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**Research Paper**

# Simplicial Complexes in Algebraic Topology

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***Abstract:***
*Simplicialcomplexes are a central tool in algebraic topology , providing a bridge between combinatorial structures and abstract simplicial complexes , the notion of realizations , the concept of triangulable spaces, and the Simplicial Approximation Theore, highlighting their role in studying topological properties.*
***Keywords:*** *providing a bridge between combinatorial structures and abstract simplicial complexes , the notion of realizations , the concept of triangulable spaces, and the Simplicial Approximation Theore.*

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## I. Introduction

Topology studies spaces through properties that remain invariant under continuous deformations. One of the most powerful tools in this field is the theory of simplicialcomplexes , which allows us to translate abstract topological problems into combinatorial ones , described in terms of finite sets of vertices and simplices .

## II. Geometric Simplicial Complexes

**Definition :**

A (geometric) simplicial complexesis a finite collection K of simplices in $R^N$satisfying :

1) if$S = \{v_0,. . . , v_n\}$ is in K and $T \prec S$ (T is a subset of S), then T is also in K;

2) forS and T in K, if $\Delta^n [S] \cap \Delta^m [T] \neq \emptyset$, then $\Delta^n [S] \cap \Delta^m [T] = \Delta^k [U]$ for some U in K, that is, if simplices of K intersect, then they do so along acommon face.

## III. Abstract Simplicial Complexes

**Definition :**

A finite collection of sets $L = \{S_\alpha | S_\alpha = \{v_{\alpha 0}, \ldots, v_{\alpha n_\alpha}\}, 1 \leq \alpha \leq N\}$ is an abstract simplicial complexesif whenever $T = \{v_{j_0}, . . . , v_{j_k}\}$ is a subset of S and S is in L, then T is also in L.

In its simplicity there is a gain in flexibility with the notion of an abstract simplicialcomplexes . We can define all sorts of combinatorial objects in this manner (see, for example, [Björner]). To maintain the connection to topology, we ask if it is possible to associate to every vertex v in an abstract simplicial complex L a point vin $R^N$in such a way that L determines a geometric simplicial complex. The answer is yes, and the proof is an exercise in linear algebra (sketched in the exercises) in which we associate a list of vectors in $R^N$in general position to each set Sin L . In fact, if the abstract simplicial complex contains a set of cardinality at most $m + 1$, then there is a geometric simplicial complex $L'$ with corresponding sets consisting of vectors in $R^{2m+1}$in general position. Another way to connect with topology is to use the combinatorial data given by an abstract simplicial complex and construct a topological space by gluing simplices together:If $L = \{S | S = \{v_o, . . . , v_n\}\}$, then the set of equivalence classes, $|L| = [\cup_{s \in L} \Delta_S^n]$, associated to the equivalence relation given by $\mathbf{p} \sim \mathbf{q}$ for $\mathbf{p} \in \Delta_s^{\mathbf{n}}$ and$\mathbf{q} \in \Delta_T^m$ if there is a shared face $U \prec S$ , $U \prec T$ and $\mathbf{p} = \mathbf{q}$ in $\Delta_U^k \subset \Delta_S^n$ and $\Delta_U^k \subset \Delta_T^m$, that is, we glue the simplicesSand T along their shared subsimplex U. We givetopology as a quotient of the disjoint union of the simplices$\Delta_S^n$. The reader should check that this quotient construction determines a space homeomorphic to the realization of a geometric simplicial complex built out of vertices in $R^N$.

## IV. The Simplicial Approximation Theorem

Given two simplicial complexes K and L and a continuous mapping $f: |K| \to |L|$, then there is a nonnegative integer r and a simplicial mapping $\emptyset: \mathrm{sd}^r\, K \to L$ with $\emptyset$ a simplicial approximation to f.

Proof:We use the fact that $|K|$ and $|L|$ are compact metric spaces. Suppose $\dim K = n$. The collection $\{f^{-1}(O_L(w)) | w \text{ a vertex in } L\}$ is an open cover of $|K|$. The cover has a Lebesgue number $\delta_K > 0$ . Iterating barycentric subdivision, we can subdivide K until

$$\mathrm{mesh}(\mathrm{sd}^r\, K) \leq \left(\frac{n}{n+1}\right)^r \mathrm{mesh}(K) < \delta_K/2.$$

This is possible because $\left(\frac{n}{n+1}\right)^r$ goes to zero as r goes to infinity. It follows that $\mathrm{sd}^r\, K$ has all simplices of diameter less than $\delta_K/2$ and so, for each $v \in \mathrm{sd}^r\, K$, the diameter of $O_k(v)$ is less than $\delta_K$. Thus each $O_K(v)$ is contained in some$f^{-1}(O_L(w))$. This determines a vertex map $\emptyset: v \to w$, which satisfies $f(O_K(v)) \subset O_L(\emptyset(v))$, a simplicial approximation.

## V. Properties and Results

Any two simplices of the same dimension are homeomorphic .-

- The realization |K| depends only on the combinatorial structure , not the embedding .
- Complexes can be subdivided ( barycentric subdivision ) without altering realizations .
- Simplicial approximations are strongly linked to homology and homotopy , enabling the computation of fundamental groups and invariants of spaces .

## VI. Conclusion

A rigorous foundation for simplicialcomplexes . Starting with their geometric and abstract definition .

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