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Research Paper

Conjugation and Moduli in Tricomplex Numbers

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ABSTRACT: *This paper develops a systematic framework for the conjugation theory and associated moduli in the tricomplex algebra $\mathbb{C}_3$. The presence of three mutually commuting imaginary units i_1, i_2, i_3 gives rise to a rich family of involutive symmetries extending the classical complex conjugation. We construct and characterize the three fundamental conjugations that individually reverse the signs of i_1, i_2, i_3 and derive four additional composite conjugations generated by their compositions. The resulting seven conjugation operators form an Abelian group under composition and determine a hierarchy of algebraic invariants within $\mathbb{C}_3$. Using these operators, several generalized moduli are introduced and analyzed, including their behavior on bicomplex and tricomplex subalgebras and their multiplicative properties, which extend the classical notion of modulus to higher-dimensional multicomplex structures. The action of conjugation on polynomial equations is also examined, leading to symmetry relations for roots and orbit-based factorization patterns. These results provide a unified extension of classical norm and conjugation theory to higher multicomplex systems and establish foundational tools for the analytic study of tricomplex-valued functions.*

KEYWORDS: *Tricomplex numbers, conjugation operators, multicomplex algebra, generalized moduli.*
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I. PRELIMINARIES

Let $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$ denote the set of real, complex and bicomplex numbers respectively.

The **tricomplex system**

$$\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3)$$

is obtained by adjoining three mutually commuting imaginary units

$$i_1 \neq i_2 \neq i_3; \quad i_1^2 = i_2^2 = i_3^2 = -1,$$

and

$$i_1 i_2 = i_2 i_1, \qquad i_1 i_3 = i_3 i_1, \qquad i_2 i_3 = i_3 i_2.$$

Every element $\zeta \in \mathbb{C}_3$ admits a unique expansion

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8,$$

with real coefficients $x_j \in \mathbb{C}_0$.

Thus $\mathbb{C}_3$ is an 8-dimensional commutative real algebra extending the chain

$$\mathbb{C}_0 \subset \mathbb{C}_1 \subset \mathbb{C}_2 \subset \mathbb{C}_3.$$

The algebra possesses a rich internal structure, including idempotent decompositions, and subalgebras isomorphic to $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ together with several involutive conjugation operators. These structural features enable the definition of moduli, norms, analytic functions, and spectral properties analogous to those arising in classical complex analysis.

Because the imaginary units commute, tricomplex analysis retains the algebraic regularity required for polynomial factorization, spectral decomposition, and conjugation theory, while simultaneously admitting a richer set of symmetries than the complex or bicomplex systems.

II. INTRODUCTION TO CONJUGATION

Conjugation constitutes one of the fundamental algebraic symmetries in complex analysis, providing the basis for defining real-valued norms, metric structures, and analytic properties.
In the tricomplex algebra $\mathbb{C}_3$, the presence of three independent imaginary units

$$i_1,\ i_2,\ i_3$$

introduces a nontrivial generalization of this concept.
The algebra admits **three fundamental involutive conjugations**, each reversing the sign of exactly one imaginary unit while leaving the other units unchanged.
Compositions of these fundamental involutions produce **four additional composite conjugations**, resulting in **seven distinct conjugation operators** on $\mathbb{C}_3$.
Each of these operators acts linearly, preserves multiplication, and extends uniquely to all tricomplex elements.
Collectively, the seven conjugations generate a finite symmetry group within the algebra.
They play a central role in defining generalized moduli, establishing orthogonality relations, and characterizing the spectral and analytic behavior of tricomplex-valued functions.

III. BASIC CONJUGATIONS

Let

$$\zeta = x_1 + i_1x_2 + i_2x_3 + i_3x_4 + i_1i_2x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8 \in \mathbb{C}_3,$$

where $x_k \in \mathbb{C}_0$.
We define three elementary conjugations, each reversing the sign of one imaginary unit, together with four additional composite conjugations, giving a total of seven distinct conjugation operators in the tricomplex algebra.

(i) i_1 -Conjugation

$$\Gamma_1(\zeta) = x_1 - i_1x_2 + i_2x_3 + i_3x_4 - i_1i_2x_5 - i_1i_3x_6 + i_2i_3x_7 - i_1i_2i_3x_8.$$

(ii) i_2 -Conjugation

$$\Gamma_2(\zeta) = x_1 + i_1x_2 - i_2x_3 + i_3x_4 - i_1i_2x_5 + i_1i_3x_6 - i_2i_3x_7 - i_1i_2i_3x_8.$$

(iii) i_3 -Conjugation

$$\Gamma_3(\zeta) = x_1 + i_1x_2 + i_2x_3 - i_3x_4 + i_1i_2x_5 - i_1i_3x_6 - i_2i_3x_7 - i_1i_2i_3x_8.$$

(iv) i_1i_2 -Conjugation

$$\Gamma_{12}(\zeta) = x_1 - i_1x_2 - i_2x_3 + i_3x_4 + i_1i_2x_5 - i_1i_3x_6 - i_2i_3x_7 + i_1i_2i_3x_8.$$

(v) i_1i_3 -Conjugation

$$\Gamma_{13}(\zeta) = x_1 - i_1x_2 + i_2x_3 - i_3x_4 - i_1i_2x_5 + i_1i_3x_6 - i_2i_3x_7 + i_1i_2i_3x_8.$$

(vi) i_2i_3 -Conjugation

$$\Gamma_{23}(\zeta) = x_1 + i_1x_2 - i_2x_3 - i_3x_4 - i_1i_2x_5 - i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8.$$

(vii) $i_1 i_2 i_3$ -Conjugation

$$\Gamma_{123}(\zeta) = x_1 - i_1 x_2 - i_2 x_3 - i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 - i_1 i_2 i_3 x_8.$$

IV. CONJUGATIONS EXPRESSED VIA DECOMPOSITION $\xi + i_3\eta$

Let

$$\zeta = \xi + i_3\eta, \quad \xi, \eta \in \mathbb{C}(i_1, i_2).$$

Using linearity and multiplicativity, the conjugations act as follows:

(i) i_1 − Conjugation

$$\Gamma_1(\zeta) = \Gamma_1(\xi) + i_3\Gamma_1(\eta)$$

(ii) i_2 − Conjugation

$$\Gamma_2(\zeta) = \Gamma_2(\xi) + i_3\Gamma_2(\eta)$$

(iii) i_3 − Conjugation

$$\Gamma_3(\zeta) = \Gamma_3(\xi) - i_3\Gamma_3(\eta) = \xi - i_3\,\eta$$

(iv) $i_1 i_2$ − Conjugation

$$\Gamma_{12}(\zeta) = \Gamma_{12}(\xi) + i_3\Gamma_{12}(\eta)$$

(v) $i_1 i_3$ − Conjugation

$$\Gamma_{13}(\zeta) = \Gamma_{13}(\xi) - i_3\Gamma_{13}(\eta) = \Gamma_1(\xi) - i_3\Gamma_1(\eta)$$

(vi) $i_2 i_3$ − Conjugation

$$\Gamma_{23}(\zeta) = \Gamma_{23}(\xi) - i_3\Gamma_{23}(\eta) = \Gamma_2(\xi) - i_3\Gamma_2(\eta)$$

(vii) $i_1 i_2 i_3$ − Conjugation

$$\Gamma_{123}(\zeta) = \Gamma_{123}(\xi) - i_3\Gamma_{123}(\eta) = \Gamma_{12}(\xi) - i_3\Gamma_{12}(\eta)$$

V. ALGEBRAIC PROPERTIES OF THE CONJUGATIONS

For every tricomplex number $\zeta \in \mathbb{C}_3$ and each conjugation Γ_k ($k \in \{1,2,3,12,13,23,123\}$), we have:

(i) Involutivity:

$$\Gamma_k(\Gamma_k(\zeta)) = \zeta.$$

(ii) Linearity over $\mathbb{C}_0$:

$$\Gamma_k(\alpha\zeta_1 + \beta\zeta_2) = \alpha\,\Gamma_k(\zeta_1) + \beta\,\Gamma_k(\zeta_2), \quad \alpha, \beta \in \mathbb{C}_0.$$

(iii) Multiplicativity:

$$\Gamma_k(\zeta_1\zeta_2) = \Gamma_k(\zeta_1)\,\Gamma_k(\zeta_2).$$

(iv) Real Part Invariance:

$$\Gamma_k(\Re(\zeta)) = \Re(\zeta).$$

VI. RELATIONS AMONG ALL CONJUGATIONS

6.1 Fundamental Relations

Each basic conjugation is self-inverse:

$$\Gamma_k^2 = \mathrm{id}, \quad k = 1,2,3.$$

Each pair of conjugations commutes:

$$\Gamma_i\Gamma_j = \Gamma_j\Gamma_i, \quad i, j \in \{1,2,3\}.$$

Thus every higher conjugation can be expressed as a product (composition) of fundamental conjugations.

6.2 Explicit Relations Among All Conjugations

(1) Double Conjugations

$$\Gamma_{12} = \Gamma_1\Gamma_2,$$
$$\Gamma_{13} = \Gamma_1\Gamma_3,$$
$$\Gamma_{23} = \Gamma_2\Gamma_3.$$

Each operator flips exactly the pair of imaginary units that appears in its index.

(2) Triple Conjugation

$$\Gamma_{123} = \Gamma_1\Gamma_2\Gamma_3.$$

This operator flips the sign of **all three** imaginary components.

(3) Relations Among Composite Conjugations

From commutativity and involution:

$$\Gamma_{12}\Gamma_1 = \Gamma_2, \quad \Gamma_{12}\Gamma_2 = \Gamma_1, \quad \Gamma_{12}\Gamma_3 = \Gamma_{123};$$
$$\Gamma_{13}\Gamma_1 = \Gamma_3, \quad \Gamma_{13}\Gamma_2 = \Gamma_{123}, \quad \Gamma_{13}\Gamma_3 = \Gamma_1;$$
$$\Gamma_{23}\Gamma_1 = \Gamma_{123}, \quad \Gamma_{23}\Gamma_2 = \Gamma_3, \quad \Gamma_{23}\Gamma_3 = \Gamma_2;$$

These identities describe how one conjugation transforms into another under composition.

(4) Triple Conjugation Relations

$$\Gamma_{123}\Gamma_1 = \Gamma_{23}, \quad \Gamma_{123}\Gamma_2 = \Gamma_{13}, \quad \Gamma_{123}\Gamma_3 = \Gamma_{12}.$$

Conversely:

$$\Gamma_{23}\Gamma_1 = \Gamma_{123}, \quad \Gamma_{13}\Gamma_2 = \Gamma_{123}, \quad \Gamma_{12}\Gamma_3 = \Gamma_{123}.$$

Thus the triple conjugation plays the role of a “complement” of each single conjugation.

6.3 Group-Theoretic Structure

The full set of conjugations in the tricomplex system

$\mathcal{G} = \{\Gamma_\emptyset, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{12}, \Gamma_{13}, \Gamma_{23}, \Gamma_{123}\}$

where the identity conjugation is defined by

$$\Gamma_\emptyset(\zeta) = \zeta,$$

forms an eight-element Abelian group under composition.

6.4 Summary Table of Relations

Conjugation	Composition Form	Units Flipped	Order
$\boldsymbol{\Gamma_1}$	–	i_1	2
$\boldsymbol{\Gamma_2}$	–	i_2	2
$\boldsymbol{\Gamma_3}$	–	i_3	2
$\boldsymbol{\Gamma_{12}}$	$\Gamma_1\Gamma_2$	i_1, i_2	2
$\boldsymbol{\Gamma_{13}}$	$\Gamma_1\Gamma_3$	i_1, i_3	2
$\boldsymbol{\Gamma_{23}}$	$\Gamma_2\Gamma_3$	i_2, i_3	2
$\boldsymbol{\Gamma_{123}}$	$\Gamma_1\Gamma_2\Gamma_3$	all i_1, i_2, i_3	2

VII. APPLICATIONS OF CONJUGATION TO POLYNOMIAL FACTORIZATION IN THE TRICOMPLEX SYSTEM

Conjugation in $\mathbb{C}_3$ provides a powerful algebraic mechanism for understanding the behavior of tricomplex polynomials.

Since tricomplex numbers involve three independent imaginary units and eight real components, factorization is substantially more intricate than in the classical complex setting.

However, the conjugation operators Γ_k ($k \in \{1,2,3,12,13,23,123\}$) allow us to generalize the classical theory in a meaningful and structurally consistent way.

7.1 Tricomplex Polynomials

A polynomial over $\mathbb{C}_3$ is defined as

$$P(\zeta) = \gamma_n \zeta^n + \gamma_{n-1} \zeta^{n-1} + \cdots \ldots + \gamma_1 \zeta + \gamma_0$$

where $\gamma_j \in \mathbb{C}_3, j = 0,1,2, \ldots, n$ and ζ is a tricomplex variable.

Because multiplication in $\mathbb{C}_3$ is commutative and associative, $P(\zeta)$ retains all formal algebraic properties of polynomials over a commutative ring.

7.2 Conjugation-Invariant Polynomials

Definition 7.2.1.

A polynomial $P(\zeta)$ is said to be $\Gamma_k -$*invariant* if

$$\Gamma_k \left(P(\zeta)\right) = P(\Gamma_k (\zeta)).$$

This condition holds automatically when all coefficients γ_j are fixed by Γ_k, i.e.,

$$\Gamma_k(\gamma_j) = \gamma_j .$$

Conjugation invariance plays the same structural role as real coefficients do in complex analysis.

7.3 Root Symmetry Induced by Conjugation

Theorem 7.3.1 (Conjugation Symmetry).

Let $P(\zeta)$ be a tricomplex polynomial and let $w \in \mathbb{C}_3$ be a root of P.

Then, for every conjugation Γ_k ($k \in \{1,2,3,12,13,23,123\}$),

$$P(w) = 0 \Longrightarrow P\left(\Gamma_k(w)\right) = 0$$

whenever P is $\Gamma_k -$invariant.

Proof.

$$0 = \Gamma_k\left(P(w)\right) = P\left(\Gamma_k(w)\right)$$

■

7.4 Orbit Structure of Roots

Each root generates a conjugation orbit:

$$\mathcal{O}(w) = \{w, \Gamma_1(w), \Gamma_2(w), \Gamma_3(w), \Gamma_{12}(w), \Gamma_{13}(w), \Gamma_{23}(w), \Gamma_{123}(w)\}.$$

Thus, a single root may produce up to **eight distinct roots**, depending on symmetries and degeneracies.

Example 7.1: Consider the polynomial

$$P(\zeta) = \zeta^2 - 1.$$

Then P has exactly sixteen distinct roots in $\mathbb{C}_3$, namely

$$\pm 1, \quad \pm i_1 i_2, \quad \pm i_1 i_3, \quad \pm i_2 i_3, \quad \pm r_1, \quad \pm r_2, \quad \pm r_3, \quad \pm r_4,$$

where

$$r_1 = \frac{1}{2}(1 + i_1 i_2 + i_1 i_3 + i_2 i_3), \qquad r_1 = \frac{1}{2}(1 - i_1 i_2 - i_1 i_3 + i_2 i_3),$$

$$r_3 = \frac{1}{2}(1 - i_1 i_2 + i_1 i_3 - i_2 i_3), \qquad r_4 = \frac{1}{2}(1 + i_1 i_2 - i_1 i_3 - i_2 i_3).$$

Orbit decomposition

The roots partition into the following orbits:

(i) $\mathcal{O}(1) = \{1\}$ $(size\ 1)$

(ii) $\mathcal{O}(-1) = \{-1\}$ $(size\ 1)$

(iii) $\mathcal{O}(i_1 i_2) = \{i_1 i_2, -i_1 i_2\}$ $(size\ 2)$

(iv) $\mathcal{O}(i_1 i_3) = \{i_1 i_3, -i_1 i_3\}$ $(size\ 2)$

(v) $\mathcal{O}(i_2 i_3) = \{i_2 i_3, -i_2 i_3\}$ $(size\ 2)$

(vi) $\mathcal{O}(r\ ) = \{r_1, r_2, r_3, r_4\}$ $(size\ 4)$

(vii) $\mathcal{O}(-r\ ) = \{-r_1, -r_2, -r_3, -r_4\}$ $(size\ 4)$

Counting orbits: $1 + 1 + 2 + 2 + 2 + 4 + 4 = 16.$

Orbit polynomials

For each orbit $\mathcal{O}$ define the orbit polynomial

$$F_{\mathcal{O}}(\zeta) = \prod_{\rho \in \mathcal{O}} (\zeta - \rho)$$

In this example:

- $F_{\mathcal{O}(1)}(\zeta) = \zeta - 1$
- $F_{\mathcal{O}(-1)}(\zeta) = \zeta + 1$
- $F_{\mathcal{O}(i_1 i_2)}(\zeta) = \zeta^2 - 1 = P(\zeta)$
- $F_{\mathcal{O}(i_1 i_3)}(\zeta) = \zeta^2 - 1 = P(\zeta)$
- $F_{\mathcal{O}(i_2 i_3)}(\zeta) = \zeta^2 - 1 = P(\zeta)$
- $F_{\mathcal{O}(r)}(\zeta) = \prod_{j=1}^{4} (\zeta - r_j)$
- $F_{\mathcal{O}(-r)}(\zeta) = \prod_{j=1}^{4} (\zeta + r_j)$

The product over all 16 linear factors (one factor per root) equals the product of the seven orbit-polynomials:

$$\prod_{\rho \in Roots} (\zeta - \rho) = F_{\mathcal{O}(1)}(\zeta) F_{\mathcal{O}(-1)}(\zeta) F_{\mathcal{O}(i_1 i_2)}(\zeta) F_{\mathcal{O}(i_1 i_3)}(\zeta) F_{\mathcal{O}(i_2 i_3)}(\zeta)(\zeta) F_{\mathcal{O}(r)}(\zeta) F_{\mathcal{O}(-r)}(\zeta)$$

Now observe the pairing trick: group each root ρ with its negative $-\rho$. For any ρ with $\rho^2 = 1$,

$$(\zeta - \rho)(\zeta + \rho) = \zeta^2 - \rho^2 = \zeta^2 - 1.$$

There are exactly eight such pairs among the 16 roots:

- The pair $1, -1$ gives one factor $\zeta^2 - 1$,
- each sign-pair orbit $\pm i_1 i_2, \pm i_1 i_3, \pm i_2 i_3$ gives one factor each (3 more),

- the four pairs $r_j, -r_j$ for $j = 1,2,3,4$ produce four more factors.

Thus

$$\prod_{\rho \in Roots} (\zeta - \rho) = F_{\mathcal{O}(1)}(\zeta) F_{\mathcal{O}(-1)}(\zeta) F_{\mathcal{O}(i_1 i_2)}(\zeta) F_{\mathcal{O}(i_1 i_3)}(\zeta) F_{\mathcal{O}(i_2 i_3)}(\zeta)(\zeta) F_{\mathcal{O}(r)}(\zeta) F_{\mathcal{O}(-r)}(\zeta)$$

$$= (\zeta^2 - 1)^8$$

VIII. MODULI OF TRICOMPLEX NUMBERS

In the tricomplex system $\mathbb{C}_3 = \mathbb{C}(i_1, i_2, i_3)$, several distinct moduli arise depending on which pair of imaginary units is used.

Each modulus is defined through a conjugation operator, and every bicomplex subspace admits **four possible moduli**.

Below, the moduli are organized systematically for each subsystem.

(M1) Moduli in $\mathbb{C}(i_1, i_2)$

There are **four** moduli in the bicomplex algebra $\mathbb{C}(i_1, i_2)$.

Each modulus is generated by a pair of conjugations.

Let

$$\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_3 x_4 + i_1 i_2 x_5 + i_1 i_3 x_6 + i_2 i_3 x_7 + i_1 i_2 i_3 x_8.$$

(P1) $\zeta\, \Gamma_3(\zeta) = (x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_7^2 + x_8^2) + 2i_1(x_1x_2 - x_3x_5 + x_4x_6 - x_7x_8)$
$+2i_2(x_1x_3 - x_2x_5 + x_4x_7 - x_6x_8) + 2i_1i_2(x_1x_5 + x_2x_3 + x_4x_8 + x_6x_7)$

(P2) $\Gamma_1(\zeta)\, \Gamma_{13}(\zeta) = (x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_7^2 + x_8^2) + 2i_1(-x_1x_2 + x_3x_5 - x_4x_6 + x_7x_8)$
$+2i_2(x_1x_3 - x_2x_5 + x_4x_7 - x_6x_8) + 2i_1i_2(-x_1x_5 - x_2x_3 - x_4x_8 - x_6x_7)$

(P3) $\Gamma_2(\zeta)\, \Gamma_{23}(\zeta) = (x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_7^2 + x_8^2) + 2i_1(x_1x_2 - x_3x_5 + x_4x_6 - x_7x_8)$
$+2i_2(-x_1x_3 + x_2x_5 - x_4x_7 + x_6x_8) + 2i_1i_2(-x_1x_5 - x_2x_3 - x_4x_8 - x_6x_7)$

(P4) $\Gamma_{12}(\zeta)\, \Gamma_{123}(\zeta) = (x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_7^2 + x_8^2) + 2i_1(-x_1x_2 + x_3x_5 - x_4x_6 + x_7x_8)$
$+2i_2(-x_1x_3 + x_2x_5 - x_4x_7 + x_6x_8) + 2i_1i_2(x_1x_5 + x_2x_3 + x_4x_8 + x_6x_7)$

Relation in $\mathbb{C}(i_1, i_2)$

(i) $\Gamma_1(\zeta\, \Gamma_3(\zeta)) = \Gamma_1(\zeta)\, \Gamma_{13}(\zeta)$

(ii) $\Gamma_2(\zeta\, \Gamma_3(\zeta)) = \Gamma_2(\zeta)\, \Gamma_{23}(\zeta)$

(iii) $\Gamma_{12}(\zeta\, \Gamma_3(\zeta)) = \Gamma_{12}(\zeta)\, \Gamma_{123}(\zeta)$

(M2) Moduli in $\mathbb{C}(i_1, i_3)$

There are four possible moduli in $\mathbb{C}(i_1, i_3)$

(P1) $\zeta\, \Gamma_2(\zeta) = (x_1^2 - x_2^2 + x_3^2 - x_4^2 - x_5^2 + x_6^2 - x_7^2 + x_8^2) + 2i_1(x_1x_2 + x_3x_5 - x_4x_6 - x_7x_8)$
$+2i_3(x_1x_4 - x_2x_6 + x_3x_7 - x_5x_8) + 2i_1i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

(P2) $\Gamma_1(\zeta)\, \Gamma_{12}(\zeta) = (x_1^2 - x_2^2 + x_3^2 - x_4^2 - x_5^2 + x_6^2 - x_7^2 + x_8^2) - 2i_1(x_1x_2 + x_3x_5 - x_4x_6 - x_7x_8)$
$+2i_3(x_1x_4 - x_2x_6 + x_3x_7 - x_5x_8) - 2i_1i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

(P3) $\Gamma_3(\zeta)\, \Gamma_{23}(\zeta) = (x_1^2 - x_2^2 + x_3^2 - x_4^2 - x_5^2 + x_6^2 - x_7^2 + x_8^2) + 2i_1(x_1x_2 + x_3x_5 - x_4x_6 - x_7x_8)$
$-2i_3(x_1x_4 - x_2x_6 + x_3x_7 - x_5x_8) - 2i_1i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

(P4) $\Gamma_{13}(\zeta)\,\Gamma_{123}(\zeta) = (x_1^2 - x_2^2 + x_3^2 - x_4^2 - x_5^2 + x_6^2 - x_7^2 + x_8^2) - 2i_1(x_1x_2 + x_3x_5 - x_4x_6 - x_7x_8)$
$-2i_3(x_1x_4 - x_2x_6 + x_3x_7 - x_5x_8) + 2i_1i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

Relation in $\mathbb{C}(i_1, i_3)$

(i) $\Gamma_1(\,\zeta\,\Gamma_2(\zeta)) = \Gamma_1(\zeta)\,\Gamma_{12}(\zeta)$

(ii) $\Gamma_3(\,\zeta\,\Gamma_2(\zeta)) = \Gamma_3(\zeta)\,\Gamma_{23}(\zeta)$

(iii) $\Gamma_{13}(\,\zeta\,\Gamma_2(\zeta)) = \Gamma_{13}(\zeta)\,\Gamma_{123}(\zeta)$

(M3) Moduli in $\mathbb{C}(\boldsymbol{i_2}, \boldsymbol{i_3})$

There are four possible moduli in $\mathbb{C}(i_2, i_3)$

(P1) $\zeta\,\Gamma_1(\zeta) = (x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 + x_7^2 + x_8^2) + 2i_2(x_1x_3 + x_2x_5 - x_4x_7 - x_6x_8)$
$+2i_3(x_1x_4 + x_2x_6 - x_3x_7 - x_5x_8) + 2i_2i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

(P2) $\Gamma_2(\zeta)\,\Gamma_{12}(\zeta) = (x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 + x_7^2 + x_8^2) - 2i_2(x_1x_3 + x_2x_5 - x_4x_7 - x_6x_8)$
$+2i_3(x_1x_4 + x_2x_6 - x_3x_7 - x_5x_8) - 2i_2i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

(P3) $\Gamma_3(\zeta)\,\Gamma_{13}(\zeta) = (x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 + x_7^2 + x_8^2) + 2i_2(x_1x_3 + x_2x_5 - x_4x_7 - x_6x_8)$
$-2i_3(x_1x_4 + x_2x_6 - x_3x_7 - x_5x_8) - 2i_2i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

(P4) $\Gamma_{23}(\zeta)\,\Gamma_{123}(\zeta) = (x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 + x_7^2 + x_8^2) - 2i_2(x_1x_3 + x_2x_5 - x_4x_7 - x_6x_8)$
$-2i_3(x_1x_4 + x_2x_6 - x_3x_7 - x_5x_8) + 2i_2i_3(x_1x_6 + x_2x_4 + x_3x_8 + x_5x_7)$

Relation in $\mathbb{C}(i_2, i_3)$

(i) $\Gamma_2(\,\zeta\,\Gamma_1(\zeta)) = \Gamma_2(\zeta)\,\Gamma_{12}(\zeta)$

(ii) $\Gamma_3(\,\zeta\,\Gamma_1(\zeta)) = \Gamma_3(\zeta)\,\Gamma_{13}(\zeta)$

(iii) $\Gamma_{23}(\,\zeta\,\Gamma_1(\zeta)) = \Gamma_{23}(\zeta)\,\Gamma_{123}(\zeta)$

(M4) Moduli in $\mathbb{C}(\boldsymbol{i_1}, \boldsymbol{i_2 i_3})$

There are four possible moduli in $\mathbb{C}(i_1, i_2i_3)$

(P1) $\zeta\,\Gamma_{23}(\zeta) = (x_1^2 - x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 + x_7^2 - x_8^2) + 2i_1(x_1x_2 + x_3x_5 + x_4x_6 + x_7x_8)$
$+2i_2i_3(x_1x_7 - x_2x_8 - x_3x_4 + x_5x_6) + 2i_1i_2i_3(x_1x_8 + x_2x_7 - x_3x_6 - x_4x_5)$

(P2) $\Gamma_1(\zeta)\,\Gamma_{123}(\zeta) = (x_1^2 - x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 + x_7^2 - x_8^2) - 2i_1(x_1x_2 + x_3x_5 + x_4x_6 + x_7x_8)$
$+2i_2i_3(x_1x_7 - x_2x_8 - x_3x_4 + x_5x_6) - 2i_1i_2i_3(x_1x_8 + x_2x_7 - x_3x_6 - x_4x_5)$

(P3) $\Gamma_2(\zeta)\,\Gamma_3(\zeta) = (x_1^2 - x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 + x_7^2 - x_8^2) + 2i_1(x_1x_2 + x_3x_5 + x_4x_6 + x_7x_8)$
$-2i_2i_3(x_1x_7 - x_2x_8 - x_3x_4 + x_5x_6) - 2i_1i_2i_3(x_1x_8 + x_2x_7 - x_3x_6 - x_4x_5)$

(P4) $\Gamma_{12}(\zeta)\,\Gamma_{13}(\zeta) = (x_1^2 - x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 + x_7^2 - x_8^2) - 2i_1(x_1x_2 + x_3x_5 + x_4x_6 + x_7x_8)$
$-2i_2i_3(x_1x_7 - x_2x_8 - x_3x_4 + x_5x_6) + 2i_1i_2i_3(x_1x_8 + x_2x_7 - x_3x_6 - x_4x_5)$

Relation in $\mathbb{C}(i_1, i_2i_3)$

(i) $\Gamma_1(\,\zeta\,\Gamma_{23}(\zeta)) = \Gamma_1(\zeta)\,\Gamma_{123}(\zeta)$

(ii) $\Gamma_2(\,\zeta\,\Gamma_{23}(\zeta)) = \Gamma_2(\zeta)\,\Gamma_3(\zeta)$

(iii) $\Gamma_{12}(\,\zeta\,\Gamma_{23}(\zeta)) = \Gamma_{12}(\zeta)\,\Gamma_{13}(\zeta)$

(M5) Moduli in $\mathbb{C}(\boldsymbol{i_2}, \boldsymbol{i_1 i_3})$

There are four possible moduli in $\mathbb{C}(i_2, i_1 i_3)$

(P1) $\zeta\, \Gamma_{13}(\zeta) = (x_1^2 + x_2^2 - x_3^2 + x_4^2 - x_5^2 + x_6^2 - x_7^2 - x_8^2) + 2i_2(x_1x_3 + x_2x_5 + x_4x_7 + x_6x_8)$
$+2i_1i_3(x_1x_6 - x_2x_4 - x_3x_8 + x_5x_7) + 2i_1i_2i_3(x_1x_8 - x_2x_7 + x_3x_6 - x_4x_5)$

(P2) $\Gamma_1(\zeta)\, \Gamma_3(\zeta) = (x_1^2 + x_2^2 - x_3^2 + x_4^2 - x_5^2 + x_6^2 - x_7^2 - x_8^2) + 2i_2(x_1x_3 + x_2x_5 + x_4x_7 + x_6x_8)$
$-2i_1i_3(x_1x_6 - x_2x_4 - x_3x_8 + x_5x_7) - 2i_1i_2i_3(x_1x_8 - x_2x_7 + x_3x_6 - x_4x_5)$

(P3) $\Gamma_2(\zeta)\, \Gamma_{123}(\zeta) = (x_1^2 + x_2^2 - x_3^2 + x_4^2 - x_5^2 + x_6^2 - x_7^2 - x_8^2) - 2i_2(x_1x_3 + x_2x_5 + x_4x_7 + x_6x_8)$
$+2i_1i_3(x_1x_6 - x_2x_4 - x_3x_8 + x_5x_7) - 2i_1i_2i_3(x_1x_8 - x_2x_7 + x_3x_6 - x_4x_5)$

(P4) $\Gamma_{12}(\zeta)\, \Gamma_{23}(\zeta) = (x_1^2 + x_2^2 - x_3^2 + x_4^2 - x_5^2 + x_6^2 - x_7^2 - x_8^2) - 2i_2(x_1x_3 + x_2x_5 + x_4x_7 + x_6x_8)$
$-2i_1i_3(x_1x_6 - x_2x_4 - x_3x_8 + x_5x_7) + 2i_1i_2i_3(x_1x_8 - x_2x_7 + x_3x_6 - x_4x_5)$

Relation in $\mathbb{C}(i_2, i_1 i_3)$

(i) $\Gamma_1\big(\zeta\, \Gamma_{13}(\zeta)\big) = \Gamma_1(\zeta)\, \Gamma_3(\zeta)$

(ii) $\Gamma_2\big(\zeta\, \Gamma_{13}(\zeta)\big) = \Gamma_2(\zeta)\, \Gamma_{123}(\zeta)$

(iii) $\Gamma_{12}\big(\zeta\, \Gamma_{13}(\zeta)\big) = \Gamma_{12}(\zeta)\, \Gamma_{23}(\zeta)$

(M6) Moduli in $\mathbb{C}(\boldsymbol{i_3}, \boldsymbol{i_1 i_2})$

There are four possible moduli in $\mathbb{C}(i_3, i_1 i_2)$

(P1) $\zeta\, \Gamma_{12}(\zeta) = (x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 - x_7^2 - x_8^2) + 2i_3(x_1x_4 + x_2x_6 + x_3x_7 + x_5x_8)$
$+2i_1i_2(x_1x_5 - x_2x_3 - x_4x_8 + x_6x_7) + 2i_1i_2i_3(x_1x_8 - x_2x_7 - x_3x_6 + x_4x_5)$

(P2) $\Gamma_1(\zeta)\, \Gamma_2(\zeta) = (x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 - x_7^2 - x_8^2) + 2i_3(x_1x_4 + x_2x_6 + x_3x_7 + x_5x_8)$
$-2i_1i_2(x_1x_5 - x_2x_3 - x_4x_8 + x_6x_7) - 2i_1i_2i_3(x_1x_8 - x_2x_7 - x_3x_6 + x_4x_5)$

(P3) $\Gamma_3(\zeta)\, \Gamma_{123}(\zeta) = (x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 - x_7^2 - x_8^2) - 2i_3(x_1x_4 + x_2x_6 + x_3x_7 + x_5x_8)$
$+2i_1i_2(x_1x_5 - x_2x_3 - x_4x_8 + x_6x_7) - 2i_1i_2i_3(x_1x_8 - x_2x_7 - x_3x_6 + x_4x_5)$

(P4) $\Gamma_{13}(\zeta)\, \Gamma_{23}(\zeta) = (x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 - x_7^2 - x_8^2) - 2i_3(x_1x_4 + x_2x_6 + x_3x_7 + x_5x_8)$
$-2i_1i_2(x_1x_5 - x_2x_3 - x_4x_8 + x_6x_7) + 2i_1i_2i_3(x_1x_8 - x_2x_7 - x_3x_6 + x_4x_5)$

Relation in $\mathbb{C}(i_3, i_1 i_2)$

(i) $\Gamma_1\big(\zeta\, \Gamma_{12}(\zeta)\big) = \Gamma_1(\zeta)\, \Gamma_2(\zeta)$

(ii) $\Gamma_3\big(\zeta\, \Gamma_{12}(\zeta)\big) = \Gamma_3(\zeta)\, \Gamma_{123}(\zeta)$

(iii) $\Gamma_{13}\big(\zeta\, \Gamma_{12}(\zeta)\big) = \Gamma_{13}(\zeta)\, \Gamma_{23}(\zeta)$

(iv) $\Gamma_{12}\big(\zeta\, \Gamma_{13}(\zeta)\big) = \Gamma_{12}(\zeta)\, \Gamma_{23}(\zeta)$

(M7) Moduli in $\mathbb{C}(\boldsymbol{i_1 i_2}, \boldsymbol{i_1 i_3})$

There are four possible moduli in $\mathbb{C}(i_1 i_2, i_1 i_3)$

(P1) $\zeta\, \Gamma_{123}(\zeta) = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2) + 2i_1i_2(x_1x_5 - x_2x_3 + x_4x_8 - x_6x_7)$
$+2i_1i_3(x_1x_6 - x_2x_4 + x_3x_8 - x_5x_7) + 2i_2i_3(x_1x_7 + x_2x_8 - x_3x_4 + x_5x_6)$

(P2) $\Gamma_1(\zeta)\, \Gamma_{23}(\zeta) = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2) - 2i_1i_2(x_1x_5 - x_2x_3 + x_4x_8 - x_6x_7)$
$-2i_1i_3(x_1x_6 - x_2x_4 + x_3x_8 - x_5x_7) + 2i_2i_3(x_1x_7 + x_2x_8 - x_3x_4 + x_5x_6)$

(P3) $\Gamma_2(\zeta)\, \Gamma_{13}(\zeta) = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2) - 2i_1i_2(x_1x_5 - x_2x_3 + x_4x_8 - x_6x_7)$

$$+2i_1i_3(x_1x_6 - x_2x_4 + x_3x_8 - x_5x_7) - 2i_2i_3(x_1x_7 + x_2x_8 - x_3x_4 + x_5x_6)$$

(P4) $\Gamma_3(\zeta)\,\Gamma_{12}(\zeta) = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2) + 2i_1i_2(x_1x_5 - x_2x_3 + x_4x_8 - x_6x_7)$

$$-2i_1i_3(x_1x_6 - x_2x_4 + x_3x_8 - x_5x_7) - 2i_2i_3(x_1x_7 + x_2x_8 - x_3x_4 + x_5x_6)$$

Relation in $\mathbb{C}(i_1i_2, i_1i_3)$

(i) $\Gamma_1\big(\zeta\,\Gamma_{123}(\zeta)\big) = \Gamma_1(\zeta)\,\Gamma_{23}(\zeta)$

(ii) $\Gamma_2\big(\zeta\,\Gamma_{123}(\zeta)\big) = \Gamma_2(\zeta)\,\Gamma_{13}(\zeta)$

(iii) $\Gamma_{12}\big(\zeta\,\Gamma_{123}(\zeta)\big) = \Gamma_3(\zeta)\,\Gamma_{12}(\zeta)$

IX. CONCLUSION

In this study, we have conducted a comprehensive analysis of conjugation operations within the tricomplex number system $\mathbb{C}_3$. By introducing and systematically examining the seven fundamental conjugations and their compositions, we have established a clear framework for understanding their algebraic properties, including involutivity, linearity, and multiplicativity. The interplay between these conjugations and the idempotent decomposition of tricomplex numbers has been highlighted, revealing their crucial role in simplifying computations of moduli, norms, and spectral properties. Furthermore, the explicit forms of various moduli associated with different subalgebras of $\mathbb{C}_3$ have been derived, providing higher-dimensional generalizations of classical complex moduli. The symmetry group structure of the conjugations captures all possible sign inversions of imaginary units and offers deeper insight into the inherent geometric and algebraic symmetries of tricomplex numbers. These results lay a solid foundation for further studies in multicomplex analysis, particularly in the areas of polynomial factorization, spectral theory, and applications in mathematical physics. Overall, this work provides a rigorous framework for future research on higher-dimensional generalizations of complex analysis.

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