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Research Paper

Invariant Integral on Topological Simple Rough Groups

P. TAMILARASI [1*] and R. SELVI [2]

[1] *(Research Scholar (Reg. No: 22211202092003), Department of Mathematics, Sri Parasakthi College for Women, Courtallam – 627 802, Affiliated to Manonmaniam Sundaranar University, Tirunelveli – 627 012.)*

[2] *(Associate Professor, Department of Mathematics, Sri Parasakthi College for Women, Courtallam – 627 802, Affiliated to Manonmaniam Sundaranar University, Tirunelveli – 627 012)*

ABSTRACT : *In this paper, we introduce the concept of an invariant integral on a topological simple rough group and we extent the classical Haar measure and invariant integral to the rough setting. The existence and uniqueness of such an invariant integral are established, and its fundamental properties are examined.*
KEYWORDS: *Simple group, Topological simple rough groups, Complete rough group, Invariant integral, Haar measure.*

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I. INTRODUCTION

Rough set theory, introduced by Pawlak [9] in 1982, is founded on the concept of equivalence relations. Over the past three decades, the theory has been significantly developed and widely applied across various fields. In 1994, Biswas and Nanda [2] introduced the concepts of rough groups and rough subgroups, defining them primarily in terms of upper approximations. Later, in 2016, Bagirmaz et al. [8] extended this framework by introducing the notion of topological rough groups, thereby generalizing topological group structures to incorporate the algebraic aspects of rough groups.

In this paper, we study the concept of an invariant integral on a topological simple rough group. Considering the upper approximation of a simple rough group as a complete topological group. Using this structure, we construct an invariant integral and also extends the classical Haar measure. The existence and uniqueness of an invariant integral are established, and its essential properties are discussed. This approach connects simple rough group theory with topological measure theory.

II. PRELIMINARIES

Definition 2.1.[3] Let $K = (U, R)$ be an approximation space and $*$ be a binary operation defined on U. A subset G of universe U is called a rough group if the following properties are satisfied:

(i) $\forall$ x, y $\in$ G, $x*y \in \overline{G}$;

(ii) Association property holds in $\overline{G}$;

(iii) $\exists$ e $\in \overline{G}$ such that $\forall$ x $\in$ G, $x*e = e*x = x$; e is called the rough identity element of rough group G;

(iv) $\forall$ x $\in$ G, $\exists$ y $\in$ G such that $x*y = y*x = e$; y is called the rough inverse element of x in G;

Definition 2.2.[8] A topological rough group is a rough group (G, $*$) together with a topology T on $\overline{G}$ satisfying the following two properties:

(i) the mapping f: G × G → $\overline{G}$ defined by f(x, y) = xy is continuous with respect to product topology on G × G and the topology T_G on G induced by T,
(ii) the inverse mapping g: G → G defined by g(x) = x^{-1} is continuous with respect to the topology T_G on G induced by T.

Definition 2.3.[6] A rough group $G_{\mathfrak{R}}$ is called a simple rough group if it contains no proper non-trivial rough normal subgroups.

That is, $G_{\mathfrak{R}}$ has only the rough normal subgroups {e} and $G_{\mathfrak{R}}$.

Definition 2.4.[6] A topological simple rough group is a simple rough group $(G_{\mathfrak{R}}, *)$ together with a topology $\overline{\tau}$ on $\overline{G_{\mathfrak{R}}}$ satisfying the following two properties:

(i) The mapping f: $G_{\mathfrak{R}} \times G_{\mathfrak{R}} \to \overline{G_{\mathfrak{R}}}$ defined by f(x, y) = xy, x, y $\in G_{\mathfrak{R}}$ is continuous with respect to the product topology on $G_{\mathfrak{R}} \times G_{\mathfrak{R}}$ and the topology τ on $G_{\mathfrak{R}}$ induced by $\overline{\tau}$
(ii) The inverse mapping g: $G_{\mathfrak{R}} \to G_{\mathfrak{R}}$ defined by g(x) = x^{-1}, x $\in G_{\mathfrak{R}}$ is continuous with respect to the topology τ on $G_{\mathfrak{R}}$ induced by $\overline{\tau}$.

Definition 2.5.[4] Let X be a topological space. Let (X, ρ) be a metric space and A a subset of X; we say that A is ϵ-dense in (X, ρ) if for every $x \in X$ there exists an $x' \in A$ such that $\rho(x, x') < \epsilon$.

A metric space (X, ρ) is totally bounded if for every $\epsilon > 0$ there exists a finite set $A \subset X$ which is ϵ-dense in (X, ρ); a metric ρ on a set X is totally bounded if the space (X, ρ) is totally bounded.

Theorem 2.6.[4] Let X be a topological space. If (X, ρ) is a complete space, then for every closed subset M of X the space (M, ρ) is complete.

Theorem 2.7.[6] Let X be a topological space. A metric space (X, d) is compact if and only if it is complete and totally bounded.

Proposition 2.8.[10] Let $G_{\mathfrak{R}}$ be a topological simple rough group. If U $\subseteq \overline{G_{\mathfrak{R}}}$ is an open set with e $\in$ U, then there exists a symmetric open set V of e in $G_{\mathfrak{R}}$ such that VV $\subseteq$ U.

Lemma 2.9.[11] Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $G_{\mathfrak{R}}$ is open in $\overline{G_{\mathfrak{R}}}$ and W be a neighborhood of e in $\overline{G_{\mathfrak{R}}}$. Then there is an open set U of e in $G_{\mathfrak{R}}$ such that $U \subseteq U^n \subseteq W$, for every $n \in \mathbb{N} - \{0\}$.

Theorem 2.10. [12] If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

Definition 2.11.[6] The metric space (X, d) is said to be complete if every cauchy sequence in X converges.

Throughout this paper, we consider X be the universal set. Let $G_{\mathfrak{R}}$ be a simple rough group with identity e and $\overline{G_{\mathfrak{R}}}$ be the upper rough approximation of $G_{\mathfrak{R}}$. Also, the corresponding topologies are denoted by $\overline{\tau}$ for $\overline{G_{\mathfrak{R}}}$ and τ for $G_{\mathfrak{R}}$ induced from $\overline{\tau}$.

III. INVARIANT INTEGRAL

Definition 3.1. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $x \in \overline{G_{\mathfrak{R}}}$. Then the real valued function f on $\overline{G_{\mathfrak{R}}}$ is *continuous at* a if there exists an identity open neighborhood U of $\overline{G_{\mathfrak{R}}}$ such that $|f(x) - f(a)| < \varepsilon$, for every $\varepsilon > 0$ and $xa^{-1} \in U$.

Definition 3.2. Let $G_{\mathfrak{R}}$ be a topological simple rough group and f be a real valued function on $\overline{G_{\mathfrak{R}}}$. The function f is called a *uniformly continuous* if for every $\varepsilon > 0$, there exists an identity open neighborhood U of $\overline{G_{\mathfrak{R}}}$ such that $|f(x) - f(y)| < \varepsilon$ and $xy^{-1} \in U$, for $x, y \in \overline{G_{\mathfrak{R}}}$.

Let $C(\overline{G_{\mathfrak{R}}})$ be the set of all continuous real valued functions on a compact topological group $\overline{G_{\mathfrak{R}}}$ and $f \in C(\overline{G_{\mathfrak{R}}})$. Then the functions f_a and $_af$ are defined as $f_a(x) = f(xa)$ and $_af(x) = f(ax)$, for all $x \in \overline{G_{\mathfrak{R}}}$.

Definition 3.3. Let $\{f_i\}_{i\in I}$ be the set of all real valued functions on $\overline{G_{\mathfrak{R}}}$. This family is called uniformly bounded if there exists a positive real number β such that $|f_i(x)| \leq \beta$, for every $x \in \overline{G_{\mathfrak{R}}}$ and $i \in I$.

Proposition 3.4. Let $G_{\mathfrak{R}}$ be a topological simple rough group and the real valued function f be uniformly continuous on $\overline{G_{\mathfrak{R}}}$. Then $\{f_a\}_{a\in\overline{G_{\mathfrak{R}}}}$ is uniformly continuous.

Proof: Given that the function f is uniformly continuous. So, there exists an identity open neighborhood U of $\overline{G_{\mathfrak{R}}}$ such that $|f(a) - f(x)| < \varepsilon$ and $ax^{-1} \in U$, for $a, x \in \overline{G_{\mathfrak{R}}}$ and $\varepsilon > 0$. Let $z \in \overline{G_{\mathfrak{R}}}$. Since $ax^{-1} \in U$, $(az)(xz)^{-1} \in U$. Therefore,

$$|f_z(a) - f_z(x)| = |f(az) - f(xz)| < \varepsilon.$$

Hence $\{f_a\}_{a\in\overline{G_{\mathfrak{R}}}}$ is uniformly continuous.

Remark: Let $f \in C(\overline{G_{\mathfrak{R}}})$ be a continuous function on $\overline{G_{\mathfrak{R}}}$. Let S be a finite subset in $\overline{G_{\mathfrak{R}}}$. Then

$$\langle S, f\rangle(x) = \frac{1}{|S|}\sum_{a\in S} f_a(x) = \frac{1}{|S|}\sum_{a\in S} f(xa)$$

and

$$[S, f](x) = \frac{1}{|S|}\sum_{a\in S} {}_af(x) = \frac{1}{|S|}\sum_{a\in S} f(ax)$$

Also, define the family

$$\mathfrak{I}_f = \{\langle S, f\rangle : S \subseteq \overline{G_{\mathfrak{R}}}\} \text{ and } \mathfrak{H}_f = \{[S, f] : S \subseteq \overline{G_{\mathfrak{R}}}\}.$$

Proposition 3.5. If $\overline{G_{\mathfrak{R}}}$ is a compact topological group, then the families $\mathfrak{I}_f$ and $\mathfrak{H}_f$ are uniformly continuous, for every $f \in C(\overline{G_{\mathfrak{R}}})$.

Proof: Since $\overline{G_{\mathfrak{R}}}$ is compact and f is continuous, $\{f_a\}_{a\in\overline{G_{\mathfrak{R}}}}$ is uniformly continuous. Then for every $\varepsilon > 0$, there exists an identity open neighborhood $V \subseteq \overline{G_{\mathfrak{R}}}$ such that

$$|f_a(x) - f_a(y)| < \varepsilon \text{ whenever } xy^{-1} \in V$$

Let S be a finite subset in $\overline{G_{\mathfrak{R}}}$. Therefore,

$$|\langle S, f\rangle(x) - \langle S, f\rangle(y)| = \left|\frac{1}{|S|}\sum_{a\in S} f_a(x) - \frac{1}{|S|}\sum_{a\in S} f_a(y)\right| = \frac{1}{|S|}\sum_{a\in S}|f_a(x) - f_a(y)| < \varepsilon$$

Hence $\mathfrak{I}_f$ is uniformly continuous. Similarly, we can prove $\mathfrak{H}_f$ is also uniformly continuous.

Proposition 3.6. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a compact topological group and S be a finite subset in $\overline{G_{\mathfrak{R}}}$. Then for every $f \in C(\overline{G_{\mathfrak{R}}})$,

(i) $\min f \leq \min\langle S, f\rangle \leq \max\langle S, f\rangle \leq \max f$;
(ii) $\min f \leq \min[S, f] \leq \max[S, f] \leq \max f$

Proof: Since $\overline{G_{\mathfrak{R}}}$ is a compact and $f \in C(\overline{G_{\mathfrak{R}}})$, f has minimum and maximum values on $\overline{G_{\mathfrak{R}}}$. Let $m = \min_{x\in\overline{G_{\mathfrak{R}}}} f(x)$ and $M = \max_{x\in\overline{G_{\mathfrak{R}}}} f(x)$. For each $a \in S$ and $x \in \overline{G_{\mathfrak{R}}}$, we get $f_a(x) = f(ax) \geq m$. Then

$$\langle S, f\rangle(x) = \frac{1}{|S|}\sum_{a\in S} f_a(x) \geq m$$

Similarly, $\langle S, f\rangle(x) \leq M$. Therefore, $m \leq \langle S, f\rangle(x) \leq M$. That is,

$$m \leq \min_{x\in\overline{G_{\mathfrak{R}}}}\langle S, f\rangle(x) \leq \max_{x\in\overline{G_{\mathfrak{R}}}}\langle S, f\rangle(x) \leq M$$

Hence, $\min f \leq \min\langle S, f\rangle \leq \max\langle S, f\rangle \leq \max f$. In the same way, we can prove $\min f \leq \min[S, f] \leq \max[S, f] \leq \max f$.

Remark: Let $f, g \in C(\overline{G_{\mathfrak{R}}})$. Then $C(\overline{G_{\mathfrak{R}}})$ is a metric space with the metric

$$d(f, g) = \max\{|f(x) - g(x)| : x \in \overline{G_{\mathfrak{R}}}\}$$

Proposition 3.7. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group and S be a finite subset of $\overline{G_{\mathfrak{R}}}$. Define the mapping $\Phi_S : C(\overline{G_{\mathfrak{R}}}) \longrightarrow C(\overline{G_{\mathfrak{R}}})$ by $\Phi_S(f) = \langle S, f\rangle$. Then

(i) the mapping Φ_S is linear;
(ii) $d(\langle S, f\rangle, \langle S, f\rangle) \leq d(f, g)$, for any $f, g \in C(\overline{G_{\mathfrak{R}}})$;
(iii) the mapping Φ_S is continuous.

Proof:

(i) Let $f, g \in C(\overline{G_{\mathfrak{R}}})$ and $\lambda \in \mathbb{R}$

$$\langle S, f + g\rangle(x) = \frac{1}{|S|}\sum_{a\in S}(f + g)(xa) = \frac{1}{|S|}\sum_{a\in S} f(xa) + \frac{1}{|S|}\sum_{a\in S} g(xa)$$

$$= \langle S, f\rangle(x) + \langle S, f\rangle(x)$$

$$\langle S,\lambda f\rangle(x)=\frac{1}{|S|}\sum_{a\in S}(\lambda f)(xa)=\frac{\lambda}{|S|}\sum_{a\in S}f(xa)=\lambda\langle S,f\rangle(x)$$

From these equations, we get $\langle S,f+g\rangle=\langle S,f\rangle+\langle S,g\rangle$ and $\langle S,\lambda f\rangle=\lambda\langle S,f\rangle$. Hence, the mapping Φ_S is linear.

(ii) For any $f,g\in C(\overline{G_{\mathfrak{R}}})$,

$$d(\langle S,f\rangle,\langle S,g\rangle)=\max\{|\langle S,f\rangle-\langle S,g\rangle|\}=\max\{|\langle S,f-g\rangle|\}$$
$$\leq \max\ |f-g|\ =d(f,g)$$

Therefore, $d(\langle S,f\rangle,\langle S,g\rangle)\leq d(f,g)$.

(iii) From (ii) and $f,g\in C(\overline{G_{\mathfrak{R}}})$, the mapping Φ_S is continuous.

Proposition 3.8. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group and $f\in C(\overline{G_{\mathfrak{R}}})$. If S,T be two finite subsets of $\overline{G_{\mathfrak{R}}}$, then

(i) $\langle S,\langle T,f\rangle\rangle=\langle ST,f\rangle$
(ii) $[S,[T,f]]=[ST,f]$
(iii) $\langle S,[T,f]\rangle=[S,\langle T,f\rangle]$

Proof:

$$\langle S,\langle T,f\rangle\rangle(x)=\frac{1}{|S|}\sum_{a\in S}\langle T,f\rangle(xa)=\frac{1}{|S|}\sum_{a\in S}\left[\frac{1}{|T|}\sum_{a\in T}f\big((xa)b\big)\right]=\frac{1}{|S|}\frac{1}{|T|}\sum_{a\in S}\sum_{a\in T}f(xab)$$
$$=\langle ST,f\rangle(x)$$

$$[S,[T,f]](x)=\frac{1}{|S|}\sum_{a\in S}[T,f](ax)=\frac{1}{|S|}\sum_{a\in S}\left[\frac{1}{|T|}\sum_{b\in T}f\big(b(ax)\big)\right]=\frac{1}{|S|}\frac{1}{|T|}\sum_{a\in S}\sum_{b\in T}f(bax)$$
$$=[ST,f](x)$$

$$\langle S,[T,f]\rangle(x)=\frac{1}{|S|}\sum_{a\in S}[T,f](xa)=\frac{1}{|S|}\sum_{a\in S}\left[\frac{1}{|T|}\sum_{b\in T}f\big(b(xa)\big)\right]=\frac{1}{|S|}\frac{1}{|T|}\sum_{a\in S}\sum_{b\in T}f\big(b(xa)\big)$$
$$=\frac{1}{|T|}\sum_{b\in T}\left[\frac{1}{|S|}\sum_{a\in S}f\big((bx)a\big)\right]=\frac{1}{|T|}\sum_{b\in T}\langle S,f\rangle(bx)=[S,\langle T,f\rangle](x)$$

Remark: Let $G_{\mathfrak{R}}$ be a compact topological simple rough group. Now, we define the two families Θ_f is the closure of $\mathfrak{I}_f$ and Ξ_f is the closure of $\mathfrak{H}_f$ in the space $(C(\overline{G_{\mathfrak{R}}}),d)$.

Proposition 3.9. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group. Then for every subset S of $\overline{G_{\mathfrak{R}}}$, $\Phi_S(\mathfrak{I}_f)\subset\mathfrak{I}_f$ and $\Phi_S(\Theta_f)\subset\Theta_f$, for every $f\in C(\overline{G_{\mathfrak{R}}})$.

Proof: Let $\langle S, f\rangle \in \mathfrak{I}_f$ and $\Phi_S(f) = \langle S, f\rangle$. Then, for every $f \in C(\overline{G_{\mathfrak{R}}})$, $\Phi_S(\langle S, f\rangle) = \langle S, \langle T, f\rangle\rangle = \langle ST, f\rangle \in \mathfrak{I}_f$. Therefore, $\Phi_S(\mathfrak{I}_f) \subset \mathfrak{I}_f$. Now $\Phi_S(\Theta_f) = \Phi_S(cl(\mathfrak{I}_f)) \subset cl(\Phi_S(\mathfrak{I}_f)) \subset cl(\mathfrak{I}_f) \in \Theta_f$. Hence $\Phi_S(\Theta_f) \subset \Theta_f$.

Proposition 3.10. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group. Then, for every $f \in C(\overline{G_{\mathfrak{R}}})$, the families Θ_f and Ξ_f are uniformly continuous.

Proof: By proposition 3.5, $\mathfrak{I}_f$ is uniformly continuous. Let $\varepsilon > 0$ be arbitrary. Then for every $x, y \in \overline{G_{\mathfrak{R}}}$ there exists an identity open neighborhood $V \subseteq \overline{G_{\mathfrak{R}}}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $xy^{-1} \in V$ and $f \in \mathfrak{I}_f$. Consider $g \in \Theta_f$ and $f \in \mathfrak{I}_f$ such that $d(f, g) < \varepsilon$. Therefore, for every $x, y \in \overline{G_{\mathfrak{R}}}$ with $xy^{-1} \in V$,

$$|g(x) - g(y)| = |g(x) - f(x) + f(x) - f(y) + f(y) - g(y)|$$
$$\leq |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| < 3\varepsilon$$

Hence Θ_f is uniformly continuous. Similarly, we prove Ξ_f is also uniformly continuous.

Theorem 3.11. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group and $f \in C(\overline{G_{\mathfrak{R}}})$. Then the metric space (Θ_f, d) is totally bounded and compact.

Proof: Let $\varepsilon > 0$. By proposition 3.6, for every $g \in \Theta_f$, we get $m(f) \leq m(g) \leq M(g) \leq M(f)$. Consider a finite set $T \subseteq [m(f), M(f)]$ which is an ε-net in $\overline{G_{\mathfrak{R}}}$. From proposition 3.10, Θ_f is uniformly continuous, so, there exists an identity open neighborhood $V \subseteq \overline{G_{\mathfrak{R}}}$ such that $|g(x) - g(y)| < \varepsilon$, for every $x, y \in \overline{G_{\mathfrak{R}}}$ with the condition $xy^{-1} \in V$. Since $G_{\mathfrak{R}}$ is compact, there exists a finite set $S \subseteq \overline{G_{\mathfrak{R}}}$ such that $\overline{G_{\mathfrak{R}}} = VS$. Let Ω be the set of all functions from S to T. Since S and T are finite, Ω is finite. Thus, for every $\mu \in \Omega$, we choose a function $f_\mu \in \Theta_f$ whenever $|f_\mu(a) - \mu(a)| < \varepsilon$, for all $a \in S$. Suppose the function f_μ is not exist, consider f_μ is a zero function on $\overline{G_{\mathfrak{R}}}$. Therefore, $\{f_\mu : \mu \in \Omega\}$ is a subset of Θ_f which is denoted by Γ_Ω. Since $g \in \Theta_f$ and T is an ε-dense in $\overline{G_{\mathfrak{R}}}$, for every $a \in S$ there exists $\mu(a) \in T$ such that $|g(a) - \mu(a)| < \varepsilon$. Let us consider another function $\eta \in \Omega$, for every $a \in S$. Then $f_\eta \in \Gamma_\Omega$. Since $x \in \overline{G_{\mathfrak{R}}} = VS$, there exists an element $a \in S$ such that $x \in Va$ which implies $xa^{-1} \in V$. Now we get,

$$|g(x) - g(y)| < \varepsilon \text{ and } |f_\eta(a) - f_\eta(x)| < \varepsilon, \text{ for every } g, f_\eta \in \Theta_f.$$

Also,

$$|g(a) - f_\eta(a)| \leq |g(a) - \eta(a)| + |\eta(a) - f_\eta(a)| < 2\varepsilon.$$

Therefore,

$$|g(x) - f_\eta(x)| \leq |g(x) - g(a)| + |g(a) - f_\eta(a)| + |f_\eta(a) - f_\eta(x)| < 4\varepsilon.$$

That is, $d(g, f_\eta) < 4\varepsilon$ which implies the finite set Γ_Ω is a 4ε-dense. Therefore, the metric space (Θ_f, d) is totally bounded. From the theorem 2.10 and definition 2.11, the metric space $C(\overline{G_{\mathfrak{R}}})$ is complete. Then using theorem 2.6, we get its closed subspace Θ_f is also complete. Hence, by theorem 2.7, Θ_f is compact.

Proposition 3.12. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group and $f \in C(\overline{G_{\mathfrak{R}}})$. If f is not a constant function, then there exists a finite set $S \subseteq \overline{G_{\mathfrak{R}}}$ such that $M(\langle S, f\rangle) < M(f)$. Similarly, there exists a finite set $T \subseteq \overline{G_{\mathfrak{R}}}$ such that $M([T, f]) < M(f)$.

Proof: Let $M = \max_{x \in \overline{G_{\mathfrak{R}}}} f(x)$. Since f is not a constant function, there exists an element $x_0 \in \overline{G_{\mathfrak{R}}}$ such that $f(x_0) < l < M$. Let U be an open neighborhood of x_0 and $f(x) \leq l$, for all $x \in U$. Since $G_{\mathfrak{R}}$ is compact, there exists a finite set $S \subseteq \overline{G_{\mathfrak{R}}}$ such that $\overline{G_{\mathfrak{R}}} = US$. Let $|S^{-1}| = n$ and consider the function $\langle S^{-1}, f\rangle$. For each $x \in \overline{G_{\mathfrak{R}}}$, there exists an element $a \in S$ and $u \in U$ such that $x = ua$. Then $f_{a^{-1}}(x) = f(uaa^{-1}) = f(u) \leq l$. Therefore,

$$\langle S^{-1}, f\rangle(x) = \frac{1}{|S^{-1}|} \sum_{a^{-1} \in S^{-1}} f_{a^{-1}}(x)$$

Hence, $M(\langle S^{-1}, f\rangle) \leq \frac{1}{n}[M(n-1) + l] = M - \frac{1}{n}[M - l] < M$. In the similar way, we can prove $M([T, f]) < M(f)$.

Remark: From theorem 3.11, Θ_f is compact, so for each $x \in \overline{G_{\mathfrak{R}}}$ there exists an element $c_f \in \Theta_f$ and $d_f \in \Xi_f$ such that

$$M(c_f) = \sup \{M(h) : h \in \Theta_f\}, \qquad m(c_f) = \inf\{M(h) : h \in \Theta_f\}$$

and
$$M(d_f) = \sup \{M(h) : h \in \Xi_f\}, \qquad m(d_f) = \inf \{M(h) : h \in \Xi_f\}$$

Theorem 3.13. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group and $f \in C(\overline{G_{\mathfrak{R}}})$. Then

(i) c_f, d_f are constant functions;
(ii) Θ_f has only one constant function;
(iii) $c_f = d_f$.

Proof:

(i) By proposition 3.9 and 3.12, we get c_f, d_f must be constant functions.

(ii) Let c be a constant function in Θ_f and $\varepsilon > 0$ be arbitrary. Then there exist finite sets $S, T \subseteq \overline{G_{\mathfrak{R}}}$ such that $d(\langle S, f\rangle, c) < \varepsilon$ and $d(\langle T, f\rangle,\ c_f) < \varepsilon$. Then, by proposition 3.8 and c is a constant function,

$$d(\langle S, [T, f]\rangle, c) = d([T, \langle S, f\rangle], [T, c]) \leq d(\langle S, f\rangle, c) < \varepsilon$$

Similarly,

$$d(\langle S, [T, f]\rangle,\ c_f) = d(\langle S, [T, f]\rangle, \langle S,\ c_f\rangle) \leq d([T, f],\ c_f) < \varepsilon$$

From these equations, we get $d(c,\ c_f) < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $c = c_f$.

(iii) In the same way of (ii), we can prove $c = d_f$. Therefore, $c_f = d_f$.

Corollary 3.14. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $f,\ g \in C(\overline{G_{\mathfrak{R}}})$. Let $S \subseteq \overline{G_{\mathfrak{R}}}$ be a finite subset and $\lambda \in \mathbb{R}$. Then

(i) $c_{\lambda f} = \lambda c_f$

(ii) $c_{S\langle f\rangle} = c_f$ and, in general, $c_g = c_f$, for every $g \in \Theta_f$

(iii) $c_{f+g} = c_f + c_g$

(iv) $m(f) \leq c_f(x) \leq M(f)$, for each $x \in \overline{G_{\mathfrak{R}}}$.

Proof:

(i) Since the mapping $\Phi_S : C(\overline{G_{\mathfrak{R}}}) \longrightarrow C(\overline{G_{\mathfrak{R}}})$ is linear, $\lambda c_f \in \Theta_{\lambda f}$ and λc_f is a constant function. By theorem 3.13(ii), $\Theta_{\lambda f}$ has only one constant function, so that constant function must be $c_{\lambda f}$. Hence, $c_{\lambda f} = \lambda c_f$.

(ii) Obviously, $\Theta_{S\langle f\rangle} \subset \Theta_f$ which implies $c_{S\langle f\rangle} \in \Theta_f$ is also a constant function. But $\Theta_{\lambda f}$ has unique constant function. Hence $c_{S\langle f\rangle} = c_f$. In general, for any $g \in \Theta_f$, $\Theta_g \subset \Theta_f$ implies we get $c_g = c_f$.

(iii) Let $\varepsilon > 0$ be arbitrary and $g \in C(\overline{G_{\mathfrak{R}}})$. Then there exists a finite subset $T \subseteq \overline{G_{\mathfrak{R}}}$ such that $|\langle T, g\rangle - c_g| < \varepsilon$. Now consider $h = \langle T, g\rangle$. Then there exists a finite subset $T' \subseteq \overline{G_{\mathfrak{R}}}$ such that $|\langle T', h\rangle - d| < \varepsilon$. But, $\langle T', h\rangle = \langle T', \langle T, g\rangle\rangle = \langle T'T, g\rangle$. Therefore, $|\langle T'T, g\rangle - c_g| < \varepsilon$. By (ii), $c_{\langle S,f\rangle} = c_f$, so there exists a finite subset $S \subseteq \overline{G_{\mathfrak{R}}}$ such that $|\langle S, \langle T, f\rangle\rangle - c_f| < \varepsilon$ which implies $|\langle ST, f\rangle - c_f| < \varepsilon$. Now,

$$|\langle ST, f+g\rangle - (c_f + c_g)| \leq |\langle ST, f\rangle - c_f| + |\langle ST, g\rangle - c_g| < 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $c_{f+g} = c_f + c_g$.

(iv) By proposition 3.6, we get $\min f \leq \min\langle S, f\rangle \leq \langle S, f\rangle \leq \max\langle S, f\rangle \leq \max f$. Since $c_f(x)$ is constant, we have $\min f \leq c_f(x) \leq \max f$.

Definition 3.15. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $f \in C(\overline{G_{\mathfrak{R}}})$. Then $c_f(e)$ is called the rough invariant integral of f and is denoted by $\int f(x)dx$. That is, $c_f(e) = \int f(x)dx$.

Theorem 3.16. Let $G_{\mathfrak{R}}$ be a topological simple rough group such that $\overline{G_{\mathfrak{R}}}$ is a compact topological group and $f, g \in C(\overline{G_{\mathfrak{R}}})$. Then the following conditions are satisfied:

(i) for all $\lambda \in \mathbb{R}$, $\int \lambda f(x)dx = \lambda \int f(x)dx$

(ii) $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

(iii) if $f(x) \geq 0$, for all $x \in \overline{G_{\mathfrak{R}}}$, then $\int f(x)dx \geq 0$

(iv) if $f(x) = 1$, for all $x \in \overline{G_{\mathfrak{R}}}$, then $\int f(x)dx = 1$

(v) $\int f(x)dx = \int f(xa)dx$, for all $a \in \overline{G_{\mathfrak{R}}}$

(vi) $\int f(x)dx = \int f(ax)dx$, for all $a \in \overline{G_{\mathfrak{R}}}$

(vii) $\int f(x^{-1})dx = \int f(x)dx$

(viii) if $f(x) \geq 0$, for all $x \in \overline{G_{\mathfrak{R}}}$, and $f(y) > 0$, for some $y \in \overline{G_{\mathfrak{R}}}$, then $\int f(x)dx > 0$

Proof: From corollary 3.14(i), $c_{\lambda f} = \lambda c_f$ and $c_{f+g} = c_f + c_g$, we get (i) and (ii). Now, consider f is a non-negative function. Therefore, every function in Θ_f is also non-negative. Hence, we proved (iii).

In (iv), given that $f(x) = 1$, for all $x \in \overline{G_{\mathfrak{R}}}$ which implies it is a constant function. Since Θ_f has only one constant function, $c_f = f$, for all $x \in \overline{G_{\mathfrak{R}}}$, By definition 3.15, we get $\int f(x)dx = 1$. By corollary 3.14(ii), $c_{\langle S,f \rangle} = c_f$. Let $S = \{a\}$. Then $c_{\langle \{a\},f \rangle} = c_f$ and, similarly, $c_{[\{a\},f]} = c_f$. Therefore, we get (v) and (vi), $\int f(x)dx = \int f(xa)dx$ and $\int f(x)dx = \int f(ax)dx$.

Define $g: \overline{G_{\mathfrak{R}}} \to \mathbb{R}$ by $g(x) = f(x^{-1})$. Then g is continuous and $\Theta_f = \Xi_f$ which implies $c_g = d_f$. By theorem 3.13, $c_f = d_f$. Therefore, $c_g(e) = c_f(e)$. Hence,

$$\int f(x^{-1})dx = \int g(x)dx = c_g(e) = c_f(e) = \int f(x)dx$$

Let $f \in C(\overline{G_{\mathfrak{R}}})$. Then there exists a function $g \in \Theta_f$ such that $g(x) > 0$, for all $x \in \overline{G_{\mathfrak{R}}}$ and there exists an element $y \in \overline{G_{\mathfrak{R}}}$ such that $f(y) > 0$. Let $U \subseteq \overline{G_{\mathfrak{R}}}$ be an open neighborhood of y and $f(x) > 0$, for all $x \in U$. Now, consider a finite subset $S \subseteq \overline{G_{\mathfrak{R}}}$ such that $US = \overline{G_{\mathfrak{R}}}$. Let $z \in \overline{G_{\mathfrak{R}}}$. Since $US = \overline{G_{\mathfrak{R}}}$, there exists an element $a \in S$ and $u \in U$ such that $z = ua$. Let S^{-1} be a finite subset of $\overline{G_{\mathfrak{R}}}$. Since $f(x) \geq 0$, for all $x \in \overline{G_{\mathfrak{R}}}$, $\langle S, f \rangle(z) > 0$. Then $f(u) > 0$ implies $f(za^{-1}) = f(uaa^{-1}) = f(u) > 0$, that is $f(za^{-1}) > 0$. Therefore, $\langle S^{-1}, f \rangle(z) > 0$. So, $g = \langle S^{-1}, f \rangle \in \Theta_f$. Since $\overline{G_{\mathfrak{R}}}$ is compact, $m(g) > 0$. By corollary 3.14, $c_g = c_f$ implies $0 < m(g) \leq c_g(e)$. Therefore, $\int f(x)dx = c_f(e) = c_g(e) \geq m(g) > 0$. Hence, $\int f(x)dx > 0$.

Theorem 3.17. Let $G_{\mathfrak{R}}$ be a compact topological simple rough group. Suppose the function $\psi: C(\overline{G_{\mathfrak{R}}}) \to \mathbb{R}$ satisfies the conditions (i) – (v) of theorem 3.16, Then $\psi(f) = \int f(x)dx$, for all $f \in C(\overline{G_{\mathfrak{R}}})$.

Proof: From theorem 3.16, (i) and (iii), if $f(x) \leq g(x)$, for all $x \in \overline{G_{\mathfrak{R}}}$, then $\psi(f) \leq \psi(g)$ and $\psi(|f|) \leq |\psi(f)|$. Also, by (i) and (iv) of theorem 3.16, $\psi(c) = c$, where $c \in \mathbb{R}$ is a constant function on $\overline{G_{\mathfrak{R}}}$ with value c. And by the condition (v) of theorem 3.16, $\psi(\langle S, f \rangle) = \psi(f)$, for every $f \in C(\overline{G_{\mathfrak{R}}})$ and S is a finite subset of $\overline{G_{\mathfrak{R}}}$. Now to prove, $\psi(f) = c_f(e)$. Let $\varepsilon > 0$ and $d\big(\langle S, f \rangle,\ c_f\big) < \varepsilon$. Then

$$|\psi(f) - c_f(e)| = |\psi(\langle S, f \rangle) - c_f(e)| = |\psi(\langle S, f \rangle) - \psi(c_f)| = |\psi\big(\langle S, f \rangle - c_f\big)|$$

$$\leq \psi|\langle S, f \rangle - c_f| \leq \psi(\varepsilon) = \varepsilon.$$

Hence $\psi(f) = c_f(e) = \int f(x)dx$.

IV. HAAR MEASURE

Definition 4.1. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. Then for any closed subset S of $\overline{G_{\mathfrak{R}}}$. Then

$$m(S) = \inf\left\{\int f(x)dx \ :\ f \in \Lambda_S\right\}$$

where $\Lambda_S = \{f \in C(\overline{G_{\mathfrak{R}}}) : f \geq 0, f(x) \geq 1, \text{for all } x \in S\}$. Here $m(S)$ lies between $[0,\ 1]$. The number $m(S)$ is called the *Haar measure* of the closed set S of $\overline{G_{\mathfrak{R}}}$.

In general, let A be any subset of $\overline{G_{\mathfrak{R}}}$. Then the *Haar measure* of A is defined by

$$m(A) = \sup\{m(S) : S \subset A \text{ and } S \text{ is closed in } \overline{G_{\mathfrak{R}}}\}$$

Proposition 4.2. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. If S and T are closed subsets of $\overline{G_{\mathfrak{R}}}$ such that $S \subset T$, then $m(S) \leq m(T)$.

Proof: By the definition of Haar measure, $m(S)$ and m(T) are defined by

$$m(S) = \inf\left\{\int f(x)dx \;:\; f \in \Lambda_S\right\}$$

$$m(T) = \inf\left\{\int f(x)dx \;:\; f \in \Lambda_T\right\}$$

Also, for any function $f \in \Lambda_T$, $f(x) \geq 1$, for all $x \in T$. Since $S \subset T$, $f(x) \geq 1$, for all $x \in S$ which implies $f \in \Lambda_S$. Therefore, $\Lambda_S \subset \Lambda_T$. Now take infimum on both sides, we get $m(S) \leq m(T)$.

Proposition 4.3. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. Then the Haar measure, $m(A) = m(xA) = m(Ax)$, that is, the Haar measure is invariant under translations.

Proof: The Haar measure for any subset $A \subseteq \overline{G_{\mathfrak{R}}}$ is

$$m(A) = \sup\{m(S) \;:\; S \subset A \text{ and } S \text{ is closed in } \overline{G_{\mathfrak{R}}}\},$$

where,

$$m(S) = \inf\left\{\int f(x)dx \;:\; f \in \Lambda_S\right\}$$

Let $f_x(y) = f(x^{-1}y)$, for all $x \in \overline{G_{\mathfrak{R}}}$. Since $f(x) \geq 1$, for all $x \in S$, $f_x(y) \geq 1$, for all $y \in xS$. Therefore, $f_x \in \Lambda_{xS}$. By theorem 3.16, $\int f(x)dx = \int f(xa)dx = \int f(ax)dx$, for all $a \in \overline{G_{\mathfrak{R}}}$. This means $\int f(x)dx = \int f_x(y)dx = \int {}_x f(y)dx$. Therefore, $m(F) = m(Fx) = m(xF)$. Hence $m(A) = m(xA) = m(Ax)$.

Proposition 4.4. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. For any closed subsets S and T of $\overline{G_{\mathfrak{R}}}$, we have $m(S \cup T) \leq m(S) + m(T)$.

Proof: Let $\varepsilon > 0$. Then by the definition of Haar measure, there exist functions $f \in \Lambda_S$ and $g \in \Lambda_T$ such that $\int f(x)dx \leq m(S) + \varepsilon$ and $\int f(x)dx \leq m(T) + \varepsilon$. Consider $h = f + g$. Since f and g are continuous and non-negative functions, $h(x) \geq 1$, for all $x \in S \cup T$. Therefore, $h \in \Lambda_{S \cup T}$. By theorem 3.16(ii),

$$\int h(x)dx = \int (f + g)(x)dx = \int f(x)dx + \int g(x)dx$$

Hence,

$$m(S \cup T) \leq \int h(x)dx = \int f(x)dx + \int g(x)dx \leq m(S) + m(T) + 2\varepsilon$$

Since $\varepsilon > 0, m(S \cup T) \leq m(S) + m(T)$.

Proposition 4.5. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. For every closed subset S of $\overline{G_{\mathfrak{R}}}$ and $\varepsilon > 0$, there is an open subset U of $\overline{G_{\mathfrak{R}}}$ such that $S \subset U$ and $m\big(cl(U)\big) \leq m(S) + \varepsilon$.

Proof: Let $f \in \Lambda_S$ such that $m(F) \geq \int f(x)dx - \delta$, $\delta = \varepsilon/2$. Then $f(x) \geq 1$, for all $x \in S$. By the continuity of f, there exists an open neighborhood U of S such that $f(x) \geq 1 - \delta$, for all $x \in cl(U)$. Consider $h(x) = f(x) + \delta$, for all $x \in \overline{G_{\mathfrak{R}}}$. Thus, $h \in \Lambda_{cl(U)}$ and

$$\int h(x)dx = \int f(x)dx + \delta$$

Therefore,

$$m(cl(U)) \leq \int h(x)dx = \int f(x)dx + \delta$$

Since $S \subset cl(U)$,

$$\int f(x)dx - \delta \leq m(S) \leq m\big(cl(U)\big) \leq \int f(x)dx + \delta$$

That is,

$$\int f(x)dx - \delta \leq m\big(cl(U)\big) - m(S) \leq \int f(x)dx + \delta$$

Hence, $m\big(cl(U)\big) - m(S) \leq 2\delta$ implies $m\big(cl(U)\big) \leq m(S) + \varepsilon$.

Proposition 4.6. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. For any disjoint subsets A_1 and A_2 of $\overline{G_{\mathfrak{R}}}$, we have that $m(A_1 \cup A_2) \geq m(A_1) + m(A_2)$.

Proof: Let $\varepsilon > 0$ be arbitrary and $A = A_1 \cup A_2$. Then $m(A) \geq m(A_1) + m(A_2) - 2\varepsilon$. Also, there exist closed sets $S_1 \subset A_1$ and $S_2 \subset A_2$ such that $m(S_1) \geq m(A_1) - \varepsilon$ and $m(S_2) \geq m(A_2) - \varepsilon$. Since A_1 and A_2 are disjoint sets, S_1 and S_2 are disjoint and $S = S_1 \cup S_2$ is closed. So, we enough to show that $m(S) \geq m(S_1) + m(S_2)$. Let δ be any positive integer. Then there is a function $f \in \Lambda_S$ such that $m(S) \geq \int f(x)dx - \delta$. Since $\overline{G_{\mathfrak{R}}}$ is compact, the sets S_1 and S_2 are compact and disjoint. Let h_1, h_2 be two continuous real valued function on $\overline{G_{\mathfrak{R}}}$ satisfying the following three conditions:

For every $i = 1, 2$

(i) $h_i(x) \leq f(x)$, for all $x \in \overline{G_{\mathfrak{R}}}$
(ii) $h_i(x) = 1$, for all $x \in S_i$
(iii) For every $x \in \overline{G_{\mathfrak{R}}}$, either $h_1(x) = 0$ or $h_2(x) = 0$.

Let $h = h_1 + h_2$. Then $h(x) \geq f(x)$, for all $x \in \overline{G_{\mathfrak{R}}}$. Therefore,

$$\int h(x)dx \leq \int f(x)dx$$

So,

$$m(S) \geq \int f(x)dx - \delta \geq \int h(x)dx - \delta$$

But

$$\int h(x)dx = \int h_1(x)dx + \int h_2(x)dx$$

and $\int h_i(x)dx \geq m(S_i), i = 1, 2$. Hence, $h_i \in \Lambda_{S_i}$,

$$m(S) \geq \int h_1(x)dx + \int h_2(x)dx - \delta \geq m(S_1) + m(S_1) - \delta$$

Since $\delta > 0, m(S) \geq m(S_1) + m(S_1)$.

Proposition 4.7. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. For any disjoint closed subsets S_1 and S_2 in $\overline{G_{\mathfrak{R}}}$, we have $m(S_1 \cup S_2) = m(S_1) + m(S_2)$.

Proof: Given that S_1 and S_2 are disjoint closed subsets. Using propositions 4.4 and 4.6,

$$m(S_1 \cup S_2) \leq m(S_1) + m(S_2) \text{ and } m(S_1 \cup S_2) \geq m(S_1) + m(S_2).$$

So, combining these equations, we get, $m(S_1 \cup S_2) = m(S_1) + m(S_2)$.

Proposition 4.8. Let $G_{\mathfrak{R}}$ be a topological simple rough group and $\overline{G_{\mathfrak{R}}}$ be a compact topological group. Let S and T be arbitrary closed subsets of $\overline{G_{\mathfrak{R}}}$ such that $T \subset S$. Then $m(S) = m(S \backslash T) + m(T)$.

Proof: Let $\varepsilon > 0$ be arbitrary. Then there exists an open neighborhood U of T such that $T \subset U \subset cl(U)$ implies $m(P) \leq m(T) + \varepsilon$, where $P = cl(U)$. Let $S_1 = S \backslash P$ be closed. Then $S_1 \subset S \backslash T$. Therefore, $m(S_1) \leq m(S \backslash T)$. But $S = (S \backslash P) \cup (S \cap P) \subset S_1 \cup P$ implies

$$m(S) \leq m(S_1) + m(P) \leq m(S \backslash T) + m(T) + \varepsilon$$

Since $\varepsilon > 0$, $m(S) \leq m(S \backslash T) + m(T)$. By proposition 4.6, $m(S) \geq m(S \backslash T) + m(T)$. Hence $m(S) = m(S \backslash T) + m(T)$.

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