear and bounded, called the analysis operator corresponding to $E^\Gamma(\Psi)$. Now, by composing the analysis and synthesis operators of two Γ -TI Bessel systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$, define another bounded operator called mixed dual-Gramian operator as follows:

$$\Theta_\Phi \Theta^*_\Psi: L^2(G) \to L^2(G); \quad f_m \mapsto \sum_{p \in \mathcal{P}} \int_\Gamma \sum_m \langle f_m, \rho(\gamma^m)(\psi_m)_p \rangle \rho(\gamma^m)(\varphi_m)_p d\mu_\Gamma(\gamma^m) \quad (2.4)$$

Definition 2.2 (see [29]). Let $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ be \Gamma-TI Bessel (frame) systems in the Γ -TI space $S^\Gamma(\Psi) = S^\Gamma(\Phi)$. Then, if the operator $\Theta_\Phi \Theta^*_\Psi$ as defined in (2.4) is the zero operator, we call $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ as pairwise orthogonal (simply, orthogonal) Γ - TI Bessel (frame) systems. In this case, we say that the Γ -TI Bessel (frame) systems satisfy the orthogonality property.

In the above definition, it should be noted that the desired equality $S^\Gamma(\Psi) = S^F(\Phi)$ also poses some conditions on the families $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$.

2.2. Pre-Gramian operator associated with Γ-TI systems.

We extend the notion of pre-Gramian of [25] to the case of LCA groups (see [29]). We set some notation and basic facts on harmonic analysis over LCA groups [9,17,18]. By $\widehat{G}$, we denote the dual group of Gwhich is the set of all continuous characters, that is, all continuous homomorphisms from G into the torus $T \cong \{z \in \mathbb{C}: |z| = 1\}$. The dual group $\widehat{G}$ forms an LCA group when equipped with an appropriate metrizable topology and possesses a Haar measure denoted by $\mu_{\widehat{G}}$. By the Pontryagin duality theorem, there exists a topological group isomorphism mapping the group $\widehat{\widehat{G}}$, that is, the dual group of $\widehat{G}$, onto G. More precisely, $\widehat{G} \cong G$(e.g. see [9]).

Let the Fourier transform $L^1(G) \to C_0(\widehat{G}); f_m \mapsto \widehat{f_m}$, be defined by $\widehat{f_m}(\xi_m) = \int_G \sum_m \quad f_m(x)\overline{\xi_m(x)}d\mu_G(x)$ for all $\xi_m \in \widehat{G}$, where $C_0(\widehat{G})$ denotes the functions on $\widehat{G}$ vanishing at infinity. The Fourier transform can be extended from $L^1(G) \cap L^2(G)$ to a surjective isometry between $L^2(G)$ and $L^2(\widehat{G})$ known as the Plancherel transform [9]. Thus, for all $f_m, g_m \in L^2(G)$, the Parseval's formula $\langle f_m, g_m\rangle = \langle \widehat{f_m}, \widehat{g_m}\rangle$ holds. Note that for a subgroup Γ of an LCA group G, the symbol $\Gamma^\perp$ denotes the annihilator of Γ, which is a subgroup of $\widehat{G}$ defined by

$$\Gamma^\perp := \{\xi_m \in \widehat{G}: \xi_m(x) = 1, \forall x \in \Gamma\}$$

It follows that $\Gamma^\perp$ is a closed subgroup in $\widehat{G}$, and if Γ is closed, then $(\Gamma^\perp)^\perp = \Gamma$. Since, $\Gamma^\perp$ is topologically isomorphic to the dual of quotient group G/Γ, that is, $\Gamma^\perp \cong \widehat{(G/\Gamma)}$, therefore, Γ is co-compact in G if, and only if, $\Gamma^\perp$ is a discrete subgroup of $\widehat{G}$.

We let Γ to be a closed and co-compact subgroup of a second countable L CA group G. Thus, $\Gamma^\perp$ will always be discrete in our case, and hence preserves a counting measure. Let $\widehat{\Gamma}$ (the dual group of Γ) be an LCA group with measure denoted by μ_Γ. Observe that there exists a topological group isomorphism mapping $\widehat{G}/\Gamma^\perp$ onto $\widehat{\Gamma}$. Hence, by choosing a measure $\mu_{G/\Gamma^\perp}$`

on $G/\Gamma^\perp$ appropriately, Weil's formula [9] can be stated as

$$\int_G \sum_m \quad \hat{f_m}(\xi_m)d\mu_{\hat{G}}(\xi_m) = \int_{\frac{\hat{G}}{\Gamma^\perp}} \sum_{\alpha\in\Gamma^\perp} \sum_m \quad \widehat{f_m}(\xi_m + \alpha)d\mu_{\frac{\hat{G}}{\Gamma^\perp}}(\xi_m + \Gamma^\perp)$$

$$= \int_{\hat{\Gamma}} \sum_{\alpha\in\Gamma^\perp} \sum_m \quad \hat{f_m}(\xi_m + \alpha)d\mu_{\hat{\Gamma}}(\xi_m) \qquad (2.5)$$

on $\widehat{G}/\Gamma^\perp$ appropriately, Weil's formula [9] can be stated asfor all $f_m \in L^1(G)$. Note that the dual group $\widehat{G} = \Omega \oplus \Gamma^\perp$, therefore, every $\xi_m \in \widehat{G}$ has a unique representation $w_m + \alpha$ for some $w_m \in \Omega$ and $\alpha \in \Gamma^\perp$. Here Ω is a μ_G -measurable subset of $\widehat{G}$ and represents a Borel section of $\Gamma^\perp$ in $\widehat{G}$, also known as a fundamental domain of $\widehat{G}/\Gamma^\perp$, whose existence is guaranteed by [8]. Moreover, it is relevant to note that every element v in $\widehat{\Gamma} \cong \widehat{G}/\Gamma^\perp$ can be thought of as an element in Ω as all cosets in $\widehat{G}/\Gamma^\perp \cong \widehat{\Gamma}$ are of the form $w_m + \Gamma^\perp$ for some (unique) $w_m \in \Omega$. For more details, see [5, Section 3].

Remark 2.3 [29]. We remark that for a countable family $\Psi = \{(\psi_m)_p\}_{p\in\mathcal{O}}$ in $L^2(G), E^\Gamma(\Psi)$ is a Γ -TI Bessel system in $S^\Gamma(\Psi)$ with bound $(\mathcal{A} + \epsilon)$ if, and only if, we have

$$\sum_{\alpha\in\Gamma^\perp} \sum_{p\in\mathcal{P}} \sum_m \quad \left|(\hat{\psi_m})_p(w_m + \alpha)\right|^2 \leq \mathcal{A} + \epsilon, \text{for a.e. } w_m \in \Omega$$

Note that the above fact can be proved easily by applying the technique used for the characterization result on frames obtained in [5].

Pre-Gramian operator: Now, we are ready to associate $E^\Gamma(\Psi)$ with a collection of 'fiber operators'. The fibers are indexed by Ω. We have the following definition:

Definition 2.4 (see [29]). For a.e. $w_m \in \Omega$, the fiber $\mathcal{J}_\mathcal{G}^\Psi(w_m)$, called the pre-Gramian operator (simply, preGramian associated with a Γ -TI Bessel system $E^\Gamma(\Psi)$, is defined by

$$\mathcal{J}_\mathcal{G}^\Psi(w_m): \ell^2(\mathcal{P}) \to \ell^2(\Gamma^\perp); \quad \eta \mapsto \left(\mathcal{J}_\mathcal{G}^\Psi(w_m)\right)\eta = \left\{\sum_{p\in\mathcal{P}} \sum_m \quad \eta(p)(\hat{\psi_m})_p(w_m + \alpha)\right\}_{\alpha\in\Gamma^\perp}$$

Note that Remark 2.3 and the upcoming calculations show that the Bessel property of $E^\Gamma(\Psi)$ plays a very important role in the well-definedness of the pre-Gramian operator. Therefore, we assume that $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ are $\Gamma -$ TI Bessel systems in $S^\Gamma(\Psi)$ with $S^\Gamma(\Psi) = S^\Gamma(\Phi)$. We mention that a family of the form $\{\mathcal{J}_\mathcal{G}^\Psi(w_m)\}_{w_m\in\Omega}$ is called a collection of fiber operators, also known as fiberization of the synthesis operator Θ_Ψ (as defined in (2.2)) corresponding to the Γ -TI Bessel system $E^\Gamma(\Psi)$. Hence, $\{\mathcal{J}_\mathcal{G}^\Psi(w_m)\}_{w_m\in\Omega}$ can be regarded

as the synthesis operator of $E^{\Gamma}(\Psi)$ represented in Fourier domain. This fact has been broadly studied for the case of $L^2(\mathbb{R}^d)$ by [25].
For a.e. $w_m \in \Omega$, clearly $\mathcal{J}_G^{\Psi}(w_m)$ is a linear operator. Further, it is well-defined and bounded since in view of Cauchy-Schwarz inequality and Remark 2.3, the following estimates hold:

$$\left\|\left(\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\right)\eta\right\|^2 = \sum_{\alpha\in\Gamma^{\perp}} \left|\sum_{p\in\mathcal{F}}\sum_{m} \eta(p)(\hat{\psi_m})_p(w_m+\alpha)\right|^2$$
$$\leq \sum_{\alpha\in\Gamma^{\perp}} \| \eta \|^2 \left(\sum_{p\in\mathcal{P}}\sum_{m} \left|(\hat{\psi_m})_p(w_m+\alpha)\right|^2\right) < \infty, for\ all\ \eta \in \ell^2(\mathcal{P})\ and\ a.e. w_m \in \Omega.$$

Furthermore, $\mathcal{J}_G^{\Psi}(w_m)$ can be associated with a matrix whose rows are indexed by $\Gamma^{\perp}$, and whose columns are indexed by $\mathcal{P}$. For each $w_m \in \Omega$, let the symbol $\mathcal{M}_{\mathcal{G}}^{\Psi}(w_m)$ denote the matrix associated to $\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)$ with $(\alpha, p) \in \Gamma^{\perp} \times \mathcal{P}$ entry defined by $(\hat{\psi_m})_p(w_m+\alpha)$, and hence $\mathcal{M}_{\mathscr{G}}^{\Psi}(w_m) = \left((\hat{\psi_m})_p(w_m+\alpha)\right)_{\alpha\in T^{\perp}, p\in\Omega}$Since the operator $\mathcal{J}_G^{\Psi}(w_m)$ is linear, well-defined, and bounded, we define the adjoint of $\mathcal{J}_G^{\dagger}(w_n)$ as

$$(\mathcal{J}_G^{\Psi}(w_m))^*: \ell^2(\Gamma^{\perp}) \to \ell^2(\mathcal{P})$$
$$\vartheta \mapsto \left((\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m))^*\right)\vartheta = \left(\left\langle \vartheta, \{(\hat{\psi_m})_p(w_m+\alpha)\}_{\alpha\in r^{\perp}}\right\rangle\right)_{p\in\mathcal{P}}$$

which is well-defined because $\{(\hat{\psi_m})_p(w_m+\alpha)\}_{\alpha\in\Gamma^{\perp}}$ belongs to $\ell^2(\Gamma^{\perp})$ for each $p \in \mathcal{P}$, and the relation is satisfied for all $\eta \in \ell^2(\mathcal{P})$. It follows that $\left(\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\right)^*$ is a bounded operator in view of CauchySchwarz inequality, Remark 2.3 and the fact that $\{(\hat{\psi_m})_p(w_m+\alpha)\}_{\alpha\in\Gamma^{\perp}} \in \ell^2(\Gamma^{\perp})$ for each $p \in \mathcal{P}$.
Further, we term the collection $\left\{\left(\mathcal{J}_{\mathscr{G}}^{\Psi}(w_m)\right)^*\right\}_{we\Omega}$ as the fiberization of the analysis operator Θ_{Ψ}^* (as defined in (2.3)) corresponding to the Γ -TI system $E^{\Gamma}(\Psi)$. Now, for each $w_m \in \Omega$, by the notation $\tilde{G}^{\psi_m}(w_m) := \left(\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\right)^* \mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)$, we denote the Gramian operatorcorresponding to $E^{\Gamma}(\Psi)$, and, by the symbol $\mathbb{G}^{\Psi,\Phi}(w_m)$, we define the mixed dual-Gramian operator associated to the Γ -TI systems $E^{T}(\Psi)$ and $E^{\Gamma}(\Phi)$ in terms of the pre-Gramain, where

$$\mathbb{G}^{\Psi,\Phi}(w_m) := \mathcal{J}_{\mathscr{G}}^{\Phi}(w_m)\left(\mathcal{J}_G^{\Phi}(w_m)\right)^*: \ell^2(\Gamma^{\perp}) \to \ell^2(\Gamma^{\perp})$$

is a bounded operator by observing that the pre-Gramian$\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)$ is bounded, and the computation

$$\|(\mathbb{G}^{\Psi,\Phi}(w_m))\vartheta\|^2 = \left\|\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\left(((\mathcal{J}_G^{\Phi}(w_m))^*)\vartheta\right)\right\|^2 < \infty, for\ all\ \vartheta \in \ell^2(\Gamma^{\perp})$$

Further, note that for all $\vartheta_1, \vartheta_2 \in \ell^2(\Gamma^{\perp})$, we can write

$$\left\langle\left(\mathbb{G}^{\Psi,\Phi}(w_m)\right)\vartheta_1, \vartheta_2\right\rangle = \sum_{m} \left\langle\left(\left(\mathcal{J}_G^{\Phi}(w_m)\right)^*\right)\vartheta_1, \left(\left(\mathcal{J}_G^{\Phi}(w_m)\right)^*\right)\vartheta_2\right\rangle$$
$$= \sum_{p\in\mathcal{O}}\sum_{m} \left\langle \vartheta_1, \{(\hat{\psi_m})_p(w_m+\alpha)\}_{\alpha\in r^{\perp}}\right\rangle \overline{\left\langle \vartheta_2, \{(\hat{\psi_m})_p(w_m+\beta)\}_{\alpha\in r^{\perp}}\right\rangle}$$

Therefore, we get

$$\langle(\mathbb{G}^{\Psi,\Phi}(w_m))\vartheta_1, \vartheta_2\rangle = \sum_{p\in\mathcal{O}}\sum_{\alpha\in\Gamma^{\perp}}\sum_{\beta\in\Gamma^{\perp}}\sum_{m} \vartheta_1(\alpha)\overline{\vartheta_2(\beta)}(\hat{\psi_m})_p(w_m+\beta)\overline{(\hat{\varphi_m})_p(w_m+\alpha)}. \quad (2.6)$$

We say that the collection $\{\mathbb{G}^{\Psi,\Phi}(w_m)\}_{w_m\in\Omega}$ is the mixed dual-Gramian fiberization of the operator $\Theta_{\Phi}\Theta_{\Psi}^*$ (as defined in (2.4)) corresponding to the Γ -TI systems $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$.

2.3. Fiberization in terms of the pre-Gramian operator.

We first focus in proving some results which are required for the fiberization of analysis and synthesis operators associated with $E^{\Gamma}(\Psi)$. Now we define the space $L^2(\hat{\Gamma}, \ell^2(\mathcal{P}))$ which appears as the image under the Fourier transform (FT) of the space $L^2(\Gamma, \ell^2(\mathcal{P}))$:

$$L^2(\hat{\Gamma}, \ell^2(\mathcal{P})) := \left\{ measurable\ \zeta_m: \hat{\Gamma} \to \ell^2(\mathcal{P})\ such\ that\ \| \zeta_m \|^2 := \int_{\hat{\Gamma}}\sum_{m} \| \zeta_m(v) \|_{\ell(\mathcal{A})}^2\, d\mu_{f_m}(v) < \infty \right\}$$

Observe that $L^2(\hat{\Gamma}, \ell^2(\mathcal{P}))$ is a Hilbert space with inner product $\langle\zeta_m^1, \zeta_m^2\rangle := \int_{\hat{\Gamma}}\sum_m \langle\zeta_m^1(v), \zeta_m^2(v)\rangle_{\ell^2(\mathcal{A})} d\mu_{\hat{\Gamma}}(v)$
Since every element v in $\hat{\Gamma} \cong \hat{G}/\Gamma^{\perp}$ can be considered as an element in a fundamental domain $\Omega \subset \hat{G}$ of the discrete subgroup $\Gamma^{\perp}$, we identify the space $L^2(\hat{\Gamma}, \ell^2(\mathcal{P}))$ with $L^2(\Omega, \ell^2(\mathcal{P}))$ for the remainder.

Proposition 2.5 (see [29]). Let $E^{\Gamma}(\Psi)$ be a Γ - TI Bessel system in $S^{T}(\Psi)$ with $\Theta^{*}_{\psi_m}$ as its analysis operator (defined in (2.3)). For each $p \in \mathcal{P}$ and f_m in $L^2(G)$, let $\left(\Theta^{*}_{\psi_m} f_m\right)_p := \left(\langle f_m, \rho(\gamma^m)(\psi_m)_p\rangle\right)_{\gamma^m \in \Gamma}$. Then, the following assertions are true:

$$(\widehat{\Theta^{*}_{\Psi} f_m})_p(v) = \sum_{\alpha\in\Gamma^{\perp}} \sum_{m} \hat{f}_m(v+\alpha)\overline{(\bar{\psi}_m)_p(v+\alpha)}.$$

Further, it is well-defined, belongs to $L^1(\Omega)$, and satisfies that

$$\left(\widehat{\Theta^{*}_{\bar{\psi}_m} f_m}\right)_p(v+\beta) = \left(\widehat{\Theta^{*}_{\psi_m} f_m}\right)_p(v), for\ all\ v \in \Omega, and\ \beta \in \Gamma^{\perp} \qquad (2.7)$$

(ii) The FT of $\Theta^{*}_{\bar{w}_m} f_m$ is given by $\widehat{\Theta^{*}_{\bar{\psi}_m}} f_m := \left(\widehat{\Theta^{*}_{\Psi} f_m}(v)\right)_{v\in\Omega}$ which is an element in $L^2(\Omega, \ell^2(\mathcal{P}))$ with

Proof. Let $f_m \in L^2(G)$ and $p \in \mathcal{P}$. For each $v \in \Omega$, let $C_p(v) := \sum_{\alpha\in T^{\perp}} \sum_m \widehat{f_m}(v+\alpha)\overline{(\bar{\psi}_m)_p(v+\alpha)}$. Then. from Weil's formula (2.5) and the relation between Ω and the dual group of Γ, it follows that

$$\int_{\Omega} \sum_{\alpha\in\Gamma^{\perp}} \sum_{m} \left|\widehat{f_m}(v+\alpha)\overline{(\bar{\psi}_m)_p(v+\alpha)}\right| d\mu_{\hat{G}}(v) = \int_{\hat{G}} \sum_{m} \left|\widehat{f_m}(v)\overline{(\bar{\psi}_m)_p(v)}\right| d\mu_{\hat{G}}(v)$$

which is finite by the Cauchy-Schwarz inequality. This implies that $\left(C_p(v)\right)_{v\in\Omega} \in L^1(\Omega)$. Note that

$$C_p(v+\beta) = \sum_{\alpha\in\Gamma^{\perp}} \sum_{m} \widehat{f_m}(v+\alpha+\beta)\overline{(\bar{\psi}_m)_p(v+\alpha+\beta)}$$

$$= \sum_{\alpha\in\Gamma^{\perp}} \sum_{m} \widehat{f_m}(v+\alpha)\overline{(\bar{\psi}_m)_p(v+\alpha)} = C_p(v), for\ all\ v \in \Omega\ and\ \beta \in \Gamma^{\perp},$$

by changing the summation variable $\alpha \to \alpha - \beta$. Now, the proof for part (i) follows from the fact that

$$\begin{aligned}
\Theta^{*}_{*}\widehat{(f_m)}^{*}_{P\,p}(v) &= \int_{\Gamma} \sum_{m} \langle f_m, \rho(\gamma^m)(\psi_m)_p\rangle \overline{v(\gamma^m)} d\mu_{\Gamma}(\gamma^m) \\
&= \int_{\Gamma} \sum_{m} \langle \widehat{f_m}, \widehat{\rho(\gamma^m)(\psi_m)_p}\rangle \overline{v(\gamma^m)} d\mu_{\Gamma}(\gamma^m) \\
&= \int_{\Gamma} \sum_{m} \left(\int_{G} \widehat{f_m}(\xi_m)\overline{\left(\widehat{\rho(\gamma^m)(\psi_m)_p}\right)(\xi_m)} d\mu_{\hat{G}}(\xi_m)\right) \overline{v(\gamma^m)} d\mu_{\Gamma}(\gamma^m), for\ a.e.\ v \in \Omega,
\end{aligned}$$

in view of the Parseval's formula. The above expression equivalently provides the following form by using Weil's formula (2.5) and the identity $\left(\widehat{\rho(\gamma^m)(\psi_m)_p}\right)(\xi_m) = \overline{\xi_m(\gamma^m)}(\hat{\psi}_m)_p(\xi_m)$, for all $\xi_m \in \hat{G}$, that means, we have

$$\begin{aligned}
\left(\widehat{\Theta^{*}_{\Psi} f_m}\right)_p(v) &= \int_{\Gamma} \sum_{m} \left(\int_{\hat{G}} \widehat{f_m}(\xi_m)\xi_m(\gamma^m)\overline{\widehat{(\psi_m)}_p(\xi_m)} d\mu_{\hat{G}}(\xi_m)\right) \overline{v(\gamma^m)} d\mu_{\Gamma}(\gamma^m) \\
&= \int_{\Gamma} \sum_{m} \left(\int_{\Omega} \sum_{\alpha\in\Gamma^{\perp}} \widehat{f_m}(\tilde{v}+\alpha)\tilde{v}(\gamma^m)\widehat{(\psi_m)}_p(\tilde{v}+\alpha) d\mu_{\hat{G}}(\tilde{v})\right) \overline{v(\gamma^m)} d\mu_{\Gamma}(\gamma^m) \\
&= \int_{\Gamma} \sum_{m} \overline{v(\gamma^m)} \left(\int_{\Omega} \tilde{v}(\gamma^m) C_p(\tilde{v}) d\mu_{\hat{G}}(\tilde{v})\right) d\mu_{\Gamma}(\gamma^m) \\
&= \int_{\Gamma} \sum_{m} \left(\mathcal{F}^{-1}(C_p)\right)(\gamma^m)\overline{v(\gamma^m)} d\mu_{\Gamma}(\gamma^m) \\
&= \mathcal{F}\left(\mathcal{F}^{-1}(C_p)\right)(v) = C_p(v), \text{ for a.e. } v \in \Omega
\end{aligned}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the FT on $L^2(\Gamma)$ and the inverse FT on $L^2(\Omega)$,

For proving part (ii), note that $\widehat{\Theta^{*}_{\Psi} f_m}$ is an element of $L^2(\Omega, \ell^2(\mathcal{P}))$ since the expression given by

$$\begin{aligned}
\int_{\Omega} \sum_{m} \left\|\left(\widehat{(\Theta^{*}_{\Psi} f_m)}_p(v)\right)_{p\in\mathcal{P}}\right\|^2 d\mu_{\hat{G}}(v) &= \int_{\Omega} \sum_{p\in\mathcal{P}} \sum_{m} \left|\widehat{\Theta^{*}_{\Psi} f_m}\right)_p(v)\Big|^2 d\mu_{\hat{G}}(v) \\
&= \int_{\Omega} \sum_{p\in\mathcal{P}} \left|\sum_{\alpha\in\Gamma^{\perp}} \sum_{m} \widehat{f_m}(v+\alpha)\overline{(\bar{\psi}_m)_p(v+\alpha)}\right|^2 d\mu_{\hat{G}}(v) \\
&\leq \int_{\Omega} \left(\sum_{m} \sum_{\alpha\in\Gamma^{\perp}} |\widehat{f_m}(v+\alpha)|^2\right) \sum_{p\in\mathcal{P}} \left(\sum_{\alpha\in\Gamma^{\perp}} \left|\widehat{(\psi_m)}_p(v+\alpha)\right|^2\right) d\mu_{\hat{G}}(v), \text{ for all } f_m \in L^2(G),
\end{aligned}$$

is finite by using Remark 2.3, Cauchy-Schwarz inequality and Weil's formula. Hence, the result.

respectively. For proving part (ii), note that $\widehat{\Theta^*_{\psi_m} f_m}$ is an element of $L^2(\Omega, \ell^2(\mathcal{P}))$ since the expression given byZrtiveis finite by using Remark 2.3, Cauchy-Schwarz inequality and Weil's formula. Hence, the result. **Proposition 2.6 (see [29]).** Let the operator Θ_Ψ be as defined in (2.2). Then for every $h \in L^2(\Gamma, \ell^2(\mathcal{P}))$,

Proof. Let $h \in L^2(\Gamma, \ell^2(\mathcal{P}))$. Then, by using the definition of the operator Θ_Ψ, we can write

$$\begin{aligned}(\widehat{\Theta_{\psi_m} h})(\xi_m) &= \int_G \sum_{p\in\mathcal{P}} \sum_m \int_\Gamma h_p(\gamma^m)(\psi_m)_p(x-\gamma^m)d\mu_\Gamma(\gamma^m)\overline{\xi_m(x)}d\mu_G(x) \\ &= \sum_{p\in\mathcal{P}} \int_G \int_\Gamma \sum_m h_p(\gamma^m)(\psi_m)_p(y)\overline{\xi_m(y+\gamma^m)}d\mu_\Gamma(\gamma^m)d\mu_G(y),\end{aligned}$$

which further equals to the expression given by

$$\begin{aligned}&\sum_{p\in\mathcal{P}} \int_G \sum_m \left(\int_\Gamma h_p(\gamma^m)\overline{\xi_m(\gamma^m)}d\mu_\Gamma(\gamma^m)\right)(\psi_m)_p(y)\overline{\xi_m(y)}d\mu_G(y) \\ &= \sum_{p\in\mathcal{P}} \sum_m \hat{h}_p(\xi_m)\left(\int_G (\psi_m)_p(y)\overline{\xi_m(y)}d\mu_G(y)\right) \\ &= \sum_{p\in\mathcal{S}} \sum_m \hat{h}_p(\xi_m)(\widehat{\psi_m})_p(\xi_m), for\ a.e.\ \xi_m \in \Omega.\end{aligned}$$

The following result establishes a relation which represents the fiberization of operators associated with the Γ - TI Bessel systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$ via the pre-Gramian operator defined in Definition 2.4.

Theorem 2.7 (see [29]). For each f_m in $L^2(G)$, and for a.e. $\kappa \in \Omega$, the following expressions hold:

$$\left((\widehat{\Theta^*_\Psi f_m})_p(\kappa)\right)_{p\in\mathcal{P}} = \left(\mathcal{J}^\Psi_\mathcal{G}(\kappa)\right)^* (\widehat{f_m}(\kappa+\alpha))_{\alpha\in\Gamma^\perp} \tag{2.8}$$

$$\left(\widehat{\Theta_\Psi} h(\kappa+\alpha)\right)_{\alpha\in\Gamma^\perp} = \mathcal{J}^\Psi_\mathcal{G}(\kappa)\left(\hat{h}_p(\kappa+\alpha)\right)_{p\in\mathcal{P}}, for\ all\ h \in L^2\left(\Gamma, \ell^2(\mathcal{P})\right), \tag{2.9}$$

and

$$\left(\left(\Theta_\Psi \widehat{\Theta^*_\Phi f_m}\right)(\kappa+\alpha)\right)_{\alpha\in\Gamma^\perp} = \mathbb{G}^{\Psi,\Phi}(\kappa)(\widehat{f_m}(\kappa+\alpha))_{\alpha\in\Gamma^\perp} \tag{2.10}$$

where for a.e. $\kappa \in \Omega$, the symbol $\mathbb{G}^{\Phi,\Phi}(\kappa) = \mathcal{J}^\Psi_G(\kappa)\left(\mathcal{J}^\Phi_\mathcal{G}\right)^*(\kappa)$ denotes the mixed dual-Gramian

$$\left(\widehat{\Theta_{\psi_m}} h(\kappa+\alpha)\right)_{\alpha\in\Gamma^\perp} = \left(\sum_{p\in\mathcal{S}} \hat{h}_p(\kappa+\alpha)(\hat{\psi_m})_p(\kappa+\alpha)\right)_{\alpha\in\Gamma\perp}, for\ all\ h \in L^2(\Gamma, \ell^2(\mathcal{P}))$$

Now, (2.10) follows by using the equalities (2.7), (2.8) and (2.9) in the following computation:

$$\left(\left(\widehat{\Theta_\Psi\left(\Theta^*_\phi f_m\right)\right)(\kappa}+\alpha)\right)_{\alpha\in\Gamma^\perp} = \mathcal{J}^\Psi_\mathcal{G}(\kappa)\left((\widehat{\Theta^*_\Phi f_m})_p(\kappa+\alpha)\right)_{p\in\mathcal{P}}$$

where for a.e. $\kappa \in \Omega$, the symbol $\mathbb{G}^{\Psi,\Phi}(\kappa) = \mathcal{J}^\Psi_\mathcal{G}(\kappa)\left(\mathcal{J}^\Phi_\mathcal{G}\right)^*(\kappa)$ denotes the mixed dual-Gramian operator corresponding to the Γ - TI Bessel systems $E^\Gamma(\Psi)$ and $E^\Gamma(\Phi)$.

Proof. Let a.e. $\kappa \in \Omega$. Then, the proof for expression (2.8) follows by using the definition of operator $(\mathcal{J}^\Phi_G(\kappa))^*$ along with Proposition 2.5. Further, (2.9) holds by observing $\mathcal{J}^\Psi_\mathcal{G}(\kappa)$ from Definition 2.4, Proposition 2.6, and the computation

$$\left(\widehat{\Theta_\Psi} h(\kappa+\alpha)\right)_{\alpha\in\Gamma^\perp} = \left(\sum_{p\in\mathcal{F}} \hat{h}_p(\kappa+\alpha)(\hat{\psi_m})_p(\kappa+\alpha)\right)_{\alpha\in\Gamma^\perp}, for\ all\ h \in L^2(\Gamma, \ell^2(\mathcal{P})).$$

Now, (2.10) follows by using the equalities (2.7), (2.8) and (2.9) in the following computation:

$$\begin{aligned}\left(\left(\Theta\widehat{\Psi}(\Theta^x_\Phi f_m)\right)(\kappa+\alpha)\right)_{\alpha\in\Gamma^\perp} &= \mathcal{J}^\Psi_\mathcal{G}(\kappa)\left((\widehat{\Theta^*_\Phi f_m})_p(\kappa+\alpha)\right)_{p\in\mathcal{P}} \\ &= \mathcal{J}^\Psi_\mathcal{G}(\kappa)\left(\left(\widehat{\Theta^*_\phi f_m}\right)_p(\kappa)\right)_{p\in\mathcal{P}} \\ &= \mathcal{J}^\Psi_\mathcal{G}(\kappa)\left(\mathcal{J}^\Phi_\mathcal{G}(\kappa)\right)^*(\widehat{f_m}(\kappa+\alpha))_{\alpha\in\Gamma^\perp}, for\ all\ f_m \in L^2(G).\end{aligned}$$

We characterize a pair of orthogonal Γ -TI Bessel (frame) systems over locally compact abelian (LCA) groups. So we use the notion of pre-Gramian operator in LCA-group setting along with the fiberization of operators associated with the Γ -TI systems which we have studied in Theorem 2.7. The next result characterizes the frame/Bessel property of a Γ -TI system in terms of the pre-Gramian operator. See[5,6] for the proof of the following result

Proposition 3.1 (see [29]). $E^{\Gamma}(\Psi)$ is a Γ -TI frame system for $S^{\Gamma}(\Psi)$ if, and only if, $\mathcal{J}_{\mathcal{C}}^{\Psi}(w_m)\left(\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\right)^*$ is uniformly bounded with uniformly bounded inverse on the range of $\mathcal{J}_{G}^{\Psi}(w_m)$ for a.e. $w_m \in \Omega$ such that ran $\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m) \neq \{0\}$. In particular, if $S^{\Gamma}(\Psi) = L^2(G)$, then the following are equivalent:
(i) The system $E^{T}(\Psi)$ is a Γ - TI frame system for $L^2(G)$,
(ii) there exist constants $0 < A \leq \mathcal{A} + \epsilon < \infty$, such that

$$\mathcal{A}I_{\ell^2(T^{\perp})} \leq \mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\left(\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\right)^* \leq (\mathcal{A}+\epsilon)I_{\ell^2(\Gamma^{\perp})}, \quad for\ a.e.\ w_m \in \Omega$$

In addition, $E^{\Gamma}(\Psi)$ is a tight $\Gamma -$ TI frame system with frame bound 1 for $L^2(G)$ if, and only if,

$$\mathcal{J}_{\mathcal{G}}^{\Psi}(w_m)\left(\mathcal{J}_{\mathcal{G}}^{\Phi}(w_m)\right)^* = I_{\ell^2(I^{\perp})}, \quad for\ a.e.\ w_m \in \Omega$$

Proof. The result follows from [5,6]. The key for the proof lies in using the computation done in (2.6) for the case of $\vartheta_1 = \vartheta_2 \in \ell^2(\Gamma^{\perp})$ along with the characterization result on frames obtained in [5].

3.1. Characterizations of pairwise orthogonal Γ -TI Bessel (frame) systems.
The following results provide necessary and sufficient conditions on a pair of Γ -TI frame systems to be orthogonal in the sense of Definition 2.2 :

Theorem 3.2 (see [29]). Let Ψ and Φ be countable subsets of $L^2(G)$. Suppose that $S^{F}(\Psi) = S^{\Gamma}(\Phi)$, and that both $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are Γ - TI frame systems for $S^{\Gamma}(\Psi)$. Then, the following are equivalent:
(i) $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ form orthogonal Γ - TI frame systems in $S^{\Gamma}(\Psi)$.
(ii) $\mathcal{J}_{\mathcal{g}}^{\Psi}(\kappa)\left(\mathcal{J}_{\mathcal{g}}^{\Phi}(\kappa)\right)^*\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.
(iii) $\mathcal{M}_{g_m}^{\Psi}(\kappa)\left(\mathcal{M}_{\mathcal{G}}^{\Phi}(\kappa)\right)^*\mathcal{M}_{\mathcal{G}}^{\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.
(iv) $\tilde{G}^{\Psi}(\kappa)\tilde{G}^{\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.
In particular, when $S^{T}(\Psi) = L^2(G), E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are pairwise orthogonal Γ - TI Bessel (frame) systems in $L^2(G)$ if, and only if, $\mathcal{J}_{\mathcal{g}}^{\Psi}(\kappa)\left(\mathcal{J}_{\mathcal{G}}^{\Phi}(\kappa)\right)^* = 0$ for a.e. $\kappa \in \Omega$.
For proving Theorem 3.2, first we need to describe the decomposition theorem for a Γ -TI space of $L^2(G)$. Hencelet $\psi_m \in L^2(G)$. We denote by $L^2(\Omega, \omega_{\psi_m})$, the space of all functions $r: \Omega \to \mathbb{C}$, which satisfy $\int_{\Omega} \sum_m \quad |r(\xi_m)|^2 \omega_{\psi_m}(\xi_m) d\mu_{\hat{G}}(\xi_m) < \infty$, where $\omega_{\psi_m}(\xi_m) = \sum\sum_m \quad |\hat{\psi_m}(\xi_m + \alpha)|^2$ for each $\xi_m \in \Omega$. Observe that by using the relation between the dual group of Γ with Ω, Weil's formula and the Plancherel theorem, we can write

$$\int_{\Omega} \sum_{\alpha \in \Gamma^{\perp}} \sum_{m} \quad |\widehat{\psi_m}(\xi_m + \alpha)|^2 d\mu_{\hat{G}}(\xi_m) = \int_{\hat{G}} \sum_{m} \quad |\widehat{\psi_m}(\xi_m)|^2 d\mu_{\hat{G}}(\xi_m) = \parallel \psi_m \parallel^2.$$

Thus, $\omega := \{\omega_{\psi_m}(\xi_m)\}_{\xi_m \in \Omega}$ is a function in $L^1(\Omega)$. Note that in this case

$$\parallel r \parallel^2_{L^2(\Omega, \omega_{\psi_m})} = \int_{\Omega} \sum_{m} \quad |r(\xi_m)|^2 \omega_{\psi_m}(\xi_m) d\mu_{\hat{G}}(\xi_m)$$

is a norm in $L^2(\Omega, \omega_{\psi_m})$. Further, we denote the support of ω_{ψ_m} by the set $\{\xi_m \in \Omega: \omega_{\psi_m}(\xi_m) \neq 0\} =: \mathbb{S}_{\psi_m}$. Here, note that the set $\mathbb{S}_{\psi_m}$ is called the spectrum of $S^{T}(\psi_m)$.
The next result shows the existence of a decomposition of a Γ -TI space of $L^2(G)$ into an orthogonal sum of spaces each of which is generated by a single function whose translates form a tight Γ -PTI frame system with frame bound 1 .

Theorem 3.3 (see [29]). Let V be a Γ - TI space of $L^2(G)$. Then, there exists a family of functions $\{(\psi_m)_n\}_{n \in \mathbb{N}}$ in V such that V can be decomposed as an orthogonal sum $V = \oplus_{n \in N} S^{\Gamma}((\psi_m)_n)$, and $E^{\Gamma}((\psi_m)_n)$ is a tight $\Gamma - PTI$ frame system in the \Gamma-PTI space $S^{\Gamma}((\psi_m)_n)$ with frame bound 1. Moreover, $f_m \in V$ if, and only if,

$$\left.\widehat{f_m}(\xi_m) = \sum_{n \in \mathbb{N}} \sum_{m} \quad r_n(\xi_m)(\hat{\psi_m})_n(\xi_m), and\ hence\ \parallel f_m \parallel^2 = \sum_{n \in \mathbb{N}} \sum_{m} \quad \|r_n\|^2_{L^2\left(\Omega \cap S_{(\psi_m)_n}, \omega_{v_v}\right)}\right) \quad (3.1)$$

where $r_n \in L^2\left(\Omega \cap \mathbb{S}_{(\psi_m)_n}, (w_m)_{(\psi_m)_n}\right)$ and $S_{(\psi_m)_n}$ is the spectrum of $S^{\Gamma}((\psi_m)_n)$, for every $n \in \mathbb{N}$.
Proof. The first part of the proof follows from [5, Theorem 5.3]. For the moreover part, let $n \in \mathbb{N}$ and $(\psi_m)_n \in L^2(G)$. Then, for each n by following the steps of [11, Proposition 2.2], we get $(f_m)_n \in S^{\Gamma}((\psi_m)_n)$ if, and only

if, $\widehat{(f_m)}_n(\xi_m) = r_n(\xi_m)\widehat{(\psi_m)}_n(\xi_m)$, for some $r_n \in L^2\big(\Omega, (w_m)_{(\psi_m)_n}\big)$. Now, if (3.1) holds, then clearly in view of the above discussion $f_m \in V$. Conversely, let P_n be the orthogonal projection onto the space $S^{\Gamma}((\psi_m)_n)$. Note that for each $f_m \in V$, we have $f_m = \sum_{n\in\mathbb{N}}\sum_m P_n f_m$. Thus, $\widehat{f_m} = \sum_{n\in\mathbb{N}}\sum_m \widehat{(P_n f_m)} = \sum_{n\in\mathbb{N}}\sum_m r_n(\hat{\psi_m})_n$, where

$r_n \in L^2\big(\Omega \cap \mathbb{S}_{(\psi_m)_n}, (w_m)_{(\psi_m)_n}\big)$ for each n. Therefore, we have

$$\| f_m \|^2 = \| \widehat{f_m} \|^2 = \sum_{n\in\mathbb{N}}\sum_m \|r_n\|^2_{L^2(\Omega\Gamma S_{(\varphi_m)_n},(w_m)_{v_n})}\big\|\widehat{(\psi_m)}_n\big\|^2 = \sum_{n\in\mathbb{N}}\|r_n\|^2_{L^2\left(\Omega\cap S_{(\psi_m)_n},(w_m)_{(\psi_m)_n}\right)},$$

in view of Plancherel's formula and the fact that $E^{\Gamma}((\psi_m)_n)$ is a tight Γ -PTI frame system in $S^{\Gamma}((\psi_m)_n)$ with frame bound 1. Hence, the result.

Proof of Theorem 3.2 (see [29]). By following Definition 2.2, $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are a pair of orthogonal Γ -TI Bessel (frame) systems for $S^{\Gamma}(\Psi)$ if, and only if, for all $f_m \in S^{\Gamma}(\Psi)$, we have $\Theta_\Psi\Theta_\Phi^* f_m = 0$, which is equivalent to saying that $\|\Theta_\Psi\Theta_\Phi^* f_m\| = 0$. Let $f_m \in L^2(G)$. Then, by Plancherel's formula and Weil's formula, the following expression holds:

$$\begin{aligned}\|\Theta_\Psi\Theta_\Phi^* f_m\|^2 &= \big\|\big(\widehat{\Theta_{(\psi_m)}\Theta_\Phi^* f_m}\big)\big\|^2 = \int_G \sum_m \big|\big(\widehat{\Theta_*\Theta_\Phi^* f_m}\big)(\xi_m)\big|^2 d\mu_{\hat{G}}(\xi_m)\\ &= \sum_{\alpha\in\Gamma^+}\int_\Omega \sum_m \big|\big(\widehat{\Theta_{(\psi_m)}\Theta_\Phi^* f_m}\big)(v+\alpha)\big|^2 d\mu_{\hat{G}}(v). \qquad (3.2)\end{aligned}$$

Therefore, $\Theta_\Psi\Theta_\Phi^* f_m = 0$ if, and only if, in view of (3.2), $\big(\widehat{\Theta_{(\psi_m)}\Theta_\Phi^* f_m}\big)(v+\alpha) = 0$ for each $v \in \Omega$ and $\alpha \in \Gamma^{\perp}$. This means, $\Theta_\Psi\Theta_\Phi^* f_m = 0$ if, and only if, we have $\Big(\big(\widehat{\Theta_\Psi}\Theta_\Phi^* f_m\big)(\kappa+\delta)\Big)_{\delta\in\Gamma^{\perp}} = 0$ for a.e. $\kappa \in \Omega$. Further, by using Theorem 2.7 observe that for every $f_m \in L^2(G)$ and for a.e. $\kappa \in \Omega$, the following relation is satisfied:

$$\Big(\big(\widehat{\Theta_\Psi}\Theta_\Phi^* f_m\big)(\kappa+\alpha)\Big)_{\alpha\in\Gamma^{\perp}} = \mathbb{G}^{\Psi,\Phi}(\kappa)(\hat{f_m}(\kappa+\alpha))_{\alpha\in\Gamma^{\perp}}$$

where $\mathbb{G}^{\Psi,\Phi}(\kappa) = \mathcal{J}_G^{\Psi}(\kappa)\big(\mathcal{J}_G^{\Phi}(\kappa)\big)^*$ is the mixed dual-Gramian operator corresponding to the $\Gamma - TI$ Bessel (frame) systems $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$. Hence, it follows that for $S^{\Gamma}(\Psi) = L^2(G)$, we get $\Theta_\Psi\Theta_\Phi^* f_m = 0$ for all $f_m \in L^2(G)$ if, and only if, $\mathbb{G}^{\Psi,\Phi}(\kappa) = 0$ for a.e. $\kappa \in \Omega$.

Now, we start for proving the equivalence of (i) and (ii). From Theorem 3.3, we observe that $f_m \in S^{\Gamma}(\Phi)$ if, and only if, the Fourier transform of f_m can be written as $\widehat{f_m} = \sum_{p\in\mathcal{F}}\sum_m r_p(\hat{\varphi_m})_p$ for some r_p in $L^2\big(\Omega \cap \mathbb{S}(\varphi m)P,(wm)(\varphi m)p$, where p belongs to a countable index set $\mathcal{P}$. Moreover, in view of the above expression of $\widehat{f_m}$ and Definition 2.4 for the pre-Gramian operator $\mathcal{J}_{g_m}^{\Phi}(\kappa)$, we get

$$(\widehat{f_m}(\kappa+\alpha))_{\alpha\in\Gamma^{\perp}} = \mathcal{J}_G^{\Phi}(\kappa)\big(r_p(\kappa+\alpha)\big)_{p\in\mathcal{P}}\ for\ a.e.\ \kappa\in\Omega$$

and hence part (i) is equivalent to the statement that for any $f_m \in S^{\Gamma}(\Phi)$, we have

$$\begin{aligned}\Big(\big(\widehat{\Theta_\Psi\Theta_\Phi^*} f_m\big)(\kappa+\alpha)\Big)_{\alpha\in\Gamma^{\perp}} &= \mathbb{G}^{\Phi,\Phi}(\kappa)(\widehat{f_m}(\kappa+\alpha))_{\alpha\in\Gamma^{\perp}}\\ &= \mathcal{J}_G^{\Psi}(\kappa)\left(\mathcal{J}_G^{\Phi}(\kappa)\right)^*\mathcal{J}_G^{\Phi}(\kappa)\left(r_p(\kappa+\alpha)\right)_{p\in\mathcal{P}}, \qquad (3.3)\end{aligned}$$

which is zero since $\Theta_\Psi\Theta_\Phi^* f_m = 0$, and hence $\mathcal{J}_G^{\Psi}(\kappa)\big(\mathcal{J}_G^{\Phi}(\kappa)\big)^*\mathcal{J}_G^{\Phi}(\kappa) = 0$, for a.e. $\kappa \in \Omega$.

Further, by using $\mathcal{M}_{g_m}^{\Psi}(\kappa)$, that is, the matrix associated with $\mathcal{J}_{g_m}^{\Psi}(\kappa)$, clearly (ii) holds if, and only if, the relation (iii) is satisfied.

Now, from (3.3), the equivalence of (ii) and (iv) follows since $E^{\Gamma}(\Phi)$ is a Γ -TI frame system in $S^{\Gamma}(\Phi)$, and hence for a.e. $\kappa \in \Omega$, $\big(\mathcal{J}_G^{\Psi}(\kappa)\big)^*$ has bounded inverse on the range of $\mathcal{J}_{\mathcal{G}}^{\Psi}(\kappa)$ by Proposition 3.1.
The next result reflects a very useful property of pairwise orthogonal frames. By using a Γ -periodic function on G and a given pair of orthogonal Γ -TI Bessel (frame) generators, we construct another pair of generators providing orthogonal frames of the same structure. Moreover, one system in the newly constructed pair of Γ -TI Bessel (frame) systems remains the same while the second system acquires some extra properties due to the effect of a Γ -periodic function. Here, note that for a co-compact subgroup Γ of an LCA group G, a bounded function on G is called Γ -periodic in the sense that for every $\gamma^m \in \Gamma$, we have $f_m(x+\gamma^m) = f_m(x)$ for all $x \in G$.

Proposition 3.4 (see [29]). Suppose that $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are orthogonal Γ - TI Bessel (frame) systems in $L^2(G)$. Let h be a complex-valued measurable function on G which is Γ -periodic, such that the collection $h\Psi$ is a subset of $L^2(G)$, where $h\Psi$ is defined by

$$h\Psi := \{h(\psi_m)_p \in L^2(G): \big(h(\psi_m)_p\big)(x) = h(x)(\psi_m)_p(x); (\psi_m)_p \in \Psi, p\in\mathcal{P}, x\in G\}$$

Then, the families $E^{\Gamma}(h\Psi)$ and $E^{\Gamma}(\Phi)$ also form orthogonal Γ - TI Bessel (frame) systems in $L^2(G)$. Proof. Given that $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ form pairwise orthogonal Γ -TI Bessel (frame) systems in $L^2(G)$. Then, for all $f_m \in L^2(G)$, we have $\Theta_\Psi \Theta_\Phi^* f_m = 0$. That means, the following holds:

$$\sum_{p\in\mathcal{P}} \int_\Gamma \sum_m \langle f_m(x), (\varphi_m)_p(x-\gamma^m)\rangle (\psi_m)_p(x-\gamma^m) d\mu_\Gamma(\gamma^m) = 0, \quad for\ all\ x \in G \tag{3.4}$$

Now, the result follows by using h as a Γ -periodic function along with (3.4) in the following relation:

$$\sum_{p\in\mathcal{T}} \int \sum_m \langle f_m(x), (\varphi_m)_p(x-\gamma^m)\rangle \big(h(\psi_m)_p\big)(x-\gamma^m) d\mu_\Gamma(\gamma^m)$$
$$= \sum_{p\in\mathcal{P}} \int \sum_m \langle f_m(x), (\varphi_m)_p(x-\gamma^m)\rangle h(x-\gamma^m)(\psi_m)_p(x-\gamma^m) d\mu_\Gamma(\gamma^m)$$
$$= h(x)\sum_{p\in\mathcal{P}} \int_\Gamma \sum_m \langle f_m(x), (\varphi_m)_p(x-\gamma^m)\rangle (\psi_m)_p(x-\gamma^m) d\mu_\Gamma(\gamma^m),$$

for all x belongs to G.

Our next proposition provides a construction of pairwise orthogonal Bessel (frame) systems by using the similarity (equivalence) property. Here, note that for a measure space (J, μ_J) with μ_J being a Haar measure on $\mathbb{J}$, we say that $X := \{x_j\}_{j\in J}$ and $Y := \{y_j\}_{j\in J}$ in the Hilbert space $\mathcal{H}$ are similar (equivalent) if there exists a bounded invertible operator $\mathbb{U}$ on $\mathcal{H}$ such that $x_j = \mathbb{U}y_j$, for all $j \in \mathbb{J}$.

Proposition 3.5 (see [29]). For a countable index set $\mathcal{P}$, let $\widetilde{\Psi} := \{(\tilde{\psi_m})_p\}_{p\in O}$ and $\tilde{\Phi} := \{\widetilde{(\varphi_m)_p}\}_{p\in\mathcal{P}}$ be subsetsin $L^2(G)$. Further, let $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ be orthogonal Γ - TI Bessel (frame) systems in $L^2(G)$, and that $E^{\Gamma}(\Psi)$ is similar to $E^{\Gamma}(\widetilde{\Psi})$, and $E^{\Gamma}(\Phi)$ is similar to $E^{\Gamma}(\widetilde{\Phi})$. Then, $E^{\Gamma}(\widetilde{\Psi})$ and $E^{\Gamma}(\widetilde{\Phi})$ also formorthogonal Γ - TI Bessel (frame) systems in $L^2(G)$.

Proof. Clearly, $E^{\Gamma}(\widetilde{\Psi})$ and $E^{\Gamma}(\widetilde{\Phi})$ are Γ -TI Bessel (frame) systems in $L^2(G)$ since similarity among the systems preserves the frame property, including the frame bounds. Further, by the definition of similarity, there exist bounded invertible operators $\mathbb{U}_1$ and U_2 on $L^2(G)$ such that we can write $\mathbb{U}_1\rho(\gamma^m)(\psi_m)_p = \rho(\gamma^m)\widetilde{(\psi_m)}_p$ and $\mathbb{U}_2\rho(\gamma^m)(\varphi_m)_p = \rho(\gamma^m)(\tilde{\varphi_m})_p$ for every $p \in \mathcal{P}$. We claim that $\Theta_{\tilde{u}} = \mathbb{U}_1\Theta_{\tilde{\psi}_m}$ and
$\Theta_{\tilde{\Phi}} = \mathbb{U}_2\Theta_\Phi$. For this, it is enough to prove that for all $h \in L^2(F, \ell^2(\mathcal{P}))$, we have

$$\Theta_{\tilde{\psi}_m} h = \sum_{p\in\mathcal{P}} \int_\Gamma \sum_m h_p(\gamma^m)\rho(\gamma^m)(\tilde{\psi_m})_p d\mu_\Gamma(\gamma^m) = \sum_{p\in\mathcal{P}} \int_\Gamma \sum_m h_p(\gamma^m)\mathbb{U}_1\rho(\gamma^m)(\psi_m)_p d\mu_\Gamma(\gamma^m)$$
$$= \mathbb{U}_1 \sum_{p\in O} \int_\Gamma \sum_m h_p(\gamma^m)\rho(\gamma^m)(\psi_m)_p d\mu_\Gamma(\gamma^m) = \mathbb{U}_1(\Theta_\Psi h),$$

and hence the result follows in view of the fact that $E^{\Gamma}(\Psi)$ and $E^{\Gamma}(\Phi)$ are orthogonal in $L^2(G)$, and

$$\Theta_{\tilde{\psi}_m}\Theta_{\tilde{\Phi}}^* = \mathbb{U}_1\Theta_\Phi(U_2\Theta_\Phi)^* = \mathbb{U}_1\Theta_\Psi\Theta_\phi^*\mathbb{U}_2^* = \mathbb{U}_1 0 U_2^* = 0$$

where 0 denotes the zero operator on $L^2(G)$.

3.2. Application of the characterization result on co-compact Gabor systems. We deduce a characterization result for the case of Gabor systems. We define these structured systems as a special case of Γ -TI systems given in Definition 1.1. Let a character χ in $\hat{G}$, and define the modulation operator $\eta(\chi)$ on $L^2(G)$ as $\eta(\chi)(f_m)(x) = \chi(x)f_m(x)$, for all $f_m \in L^2(G)$ and $x \in G$, and observe that it is associated with the translation operator on $L^2(\hat{G})$ by

$$(\widehat{\eta(\chi)f_m})(\xi_m) = \int_G \sum_m \chi(x)f_m(x)\overline{\xi_m(x)}d\mu_G(x) = \int_G \sum_m f_m(x)\overline{(\xi_m-\chi)(x)}d\mu_G(x)$$
$$= \widehat{f_m}(\xi_m - \chi) = \rho(\chi)\widehat{f_m}(\xi_m), for\ a.e.\ \xi_m \in \hat{G}$$

Let Γ and Λ be respectively, co-compact subgroups of G and $\hat{G}$. For an index set $J \subset \mathbb{Z}$, let $\mathcal{A} := \{(f_m)_j\}_{j\in J}$ be a subset in $L^2(G)$. Then the collection $\mathcal{G}(\mathcal{A}, \Gamma, \Lambda)$ defined by

$$\mathcal{G}(\mathcal{A}, \Gamma, \Lambda) := \{\rho(\gamma^m)\eta(\chi)(f_m)_j : \gamma^m \in \Gamma, \chi \in \Lambda, j \in J\} \tag{3.5}$$

is called the Gabor system generated by $\mathcal{A}$. Note that $\mathcal{G}(\mathcal{A}, \Gamma, \Lambda)$ is a frame for $L^2(G)$ if, and only if, $\{\eta(\chi)\rho(\gamma^m)(f_m)_j : \gamma^m \in \Gamma, \chi \in \Lambda, j \in J\}$ is a frame for $L^2(G)$, where the later system is termed as a co-compact Gabor system in [20]. Further, observe that $\mathcal{G}(\mathcal{A}, \Gamma, \Lambda)$ is a Γ -TI system of form $\{\rho(\gamma^m)(\psi_m)_j : \gamma^m \in \Gamma, \psi mj \in \Psi, j \in J$ defined in Definition 1.1, with $\psi mj = \eta(\chi) fmj$, where $(j,\chi) \in J \times \Lambda$.

Let $\mathcal{A}=\{(f_m)_j\}_{j\in J}$ and $(\mathcal{A}+\epsilon):=\{h_j\}_{j\in J}$ be countable subsets of $L^2(G)$. Then, our next result gives a characterization of $\mathcal{A}$ and $(\mathcal{A}+\epsilon)$ such that $\mathcal{G}(\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{G}(\mathcal{A}+\epsilon,\Gamma,\Lambda)$ form a pair of orthogonal Bessel families (frames) in $L^2(G)$, we call as a pair of co-compact Gabor orthogonal Bessel (frame) systems over LCA groups.

Proposition 3.6 (see [29]). Suppose that $\mathcal{G}(\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{G}(\mathcal{A}+\epsilon,\Gamma,\Lambda)$ are co-compact Gabor Bessel (frame) systems in $L^2(G)$. Then, the following assertions are equivalent:

(i) $\mathcal{G}(\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{G}(\mathcal{A}+\epsilon,\Gamma,\Lambda)$ form a pair of co-compact Gabor orthogonal Bessel (frame) systems in $L^2(G)$ in the sense of Definition 2.2.

(ii) For each $\alpha,\beta\in\Gamma^{\perp}$ and $\chi\in\Lambda$, we have $\sum_{j\in J}\sum_m \widehat{(f_m)}_j(\kappa+\alpha-\chi)\overline{\widehat{h_j}(\kappa+\beta-\chi)}=0$, for a.e. $\kappa\in\Omega$.

Proof. In view of computation (2.6) and Theorem 3.2, (i) holds if, and only if, for each $\chi\in\Lambda$, we have

$$\sum_{\alpha\in\Gamma^{\perp}}\vartheta_1(\alpha)\sum_{\beta\in\Gamma^{\perp}}\sum_m \overline{\partial_2(\beta)}\left(\sum_{j\in J}\left(\eta(\chi)\widehat{(f_m)}_j\right)(\kappa+\alpha)\overline{\left(\widehat{\eta(\chi)h_j}\right)(\kappa+\beta)}\right)=0, for\ a.e.\kappa\in\Omega$$

and for all $\vartheta_1,\vartheta_2\in\ell^2(\Gamma^{\perp})$. Hence, the result follows since ϑ_1,ϑ_2 are arbitrary elements of $\ell^2(\Gamma^{\perp})$. Note that Proposition 3.6 can be used to derive various results on a pair of co-compact Gabor orthogonal Bessel (frame) systems by letting different situations on Γ,Λ and G, etc. Example 3.7. Let $G=\mathbb{Z}^d$, and let $\Gamma=A\mathbb{Z}^d$ and $\Lambda=(A+\epsilon)\mathbb{Z}^d$ be uniform lattices in $\mathbb{Z}^d$ for some invertible $d\times d$ matrices A and $(A+\epsilon)$ with integer entries. Then, $\Gamma^{\perp}=\tilde{A}\mathbb{Z}^d$, where $\tilde{A}=(A^t)^{-1}$, that is, inverse of the transpose of matrix A. In this case, (3.5) reduces to the following collection:

$$\mathcal{G}(\mathcal{A},A\mathbb{Z}^d,(A+\epsilon)\mathbb{Z}^d):=\{\rho(\gamma^m)\eta(\chi)(f_m)_j:\gamma^m\in A\mathbb{Z}^d,\chi\in(A+\epsilon)\mathbb{Z}^d,j\in J\}$$

Hence, $\mathcal{G}(\mathcal{A},A\mathbb{Z}^d,(A+\epsilon)\mathbb{Z}^d)$ and $\mathcal{G}(\mathcal{A}+\epsilon,A\mathbb{Z}^d,(A+\epsilon)\mathbb{Z}^d)$ form a pair of orthogonal frames in $\ell^2(\mathbb{Z}^d)$ if, and onlyif, for $m,n,p\in\mathbb{Z}^d$, we have $\sum_{j\in J}\widehat{(f_m)}_j(\kappa+\tilde{A}m-(A+\epsilon)p)\overline{\widehat{h}_j(\kappa+\tilde{A}n-(A+\epsilon)p)}=0$, for a.e. $\kappa\in\tilde{A}([0,1]^d)$.

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