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Research Paper

On An Operator-Valued $T1$ Theory of Densely Kernels for Symmetric Czos

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Abstract
We setup and follow the perfect study of the pioneer authors [63] who provide a natural BMO-criterion for the L_2-boundedness of Calderón-Zygmund operators with operator-valued densely kernels satisfying a symmetric property. Our arguments involve both classical and quantum probability theory. They give a proof of the L_2-boundedness of the commutators $[R_j, b]$ whenever b belongs to the Bourgain's vector-valued BMO space, where R_j is the j-th Riesz transform. A common ingredient is the operator-valued Haar multiplier studied by Blasco and Pott.
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I. Introduction

An improvement of the generalization of the classical Calderón-Zygmund singular integral theories to the operator-valued (or d by d matrix-valued) setting are establish. The situation is quite subtle and many straightforward generalizations are turned out to be under interest. For example, (see [51, 14]) that, for each $0 < \epsilon < \infty$, there exists a scalar-valued Fourier multiplier T that is bounded on $L_{1+\epsilon}(\mathbb{R})$ but $T \otimes id_{S_{1+\epsilon}}$ is not bounded on $L_{1+\epsilon}(\mathbb{R}, S_{1+\epsilon})$. Here, $S_{1+\epsilon}$ denotes the Schatten- $(1+\epsilon)$ classes and $L_{1+\epsilon}(\mathbb{R}, S_{1+\epsilon})$ denotes the space of $S_{1+\epsilon}$-valued $(1+\epsilon)$-integrable functions. Another example is the dyadic paraproduct

$$\pi(b, f_s) = \sum_{n>0} \sum_{s} \quad d_n b E_{n-1} f_s$$

Here, E_n denotes the conditional expectation with respect to the usual dyadic filtration on the real line $\mathbb{R}$ and d_n is the difference $E_n - E_{n-1}$. It is well known that π maps $L_2(\mathbb{R}) \times L_2(\mathbb{R})$ to $L_1(\mathbb{R})$, and this extends to the vector valued setting that π maps $L_2(\mathbb{R}, \ell_2) \times L_2(\mathbb{R}, \ell_2)$ to $L_1(\mathbb{R}, \ell_1)$. However, π fails to map $L_2(\mathbb{R}, S_2) \times L_2(\mathbb{R}, S_2)$ to $L_1(\mathbb{R}, S_1)$, see [41] and [45], [46]. This pathological property of π prevents a desirable operator-valued $T1$-theory with a natural BMO testingcondition.

[63] notice that this kind of pathological property could be rectified for operators T with a dense "symmetric" kernel $K(x, x+\epsilon)$ s.t. $K(x, x+\epsilon) = K(x+\epsilon, x)$, including the Beuling transforms, the Haar multipliers, and the commutator $[R_j, b]$ where R_j is the j-Riesz transform. We formulate a $T1$ theory with a natural BMO test condition for operator valued Calderón-Zygmund operators T satisfying the symmetric property $(T1)^* = T^*1$.

[25], Hytönen and Weiss already established an operatorvalued $T1$ theory in a quite general setting, i.e. for operator valued singular integral operators on vector valued function space $L_{1+\epsilon}(\mathbb{R}, X)$. Their BMO space seems to be quite complicated and does not contain the space of uniformly bounded $\mathcal{B}(\ell_2)$ valued functions in the most interesting case $X = \ell_2$. This is necessary because of the bad behavior of operator valued paraproducts mentioned above. The authors hope that this work may complement Hytönen and Weis' work for the case of symmetric singular integrals. On the other hand, even the commutator $[R_j, b]$ is not a singular integral operator, we are still able to show its L_2-boundedness whenever b satisfies a natural BMO test condition in the samespirit. This result might be essentially known to experts, and we will provide a proof in the Appendix.

We investigate noncommutative $T1$ theorem in the semicommutative case, which would provide ideas or insights in searching for $T1$ type theorem in the more general noncommutative setting such as on quantum Euclidean spaces, where a $T1$-theory is in high demand but still missing (see [60], [61], [13], [58], [39]). The commutative $T1$ theorem due to [10] is a revolutionary result and finds many applications in classical harmonic analysis [7],[9]. Let $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x): x \in \mathbb{R}^n\} \to \mathbb{C}$ be a densely kernel satisfying the standard assumptions:

$$|K(x, x+\epsilon)| \lesssim \frac{1}{|\epsilon|^n}, \forall \epsilon \neq 0; \tag{1.1}$$

$$|K(x, x+\epsilon) - K(x', x+\epsilon)| + |K(x+\epsilon, x) - K(x+\epsilon, x')| \lesssim \frac{|x-x'|^\alpha}{|\epsilon|^{n+\alpha}} \tag{1.2}$$

$\forall |\epsilon| \geq 2|x-x'|$, with some $\alpha \in (0,1]$. Here$A \lesssim B$ means that there exists an absolute constant $C > 0$ such that $A \leq CB$. A linear operator T initially defined on "nice" functions is called a Calderón-Zygmund operator (CZO) associated with K, if T satisfies the densely kernel representation, for a.e.$x \notin \mathrm{supp} f_s$,

$$Tf_s(x) = \int_{\mathbb{R}^n} \sum_s K(x, x+\epsilon) f_s(x+\epsilon) d(x+\epsilon)$$

The $T1$ theorem states that T extends to a bounded operator on $L_{1+\epsilon}(\mathbb{R}^n)$ for one (or equivalently all) $0 < \epsilon < \infty$ if and only if

$$T1, T^*1 \in \mathrm{BMO}(\mathbb{R}^n), \text{ and} \tag{1.3}$$

$$T \text{ has the Weak Boundedness Property } \sup_{I \text{ cube}} \frac{1}{|I|} |\langle 1_I, T1_I \rangle| < \infty. \tag{1.4}$$

Along the current research line of noncommutative harmonic analysis, we devoted to the study of a matrix (operator)-valued $T1$ theorem. We are interested in the matrix-valued densely kernels densely$K: \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x): x \in \mathbb{R}^n\} \to \mathcal{B}(\ell_2)$ verifying natural assumptions:

$$\|K(x, x+\epsilon)\|_{\mathcal{B}(\ell_2)} \lesssim \frac{1}{|\epsilon|^n}, \forall \epsilon \neq 0 \tag{1.5}$$

$$\|K(x, x+\epsilon) - K(x', x+\epsilon)\|_{\mathcal{B}(\ell_2)} + \|K(x+\epsilon, x) - K(x+\epsilon, x')\|_{\mathcal{B}(\ell_2)} \lesssim \frac{|x-x'|^\alpha}{|\epsilon|^{n+\alpha}}, \tag{1.6}$$

$\forall |\epsilon| \geq 2|x-x'|$. We are interested in operators T such that, for all $S_{\mathcal{B}(\ell_2)}$-valued step functions f_s and a.e. $x \notin \mathrm{supp} f_s$,

$$Tf_s(x) = \int_{\mathbb{R}^n} \sum_s K(x, x+\epsilon) f_s(x+\epsilon) d(x+\epsilon) = \int_{\mathbb{R}^n} \sum_{i,j} \left(\sum_k \sum_s K_{ik}(x, x+\epsilon)(f_s)_{kj}(x+\epsilon) \right) \otimes e_{i,j} d(x+\epsilon) \tag{1.7}$$

Here $S_{\mathcal{B}(\ell_2)}$ denotes the set of all the elements with finite trace support in $\mathcal{B}(\ell_2)$. We aim to find a natural BMO condition such as (1.3), (1.4) such that T extends toa bounded operator on the noncommutative $L_{1+\epsilon}$ spaces. Here, the noncommutative $L_{1+\epsilon}$ spaces are associated to the von Neumann algebra

$$\mathcal{A} = L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{B}(\ell_2)$$

which consists of all essentially bounded functions $f_s: \mathbb{R}^n \to \mathcal{B}(\ell_2)$. See[5], [3], [6], [2] for more information on noncommutative $L_{1+\epsilon}$ spaces.

The modern development of quantum probability and noncommutative harmonic analysis begun by [52], where noncommutative Burkholder-Gundy inequality and Fefferman-Stein duality were established. Many inequalities in classical martingale theory have been transferred into the noncommutative setting [26],[35],[36], [55], [56], [20], [15], [16], [34], etc. Meanwhile, noncommutative harmonic analysis has gained rapid developments ranging from the noncommutative H^∞-calculus [27], [11], operator-valued harmonic analysis [42],[21], [17], [48] to Riesz transform/Fourier multipliers on group von Neumann algebras [30],[31],[28], hypercontractivity of quantum Markov semigroups [33],[32],[57], and harmonic analysis on quantum Euclidean spaces/torus [6], [61], [13].

The operator-valued (or semi-commutative) harmonic analysis often provides deep insights in harmonic analysis in the general noncommutative setting, and sometimes plays essential role based on the transference principles. The main ideas of [30], [8], [61], [13] are to reduce the problems in their setting to the corresponding problems in the operator-valued setting.

An interesting case is that the functions f_s are ℓ_2-valued. This case has been extensively studied in [59], [46], [49],[12],[45],etc since 97's. In these works, many results in classical harmonic analysis such as weighted norm inequalities, Carleson embedding theorem, Hankel operators, commutators, paraproducts have been extended to the matrix-valued setting. A common character of all these results is that the behavior depends on the dimension of the underlying matrix. In [45], among many other related results, the authors consider the dyadic paraproduct with symbol in noncommutative BMO acting on $\mathbb{C}^d$-valued functions and show that the bound of the paraproduct operator is of order $O(\log d)$. Since we will not work with noncommutative BMO space $\mathrm{BMO}^{cr}(\mathcal{A})$, see[42] for the definition and properties.

Now, let $\mathcal{D}$ be the collection of dyadic intervals in $\mathbb{R}$. For any dyadic interval $I \in \mathcal{D}$, let $h_I := |I|^{-1/2}(1_{I_+} - 1_{I_-})$be the associated Haar function, where $I_+, I_-$are left and right halves of the interval I. Let b be a $d \times d$-matrix-valued function on $\mathbb{R}$ and f_s be a $\mathbb{C}^d$-valued function on $\mathbb{R}$, the paraproduct is defined as

$$\pi_b(f_s) := \sum_{I\in\mathcal{D}} \sum_s \mathbb{D}_I(b)\mathbb{E}_I(f_s)$$

where $\mathbb{D}_I(b) := \langle h_I, b\rangle h_I = \int_{\mathbb{R}} b(x)h_I(x)dx h_I$ is a $d \times d$-matrix-valued function on $\mathbb{R}$ and $\mathbb{E}_I(f_s) := \left\langle \frac{1_I}{|I|}, f_s \right\rangle 1_I = (f_s)_I f_s(x)dx 1_I$ is a $\mathbb{C}^d$-valued function on $\mathbb{R}$. [45], [41], showed that it may happen

$$\|\pi_b\|_{L_2(\mathbb{R};\mathbb{C}^d)\to L_2(\mathbb{R};\mathbb{C}^d)} \gtrsim \|b\|_{L_\infty(\mathcal{A})} \log d \tag{1.8}$$

This tells us that a naive generalization of classical $T1$ theorem in the semicommutative setting is not true, that is, $T1, T^*1 \in \mathrm{BMO}^{cr}(\mathcal{A})$ can not guarantee the boundedness of matrix-valued CZOs since the paraproduct is a typical example ofperfect dyadic CZOs and $L_\infty(\mathcal{A})$ is contained in $\mathrm{BMO}^{cr}(\mathcal{A})$. A CZO on $\mathbb{R}$ being perfect dyadic means its densely kernel satisfies the condition (instead of (1.6))

$$\|K(x, x+\epsilon) - K(x', x+\epsilon)\|_{\mathcal{B}(\ell_2)} + \|K(x+\epsilon, x) - K(x+\epsilon, x')\|_{\mathcal{B}(\ell_2)} = 0, \tag{1.9}$$

whenever $x, x' \in I$ and $(x+\epsilon) \in J$ for some disjoint dyadic intervals I and J. Perfect dyadic densely kernels were introduced in [1] and include martingale transforns, as well as paraproducts and their adjoints.

[22], [25], have proven an operator valued $T1$-theorem. However, the BMO-space in their work is a bit artificial and it may not contain L_∞-functions, though this is necessary due to the abnormality of matrix-valued paraproducts.

The first result is that under the symmetric assumption $(T1)^* = T^*1$, the perfect dyadic CZOs T are bounded on $L_2(\mathcal{A})$ provided $T1 \in \mathrm{BMO}(\mathcal{A}, \Sigma_{\mathcal{A}})$, the usual dyadic vector-valued BMO spaces which contains $L_\infty(\mathcal{A})$. Here " 1 " means the identity of the algebra $\mathcal{A}$, and the BMO space $\mathrm{BMO}(\mathcal{A}, \Sigma_{\mathcal{A}})$ is the dyadic version of the one first studied by Bourgain[4], whose norm of an operator-valued function g_s on $\mathbb{R}$ is defined as

$$\|g_s\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})} = \sup_{I\in\mathcal{D}} \sum_s \left(\fint_I \|g_s(x) - (g_s)_I\|^2_{\mathcal{B}(\ell_2)} dx \right)^{\frac{1}{2}}$$

On the other hand, providing suitable analogue of (1.4) for $L_{1+\epsilon}(\mathcal{A})$-boundedness of matrix-valued CZOs when $\epsilon \neq 1$ is also subtle, since there are some noncommutative martingale transforms with noncommuting coefficients-another type of examples of perfect dyadic CZOs with $T^*1 = T1 = 0$-failing $L_{1+\epsilon}(\mathcal{A})$-boundedness for $\epsilon \neq 1$, see [48]. That implies that a natural Weak Boundedness Property

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \|\langle 1_I, T1_I\rangle\|_{\mathcal{B}(\ell_2)} < \infty$$

can not guarantee the $L_{1+\epsilon}(\mathcal{A})$-boundedness of matrix-valued CZOs for $\epsilon \neq 1$. We are content with the second best- showing the boundedness between $L_{1+\epsilon}(\mathcal{A})$ and noncommutative Hardy spaces under the natural Weak Boundedness Property.

Assuming the symmetric condition, we build a weakened form of $T1$ theorem first for the toy model-matrix-valued perfect dyadic CZOs.

Theorem 1.1 (see [63]). Let T be an operator-valued perfect dyadic CZO satisfying

$$\text{Symmetric condition} : (T1)^* = T^*1; \tag{1.10}$$

$$\text{BMO condition} : T1 \in \mathrm{BMO}(\mathcal{A}, \Sigma_{\mathcal{A}}); \tag{1.11}$$

$$\text{WBP condition} : \sup_{I\in\mathcal{D}} \frac{1}{|I|} \|\langle 1_I, T1_I\rangle\|_{\mathcal{B}(\ell_2)} < \infty. \tag{1.12}$$

Then T is bounded on $L_2(\mathcal{A})$. Moreover,

- T is bounded from $L_{2+\epsilon}(\mathcal{A})$ to $\mathrm{H}^c_{2+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ whenever $0 < \epsilon < \infty$;
- T is bounded from $\mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ to $L_{1+\epsilon}(\mathcal{A})$ whenever $0 < \epsilon < 1$.

Here $\mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ is the noncommutative martingale Hardy spaces that. A useful observation in the proof is Lemma 2.3, which states that dyadic martingale transforms, dyadic paraproducts or their adjoints are essentially the only perfect dyadic CZOs. Then we are reduced to show the boundedness of noncommutative Haarmulitplier-the sum of paraproduct and its adjoint-in Lemma 2.2 where the symmetry is exploited, and the boundedness of noncommutative martingale transform in Lemma 2.1.

The proof of this toy model is relatively easy but essential for the understanding of our arguments for (higher-dimensional) general CZOs and commutators.

For continuous CZO, that is the general singular integrals satisfying (1.7) with densely kernels verifying the standard size and smooth conditions (1.5) (1.6), we establish a similar result.

Theorem 1.2. Let T be a continuous CZO on $\mathbb{R}^n$ satisfying

$$\text{Symmetric condition} : (T1)^* = T^*1; \tag{1.13}$$

$$\text{BMO condition} : T1 \in \mathrm{BMO}\big(\mathbb{R}^n; \mathcal{B}(\ell_2)\big); \tag{1.14}$$

$$\text{WBP condition} : \sup_{I \text{ cube}} \frac{1}{|I|} \|\langle 1_I, T1_I\rangle\|_{\mathcal{B}(\ell_2)} < \infty. \tag{1.15}$$

Then T is bounded on $L_2(\mathcal{A})$. Moreover,

- T is bounded from $L_{2+\epsilon}(\mathcal{A})$ to $\mathrm{H}^c_{2+\epsilon}\big(\mathbb{R}^n; \mathcal{B}(\ell_2)\big)$ whenever $0 < \epsilon < \infty$;
- T is bounded from $\mathrm{H}^c_{1+\epsilon}\big(\mathbb{R}^n; \mathcal{B}(\ell_2)\big)$ to $L_{1+\epsilon}(\mathcal{A})$ whenever $0 < \epsilon < 1$.

Here, the BMO and Hardy spaces are the continuous version of the dyadic spaces in the toy model case that we will recall in the body of the proof. Decompose $T = T_e + T_o$ as the sum of even and odd parts associated with the densely kernels

$$K_e(x, x+\epsilon) = \frac{K(x, x+\epsilon) + K(x+\epsilon, x)}{2}, K_o(x, x+\epsilon) = \frac{K(x, x+\epsilon) - K(x+\epsilon, x)}{2}.$$

It is easy to see that T_e satisfies our symmetric assumption $(T_e 1)^* = T_e^* 1$. We then reduce the L_2-boundedness of T to $T_e 1 \in \mathrm{BMO}\big(\mathbb{R}^n; \mathcal{B}(\ell_2)\big)$ and the L_2-boundedness of T_o. In particular, together with Remark 1.37 in [25], we get the following corollary (see [63]).

Corollary 1.3. Let T be a continuous CZO on $\mathbb{R}^n$ satisfyingSymmetric condition:

$$K_o \in L_2\big(\mathbb{R}^{2n}; \mathcal{B}(\ell_2)\big) \text{ or } T_o 1, T_o^* 1 \in \mathrm{BMO}\big(\mathbb{R}^n; S_{1+\epsilon}(\ell_2)\big) 0 < \epsilon < \infty; \tag{1.16}$$

$$\text{BMO condition} : T_e 1 \in \mathrm{BMO}\big(\mathbb{R}^n; \mathcal{B}(\ell_2)\big); \tag{1.17}$$

$$\text{WBP condition} : \sup_{I \text{ cube}} \frac{1}{|I|} \|\langle 1_I, T1_I\rangle\|_{\mathcal{B}(\ell_2)} < \infty. \tag{1.18}$$

Then T is bounded on $L_2(\mathcal{A})$.

Theorem 1.1. and Corollary 1.3 hold for general operator-valued functions, e.g. replacing $\mathcal{B}(\ell_2)$ by any semifinite von Neumann algebra $\mathcal{M}$. Our proof will be written in this general framework.

As in classical harmonic analysis [1, from the result in the dyadic settingTheorem 1.1 it is usually not difficult to guess similar result-Theorem 1.2-in thecontinuous setting. In scalar-valued harmonic analysis, we can realize this passage from the dyadic setting to the continuous one by dealing with issues such as rapidly decreasing tails or using the Vitali covering lemma. In the case of vector-valued harmonic analysis, this passage requires deep understanding on the connection between martingale theory and harmonic analysis as done in [4], [5],[6], [18], [19],[62],[38], etc. In noncommutative harmonic analysis, in addition to the idea or the techniques developed in vector-valued theory, new idea, techniques or tools developed in noncommutative analysis are usually needed to realize this passage such as in [42], [21]. The main idea or technique from vector-valued theory we need is the method of random dyadic cubes firstly introduced in [47], later modified in [24, 23].

We will show Theorem 1.1 and 1.2 in Section 2 and 3 respectively. The definitions of BMO spaces and Hardy spaces as well as the method of random dyadic cubes will be properly recalled. In the Appendix, we will show that the commutator $\big[R_j, b\big]$ is L_2-bounded whenever b belongs to Bourgain's vector-valued BMO space BMO $(\mathbb{R}^n; \mathcal{M})$. This result might be essentially known to experts, (see the Appendix).

II. Perfect dyadic CZOs: proof of Theorem 1.1

For$\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Consider the algebra of essentially bounded functions $\mathbb{R} \to \mathcal{M}$ equipped with the n.s.f. trace

$$\varphi(f_s) = \int_{\mathbb{R}} \sum_s \tau(f_s(x)) dx$$

Its weak-operator closure is a von Neumann algebra $\mathcal{A}$. If $0 \le \epsilon \le \infty$, we write $L_{1+\epsilon}(\mathcal{M})$ and $L_{1+\epsilon}(\mathcal{A})$ for the noncommutative $L_{1+\epsilon}$ spaces associated to the pairs $(\mathcal{M}, \tau)$ and $(\mathcal{A}, \varphi)$. The set of all the elements with finite trace support in $\mathcal{M}$ is written as $S_{\mathcal{M}}$. The set of dyadic intervals in $\mathbb{R}$ is denoted by $\mathcal{D}$ and we use $\mathcal{D}_k$ for the k-th generation, formed by intervals I with side length $\ell(I) = 2^{-k}$. We consider the associated filtration $\big(L_\infty(\mathcal{D}_k) \bar{\otimes} \mathcal{M}\big)_{k\in\mathbb{Z}}$ of $\mathcal{A}$, which will be simplified as $\Sigma_{\mathcal{A}} = (\mathcal{A}_k)_{k\in\mathbb{Z}}$. For$\mathbb{E}_k$ and $\mathbb{D}_k$ denote the corresponding conditional expectations and martingale difference operators.

2.1. Two auxiliary results. We first show two auxiliary results with respect to the following two kinds of operators:

(a) Noncommuting martingale transforms

$$M_\xi f_s = \sum_{k\in\mathbb{Z}} \sum_s \xi_{k-1} \mathbb{D}_k(f_s)$$

(b) Haar multipliers with noncommuting symbol

$$\Lambda_b(f_s) = \sum_{k\in\mathbb{Z}} \sum_s \mathbb{D}_k(b)\mathbb{E}_k(f_s)$$

Here $\xi_k \in \mathcal{A}_k$ is an adapted sequence. The symbols ξ and b do not necessarily commute with the function. Our arguments on the operator-valued Λ_b follow the ideas from [2,3] and [42].

For noncommutative martingale Hardy spaces. Let $0 \le \epsilon < \infty$. The column Hardy space $\mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ is defined to be the completion of all finite $L_{1+\epsilon}$-martingales under the norm

$$\|f_s\|_{\mathrm{H}^c_{1+\epsilon}(\mathcal{A},\Sigma_{\mathcal{A}})} := \left\| \left(\sum_{k\in\mathbb{Z}} \sum_s \mathbb{D}_k f_s^* \mathbb{D}_k f_s \right)^{1/2} \right\|_{1+\epsilon}$$

Taking adjoint - so that the * switches from left to right - we find the row-Hardy space norm and the space. The noncommutative Hardy space $\mathrm{H}_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$, defined through column and row spaces differently for $0 \le \epsilon < 1$ and $\epsilon > 0$, was introduced in [52]. And similarly also introduced noncommutative martingale BMO spaces, and show the noncommutative Burkholder-Gundy inequality and Fefferman-Stein duality. According to 44 it has the expected interpolation behavior in the scale of noncommutative $L_{1+\epsilon}$ spaces.

Vector-valued BMO spaces. The $\mathcal{M}$-valued martingale BMO space BMO ($\mathcal{A}, \Sigma_{\mathcal{A}}$) is defined to the set of $\mathcal{M}$-valued locally integrable functions with norm

$$\|f_s\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})} := \sup_{k\in\mathbb{Z}} \sum_s \|\mathbb{E}_k \|f_s - \mathbb{E}_{k-1} f_s\|_{\mathcal{M}}^2 \|_{L_\infty(\mathbb{R})}^{1/2}$$

This space is related to the vector-valued Hardy space $\mathrm{H}^m_1(\mathcal{A}, \Sigma_{\mathcal{A}})$ whose norm is defined as

$$\|f_s\|_{H^m_1(\mathcal{A},\Sigma_{\mathcal{A}})} := \left\| \sup_{k\in\mathbb{Z}} \sum_s \left\| \mathbb{E}_k f_s \right\|_{L_1(\mathcal{M})} \right\|_{L_1(\mathbb{R})}$$

In fact, [4] and Garcia-Cuerva proved independently that $\mathrm{BMO}(\mathcal{A}, \Sigma_{\mathcal{A}})$ embeds continuously into the dual of $\mathrm{H}^m_1(\mathcal{A}, \Sigma_{\mathcal{A}})$. That is

$$\left| \sum_s \varphi(f_s^* g_s) \right| \lesssim \sum_s \|f_s\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})} \|g_s\|_{\mathrm{H}^m_1(\mathcal{A},\Sigma_{\mathcal{A}})} \tag{2.1}$$

We also need the following Doob's inequality for $L_{1+\epsilon}(\mathcal{M})$-valued function: For all $0 < \epsilon \le \infty$ and $f_s \in L_{1+\epsilon}(\mathcal{A})$

$$\left\| \sup_{k\in\mathbb{Z}} \left\| \sum_s \mathbb{E}_k f_s \right\|_{L_{1+\epsilon}(\mathcal{M})} \right\|_{L_{1+\epsilon}(\mathbb{R})} \lesssim \sum_s \|f_s\|_{L_{1+\epsilon}(\mathcal{A})} \tag{2.2}$$

Proposition 2.1 (see [63]). If $\sup_k \|\xi_k\|_{\mathcal{A}} < \infty$, then

- M_ξ is bounded from $L_{1+\epsilon}(\mathcal{A})$ to $\mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ whenever $0 \le \epsilon < \infty$;
- M_ξ is bounded from $\mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ to $L_{1+\epsilon}(\mathcal{A})$ whenever $0 < \epsilon \le 1$.

Proof. We only give the proof of the case $0 \le \epsilon < \infty$, another case can be shown similarly. Let $f_s \in L_{1+\epsilon}(\mathcal{A})$. Using the fact $(a^s)^* c a^s \le (a^s)^* a^s \|c\|_\infty$ for any $a^s \in \mathcal{A}$ and $c \in \mathcal{A}^+$ and $\xi_k \in \mathcal{A}_k$, it is easy to check

$$\left\| \sum_s M_\xi(f_s) \right\|_{\mathrm{H}^c_{1+\epsilon}(\mathcal{A},\Sigma_{\mathcal{A}})} = \left\| \sum_s \left(\sum_{k\in\mathbb{Z}} \mathbb{D}_k f_s^* \xi_{k-1}^* \xi_{k-1} \mathbb{D}_k f_s \right)^{1/2} \right\|_{1+\epsilon}$$

$$\le \sup_k \sum_s \|\xi_k\|_{L_\infty(\mathcal{A})} \left\| \left(\sum_{k\in\mathbb{Z}} \mathbb{D}_k f_s^* \mathbb{D}_k f_s \right)^{1/2} \right\|_{1+\epsilon} \lesssim \sum_s \|f_s\|_{L_{1+\epsilon}(\mathcal{A})}$$

We have used the Hölder inequality and Burkholder-Gundy inequality in the inequalities.

Proposition 2.2 (see [63]). If $b \in \mathrm{BMO}(\mathcal{A}, \Sigma_{\mathcal{A}})$, then we have

- Λ_b is bounded from $L_{1+\epsilon}(\mathcal{A})$ to $\mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ whenever $0 \le \epsilon < \infty$;
- Λ_b is bounded from $\mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$ to $L_{1+\epsilon}(\mathcal{A})$ whenever $0 < \epsilon \le 1$.

Proof. We only provide the proof of the case $0 \le \epsilon < \infty$, since another case can be shown similarly. Let $(1+\epsilon)$ be the conjugate index of $(1+\epsilon)$. Let $f_s \in L_{1+\epsilon}(\mathcal{A})$, and $g_s \in \mathrm{H}^c_{1+\epsilon}(\mathcal{A}, \Sigma_{\mathcal{A}})$. By duality, it suffices to show

$$\left| \sum_s \varphi(\Lambda_b(f_s) g_s^*) \right| \lesssim \sum_s \|f_s\|_{L_{1+\epsilon}(\mathcal{A})} \|g_s\|_{\mathrm{H}^c_{1+\epsilon}(\mathcal{A},\Sigma_{\mathcal{A}})}$$

Using the assumption that $\mathbb{D}_k(b)\mathbb{D}_k(f_s) \in \mathcal{A}_{k-1}$ for each k, we have

$$\left|\sum_{s} \varphi(\Lambda_b(f_s)g_s^*)\right| = \left|\sum_{k\in\mathbb{Z}}\sum_{s} \varphi(\mathbb{D}_k(b)\mathbb{E}_k(f_s)g_s^*)\right|$$
$$= \left|\sum_{k\in\mathbb{Z}}\sum_{s} \varphi(\mathbb{D}_k(b)\mathbb{E}_{k-1}(f_s)g_s^* + \mathbb{D}_k(b)\mathbb{D}_k(f_s)g_s^*)\right|$$
$$= \left|\sum_{k\in\mathbb{Z}}\sum_{s} \varphi\big(\mathbb{D}_k(b)\mathbb{E}_{k-1}(f_s)\mathbb{D}_k(g_s^*) + \mathbb{D}_k(b)\mathbb{D}_k(f_s)\mathbb{E}_{k-1}(g_s^*)\big)\right|$$
$$= \left|\varphi\left(b\sum_{k\in\mathbb{Z}}\sum_{s} \big(\mathbb{E}_{k-1}(f_s)\mathbb{D}_k(g_s^*) + \mathbb{D}_k(f_s)\mathbb{E}_{k-1}(g_s^*)\big)\right)\right|$$

Hence by duality between vector-valued BMO space and Hardy space, we have

$$\left|\sum_{s} \varphi(\Lambda_b(f_s)g_s^*)\right|$$
$$\lesssim \sum_{s} \|b\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})}\left\|\sum_{k\in\mathbb{Z}} \mathbb{E}_{k-1}(f_s)\mathbb{D}_k(g_s^*) + \sum_{k\in\mathbb{Z}} \mathbb{D}_k(f_s)\mathbb{E}_{k-1}(g_s^*)\right\|_{\mathrm{H}_1^m(\mathcal{A},\Sigma_{\mathcal{A}})}$$
$$= \|b\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})}\int_{\mathbb{R}}\sum_{s} \sup_{\ell\in\mathbb{Z}}\left\|\sum_{k=-\infty}^{\ell} \mathbb{E}_{k-1}(f_s)\mathbb{D}_k(g_s^*) + \sum_{k=-\infty}^{\ell} \mathbb{D}_k(f_s)\mathbb{E}_{k-1}(g_s^*)\right\|_{L_1(\mathcal{M})} dx.$$

Using the identity for each $\ell \in \mathbb{Z}$

$$\sum_{k=-\infty}^{\ell}\sum_{s} \mathbb{E}_{k-1}(f_s)\mathbb{D}_k(g_s^*) + \sum_{k=-\infty}^{\ell}\sum_{s} \mathbb{D}_k(f_s)\mathbb{E}_{k-1}(g_s^*)$$
$$= \sum_{s} \mathbb{E}_\ell(f_s)\mathbb{E}_\ell(g_s^*) - \sum_{k=-\infty}^{\ell}\sum_{s} \mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)$$

we are reduced to show

$$\int_{\mathbb{R}}\sum_{s} \sup_{\ell\in\mathbb{Z}}\|\mathbb{E}_\ell(f_s)\mathbb{E}_\ell(g_s^*)\|_{L_1(\mathcal{M})}dx \lesssim \sum_{s} \|f_s\|_{L_{1+\epsilon}(\mathcal{A})}\|g_s\|_{H^c_{1+\epsilon}(\mathcal{A},\Sigma_{\mathcal{A}})} \tag{2.3}$$

and

$$\int_{\mathbb{R}}\sup_{\ell\in\mathbb{Z}}\left\|\sum_{k=-\infty}^{\ell}\sum_{s} \mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\right\|_{L_1(\mathcal{M})} dx \lesssim \sum_{s} \|f_s\|_{L_{1+\epsilon}(\mathcal{A})}\|g_s\|_{\mathrm{H}^c_{1+\epsilon}(\mathcal{A},\Sigma_{\mathcal{A}})} \tag{2.4}$$

The first estimate is relatively easy to handle. Using twice the Hölder inequalities and vector-valued Doob's inequality (2.2), the left hand side of (2.3) is controlled by

$$\le \sum_{s} \left(\int_{\mathbb{R}}\sup_{\ell\in\mathbb{Z}}\|\mathbb{E}_\ell(f_s)\|^{1+\epsilon}_{L_{1+\epsilon}(\mathcal{M})}dx\right)^{1/1+\epsilon}\left(\int_{\mathbb{R}}\sup_{\ell\in\mathbb{Z}}\|\mathbb{E}_\ell(g_s)\|^{1+\epsilon}_{L_{1+\epsilon}(\mathcal{M})}dx\right)^{1/1+\epsilon}$$
$$\lesssim \sum_{s} \|f_s\|_{L_{1+\epsilon}(\mathcal{A})}\|g_s\|_{L_{1+\epsilon}(\mathcal{A})} \lesssim \sum_{s} \|f_s\|_{L_{1+\epsilon}(\mathcal{A})}\|g_s\|_{\mathrm{H}^c_{1+\epsilon}(\mathcal{A},\Sigma_{\mathcal{A}})}$$

where we used noncommutative Burkholder-Gundy inequality for $\epsilon \le 1$ in the last inequality.
To show the second estimate (2.4), we only need to show for any ℓ

$$\left\|\sum_{k=-\infty}^{\ell}\sum_{s} \mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\right\|_{L_1(\mathcal{M})} \le \sum_{s} \left\|\left(\sum_{k\in\mathbb{Z}} |\mathbb{D}_k(f_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})}\left\|\left(\sum_{k\in\mathbb{Z}} |\mathbb{D}_k(g_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})},$$

since then we can follow similar arguments as in the (2.3). By duality and the Hölder inequality,

$$\left\|\sum_{k=-\infty}^{\ell}\sum_{s}\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\right\|_{L_1(\mathcal{M})}$$
$$=\sup_{u,\|u\|_{\mathcal{M}}\leq 1}\left|\tau\left(u\sum_{k=-\infty}^{\ell}\sum_{s}\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\right)\right|=\sup_{u,\|u\|_{\mathcal{M}}\leq 1}\left|\tau\left(\sum_{k=-\infty}^{\ell}\sum_{s}\mathbb{D}_k(g_s^*)(u\mathbb{D}_k(f_s))\right)\right|$$
$$=\sup_{u,\|u\|_{\mathcal{M}}\leq 1}\sum_{s}\left|\tau\otimes\mathrm{tr}\left(\left(\sum_{k=-\infty}^{\ell}\mathbb{D}_k(g_s^*)\otimes e_{1k}\right)\left(\sum_{k=-\infty}^{\ell}u\mathbb{D}_k(f_s)\otimes e_{k1}\right)\right)\right| \qquad \square$$
$$\leq\sup_{u,\|u\|_{\mathcal{M}}\leq 1}\sum_{s}\left\|\sum_{k=-\infty}^{\ell}\mathbb{D}_k(g_s^*)\otimes e_{1k}\right\|_{L_{1+\epsilon}\left(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\right)}\left\|\sum_{k=-\infty}^{\ell}u\mathbb{D}_k(f_s)\otimes e_{k1}\right\|_{L_{1+\epsilon}\left(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\right)}$$
$$\leq\sum_{s}\left\|\left(\sum_{k\in\mathbb{Z}}|\mathbb{D}_k(f_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})}\left\|\left(\sum_{k\in\mathbb{Z}}|\mathbb{D}_k(g_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})}.$$

2.2. Representation of perfect dyadic CZOs. The notion of perfect dyadic CZO was defined in [1]. Classical perfect dyadic CZOs include Haar multipliers/martingale transforms and dyadic paraproducts or their adjoints. They also show that these operators and their combinations are the only perfect dyadic CZOs. That is, any operator-valued perfect dyadic CZO is a sum of one noncommutative dyadic martingale transform, one noncommutative dyadic paraproduct and its adjoint.
Nowif $f_s\colon\mathbb{R}\to\mathcal{M}$ is integrable on $I\in\mathcal{D}$, we set the average

$$(f_s)_I=⨍_I\sum_{s}f_s(x)dx$$

If $0\leq\epsilon\leq\infty$ and $f_s\in L_{1+\epsilon}(\mathcal{A})$

$$\mathbb{E}_k(f_s):=\sum_{I\in\mathcal{D}_k}\sum_{s}\mathbb{E}_I(f_s):=\sum_{I\in\mathcal{D}_k}\sum_{s}(f_s)_I1_I,\mathbb{D}_k(f_s):=\sum_{I\in\mathcal{D}_{k-1}}\sum_{s}\mathbb{D}_I(f_s):=\sum_{I\in\mathcal{D}_{k-1}}\sum_{s}\langle h_I,f_s\rangle h_I,$$

where $h_I:=|I|^{-1/2}\left(1_{I_+}-1_{I_-}\right)$is the Haar function and $\langle\cdot,\cdot\rangle$ denotes the operator-valued inner product anti-linear in first coordinate. We will use $\langle\langle\cdot,\cdot\rangle\rangle$ to denote the inner product in $L_2(\mathcal{A})$ anti-linear in first coordinate.
Lemma 2.3 (see [63]). Let T be an operator-valued perfect dyadic CZO. Then for $f_s,g_s\in\mathcal{S}(\mathbb{R})\otimes S_{\mathcal{M}}$,

$$\langle\langle g_s,T(f_s)\rangle\rangle=\sum_{s}\left\langle\left\langle g_s,\sum_{I\in\mathcal{D}}\langle h_I,T(h_I)\rangle\langle h_I,f_s\rangle h_I\right\rangle\right\rangle$$
$$+\sum_{s}\left\langle\left\langle g_s,\sum_{I\in\mathcal{D}}\mathbb{D}_I((T^*1)^*)\mathbb{D}_I(f_s)\right\rangle\right\rangle+\sum_{s}\left\langle\left\langle g_s,\sum_{I\in\mathcal{D}}\mathbb{D}_I(T1)\mathbb{E}_I(f_s)\right\rangle\right\rangle \tag{2.5}$$

This representation (2.5) has been essentially verified in [1] using the language of wave package. Here, we give a proof using an alternate approach due to [6], which motivates us to deduce a similar representation formula for general matrix-valued Calderón-Zygmund operators.
Proof. Without loss of generality, we can assume that both f_s and g_s are of the form $h\otimes m$ with h being scalar-valued function and m being an operator. Then the convergence of $\mathbb{E}_k(h)$ to h as $k\to\infty$ and to 0 as $k\to-\infty$ (both a.e. and in $L_{1+\epsilon}(\mathbb{R})$) leads to Figiel's representation of T as the telescopic series

$$\langle\langle g_s,Tf_s\rangle\rangle=\sum_{k=-\infty}^{\infty}\sum_{s}\left(\langle\langle\mathbb{E}_kg_s,T\mathbb{E}_kf_s\rangle\rangle-\langle\langle\mathbb{E}_{k-1}g_s,T\mathbb{E}_{k-1}f_s\rangle\rangle\right)$$
$$=\sum_{k=-\infty}^{\infty}\sum_{s}\left(\langle\langle\mathbb{D}_kg_s,T\mathbb{D}_kf_s,\rangle\rangle+\sum_{s}\langle\langle\mathbb{E}_{k-1}g_s,T\mathbb{D}_kf_s\rangle\rangle+\sum_{s}\langle\langle\mathbb{D}_kg_s,T\mathbb{E}_{k-1}f_s\rangle\rangle\right)$$
$$:=A+B+C,$$

where, upon expanding in terms of the Haar functions,

$$A=\sum_{m\in\mathbb{Z}}\sum_{I\in\mathcal{D}}\sum_{s}\quad \langle\langle g_s,\langle h_{I\dot{+}m},Th_I\rangle\langle h_I,f_s\rangle h_{I\dot{+}m}\rangle\rangle,$$

$$B=\sum_{m\in\mathbb{Z}}\sum_{I\in\mathcal{D}}\sum_{s}\quad \left\langle\left\langle g_s,\left\langle\frac{1_{I\dot{+}m}}{|I\dot{+}m|},Th_I\right\rangle\langle h_I,f_s\rangle\frac{1_{I\dot{+}m}}{|I_{I\dot{+}m}|}\right\rangle\right\rangle,$$

and

$$C=\sum_{m\in\mathbb{Z}}\sum_{I\in\mathcal{D}}\sum_{s}\quad \left\langle\left\langle g_s,\left\langle h_I,T\frac{1_{I\dot{+}m}}{|I_{I\dot{+}m}|}\right\rangle(f_s)_{I\dot{+}m}h_I\right\rangle\right\rangle.$$

Here $I\dot{+}m:=I+\ell(I)m$ is the translation of a dyadic interval I by $m\in\mathbb{Z}$ times its sidelength $\ell(I)$. Now by the perfect property of the densely kernel (1.9)- $T1_J$ is supported in J for any dyadic interval J, we see that only the term $m=0$ in the summation contributes. Then observing that $|h_I|^2=1_I/|I|$, we see clearly

$$B=\sum_{s}\quad \left\langle\left\langle g_s,\sum_{I\in\mathcal{D}}\mathbb{D}_I((T^*1)^*)\mathbb{D}_I(f_s)\right\rangle\right\rangle$$

and

$$C=\sum_{s}\quad \left\langle\left\langle g_s,\sum_{I\in\mathcal{D}}\mathbb{D}_I(T1)\mathbb{E}_I(f_s)\right\rangle\right\rangle$$

finishing the proof.

2.3. Proof of Theorem 1.1 (see [63]). From the representation (2.5), using the symmetric condition (1.10), we clearly have $T(f_s)=M_\xi(f_s)+\Lambda_b(f_s)$ with

$$\xi_k=\sum_{I\in\mathcal{D}_k}\langle h_I,T(h_I)\rangle 1_I,\ b=T1$$

Then observing that $\|\cdot\|_{H_2^c(\mathcal{A},\Sigma_{\mathcal{A}})}=\|\cdot\|_{L_2(\mathcal{A})}$, we finish the proof using Proposition (2.1) and (2.2) since WBP condition (1.12) ensures $\sup_k\|\xi_k\|_{\mathcal{A}}<\infty$, while BMO condition (1.11) ensures $b\in \mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})$.

III. General CZOs: proof of Theorem 1.2 (see [63])

As in the proof of classical $T1$ theorem, the most difficult part of Theorem 1.2 is the case $\epsilon=1$, and other cases will follow by standard arguments. We will summarize the proof at the end. For the case $\epsilon=1$, as mentioned, we will use the method of random dyadic cubes first introduced in [47], later modified in [23]. **3.1. Radom dyadic system [63].** Let $\mathcal{D}^0:=\bigcup_{j\in\mathbb{Z}}\mathcal{D}_j^0, \mathcal{D}_j^0:=\{2^{-j}([0,1)^n+m):m\in\mathbb{Z}^n\}$ be the standard system of dyadic cubes-the one in the previous section when $n=1$. For every $\beta=(\beta_j)_{j\in\mathbb{Z}}\in(\{0,1\}^n)^{\mathbb{Z}}$, consider the dyadic system $\mathcal{D}^\beta=\{I+\beta: I\in\mathcal{D}^0\}$ where $I+\beta:=I+\sum_{i:2^{-i}<\ell(I)}2^{-i}\beta_i$.

The product probability $\mathbb{P}_\beta$ on $(\{0,1\}^n)^{\mathbb{Z}}$ induces a probability on the family of all dyadic systems $\mathcal{D}^\beta$. Consider for a moment a fixed dyadic system $\mathcal{D}=\mathcal{D}^\beta$ for some β. A cube $I\in\mathcal{D}$ is called 'bad' (with parameters $r\in\mathbb{Z}_+$and $\gamma\in(0,1)$) if there holds

$$\mathrm{dist}(I,J^c)\le\ell(I)^\gamma\ell(J)^{1-\gamma}\text{ for some }J=I^{(k)},k\ge r,$$

where $I^{(k)}$ denotes the k-th dyadic ancestor of I. Otherwise, I is said to be 'good'. Fixing a $I\in\mathcal{D}^0$, consider the random event that its shift $I+\beta$ is bad in $\mathcal{D}^\beta$. Because of the symmetry it is obvious that the probability $\mathbb{P}_\beta(I+\beta\text{ is bad })$ is independent of the cube I, and we denote it by π_{bad} ; similarly one defines $\pi_{\text{good}}=1-\pi_{\text{bad}}$. The only thing that is needed about this number in the present as in [47],[24] is that $\pi_{\text{bad}}<1$, and hence $\pi_{\text{good}}>0$, as soon as r is chosen sufficiently large. We henceforth consider the parameters γ and r being fixed in such a way.

Note that

$$\pi_{\text{good}}=\mathbb{P}_\beta(I+\beta\text{ is good })=\mathbb{E}_\beta 1_{\text{good}}(I+\beta)$$

which is independent of the particular cube I. Then as in [23], using the fact that the event that $I+\beta$ is good is independent of the position of the cube $I+\beta$, hence of the function $\phi(I+\beta)$, for $\phi(I)$ defined on all the cubes, we have

$$\pi_{\text{good}}\,\mathbb{E}_\beta \sum_{I\in\mathcal{D}^\beta} \phi(I) = \sum_{I\in\mathcal{D}^0} \mathbb{E}_\beta 1_{\text{good}}(I+\beta)\mathbb{E}_\beta\phi(I+\beta)$$
$$= \sum_{I\in\mathcal{D}^0} \mathbb{E}_\beta\big(1_{\text{good}}(I+\beta)\phi(I+\beta)\big) = \mathbb{E}_\beta \sum_{I\in\mathcal{D}^\beta_{\text{good}}} \phi(I) \tag{3.1}$$

This identity is the only thing from the probabilistic approach that we will use.

3.2. Representation of general CZOs (see [63]). Fix a $\beta \in (\{0,1\}^n)^{\mathbb{Z}}$. For $\mathcal{D} = \mathcal{D}^\beta$, let $\mathbb{E}_k$ be the associated conditional expectation with respect to $\mathcal{D}_k$, and $\mathbb{D}_k := \mathbb{E}_k - \mathbb{E}_{k-1}$. These operators can be represented by the Haar functions $h_I^\theta, \theta \in \{0,1\}^n$, which is defined as follows: When $n = 1$,

$$h_I^0 := |I|^{\frac{1}{2}} 1_I,\, h_I^1 := |I|^{\frac{1}{2}}\big(1_{I_+} - 1_-\big)$$

When $n \geq 2$,

$$h_I^\theta(x) := h^{(\theta_1,\cdots,\theta_n)}_{I_1\times\cdots\times I_n}(x_1,\cdots,x_n) = \prod_{i=1}^n h_{I_i}^{\theta_i}(x_i)$$

Then

$$\mathbb{E}_k(f_s) = \sum_{I\in\mathcal{D}_k}\sum_s h_I^0\langle h_I^0, f_s\rangle, \mathbb{D}_k(f_s) = \sum_{I\in\mathcal{D}_{k-1}}\sum_{\theta\in\{0,1\}^n\backslash\{0\}}\sum_s h_I^\theta\langle h_I^\theta, f_s\rangle.$$

The translation of a dyadic cube I by $m \in \mathbb{Z}^n$ times its sidelength $\ell(I)$, is defined similarly as $I\dot{+}m := I + m\ell(I)$.

As in Lemma 2.3, we also have Figiel's representation of an operator-valued Calderón-Zygmund operator. Let $f_s, g_s \in \mathcal{S}(\mathbb{R}^n) \otimes S_{\mathcal{M}}$.

$$\langle\langle g_s, Tf_s\rangle\rangle = \sum_{k=-\infty}^{\infty}\sum_s \left(\langle\langle \mathbb{D}_k g_s, T\mathbb{D}_k f_s\rangle\rangle + \sum_s \langle\langle \mathbb{E}_{k-1} g_s, T\mathbb{D}_k f_s\rangle\rangle + \sum_s \langle\langle \mathbb{D}_k g_s, T\mathbb{E}_{k-1} f_s\rangle\rangle\right)$$
$$=: A + B + C \tag{3.2}$$

where

$$A = \sum_{\eta,\theta\in\{0,1\}^n\backslash\{0\}}\sum_{m\in\mathbb{Z}^n}\sum_{I\in\mathcal{D}}\sum_s \left\langle\left\langle g_s, \langle h^\eta_{I\dot{+}m}, Th_I^\theta\rangle\langle h_I^\theta, f_s\rangle h^\eta_{I\dot{+}m}\right\rangle\right\rangle;$$

$$B = \sum_{\theta\in\{0,1\}^n\backslash\{0\}}\sum_{m\in\mathbb{Z}^n}\sum_{I\in\mathcal{D}}\sum_s \left\langle\left\langle g_s, \langle h^0_{I\dot{+}m}, Th_I^\theta\rangle\langle h_I^\theta, f_s\rangle h^0_{I\dot{+}m}\right\rangle\right\rangle$$
$$= \sum_{\theta\in\{0,1\}^n\backslash\{0\}}\sum_{m\in\mathbb{Z}^n}\sum_{I\in\mathcal{D}}\sum_s \left\langle\left\langle g_s, \langle h^0_{I\dot{+}m}, Th_I^\theta\rangle\langle h_I^\theta, f_s\rangle\big(h^0_{I\dot{+}m} - h_I^0\big)\right\rangle\right\rangle$$
$$+ \sum_{\theta\in\{0,1\}^n\backslash\{0\}}\sum_{I\in\mathcal{D}}\sum_s \left\langle\left\langle g_s, \langle h_I^\theta, (T^*1)^*\rangle\langle h_I^\theta, f_s\rangle 1_I/|I|\right\rangle\right\rangle =: B^0 + P;$$

and

$$C = \sum_{\theta\in\{0,1\}^n\backslash\{0\}}\sum_{m\in\mathbb{Z}^n}\sum_{I\in\mathcal{D}}\sum_s \left\langle\left\langle g_s, \langle h_I^\theta, Th^0_{I\dot{m}m}\rangle\langle h^0_{I\dot{m}m}, f_s\rangle h_I^\theta\right\rangle\right\rangle$$
$$= \sum_{\theta\in\{0,1\}^n\backslash\{0\}}\sum_{m\in\mathbb{Z}^n}\sum_{I\in\mathcal{D}}\sum_s \left\langle\left\langle g_s, \langle h_I^\theta, Th^0_{I\dot{m}}\rangle\langle h^0_{I\dot{m}} - h_I^0, f_s\rangle h_I^\theta\right\rangle\right\rangle$$
$$+ \sum_{\theta\in\{0,1\}^n\backslash\{0\}}\sum_{I\in\mathcal{D}}\sum_s \left\langle\left\langle g_s, \langle h_I^\theta, T1\rangle\langle 1_I/|I|, f_s\rangle h_I^\theta\right\rangle\right\rangle =: C^0 + Q$$

Taking integral $\mathbb{E}_\beta$ on both sides of identity (3.2), and then using the identity (3.1), we get

$$\langle\langle g_s, Tf_s\rangle\rangle = \frac{1}{\pi_{\text{good}}}\mathbb{E}_\beta\big(A_{\text{good}} + B^0_{\text{good}} + C^0_{\text{good}}\big) + \mathbb{E}_\beta(P + Q),$$

where for instance

$$A_{\text{good}} = A^\beta_{\text{good}} = \sum_{\eta,\theta\in\{0,1\}^n\backslash\{0\}}\sum_{m\in\mathbb{Z}^n}\sum_{I\in\mathcal{D}^\beta_{\text{good}}}\sum_s \left\langle\left\langle g_s, \langle h^\eta_{I\dot{+}m}, Th_I^\theta\rangle\langle h_I^\theta, f_s\rangle h^\eta_{I\dot{+}m}\right\rangle\right\rangle.$$

The desired estimate

$$\sum_{s} |\langle\langle g_s, Tf_s\rangle\rangle| \lesssim \sum_{s} \|f_s\|_{L_2(\mathcal{A})}\|g_s\|_{L_2(\mathcal{A})}$$

is reduced to the corresponding uniform estimate (in β) for A_{good} , B^0_{good} , C^0_{good} and $P+Q$.

Estimate of A_{good} [63]. This term can be estimated directly since $\{h^\eta_{I+m}\}_{I\in\mathcal{D}}$ form a martingale difference sequence for fixed η, m.

$$|A_{\text{good}}| \leq \sum_{\eta,\theta\in\{0,1\}^n\setminus\{0\}} \sum_{m\in\mathbb{Z}^n} \sum_{s} \left|\left\langle\left\langle g_s, \sum_{I\in\mathcal{D}_{\text{good}}} \langle h^\eta_{I+m}, Th^\theta_I\rangle\langle h^\theta_I, f_s\rangle h^\eta_{I+m}\right\rangle\right\rangle\right|$$

$$\leq \sum_{s} \|g_s\|_{L_2(\mathcal{A})} \sum_{\eta,\theta\in\{0,1\}^n\setminus\{0\}} \sum_{m\in\mathbb{Z}^n} \left\|\sum_{I\in\mathcal{D}_{\text{good}}} \langle h^\eta_{I+m}, Th^\theta_I\rangle\langle h^\theta_I, f_s\rangle h^\eta_{I+m}\right\|_{L_2(\mathcal{A})}$$

$$\leq \sum_{s} \|g_s\|_{L_2(\mathcal{A})} \sum_{\eta,\theta\in\{0,1\}^n\setminus\{0\}} \sum_{m\in\mathbb{Z}^n} \sup_{\text{cubes}} \|\langle h^\eta_{I\dot{m}}, Th^\theta_I\rangle\|_{\mathcal{M}} \left(\tau\left(\sum_{I\in\mathcal{D}_{\text{good}}} |\langle h^\theta_I, f_s\rangle|^2\right)\right)^{\frac{1}{2}}$$

$$\lesssim \sum_{s} \|g_s\|_{L_2(\mathcal{A})} \sum_{\eta,\theta\in\{0,1\}^n\setminus\{0\}} \sum_{m\in\mathbb{Z}^n} (1+|m|)^{-n-\alpha}\|f_s\|_{L_2(\mathcal{A})} \lesssim \sum_{s} \|g_s\|_{L_2(\mathcal{A})}\|f_s\|_{L_2(\mathcal{A})}.$$

Here we used the fact

$$\sup_{\eta,\theta\in\{0,1\}^n} \sup_{I \text{ cubes } I+m} \|\langle h^\eta_{I\dot{m}}, Th^\theta_I\|_{\mathcal{M}} \lesssim (1+|m|)^{-n-\alpha} \tag{3.3}$$

which was essentially observed by Figiel following from the size condition (1.5), the smooth condition (1.6) and the Weak Boundedness Condition (1.15).

Estimates of B^0_{good} [63]. This term can not be estimated directly since $h^0_{I\dot{+}m} - h^0_I$ do not form a martingale difference sequence when I runs over all elements in D_{good} , but can be achieved when I runs over elements in some subcollections which partition $\mathcal{D}_{\text{good}}$ as in [23].

For each m, let $M = M(m) := \max\{r, \lceil(1-\gamma)^{-1}\log_2^+ |m|\rceil\}$. Let then $a^s(I) := \log_2 \ell(I) \text{mod} M + 1$, and define $b(I)$ to be alternatingly 0 and 1 along each orbit of the permutation $I \to I\dot{+}m$ of $\mathcal{D}$. It has been proved in [23] if $(a^s(I), b(I)) = (a^s(J), b(J))$ for two different cubes $I, J \in \mathcal{D}_{\text{good}}$, then the cubes satisfy the following m-compatibility condition: either the sets $I\cup(I\dot{+}m)$ and $J\cup(J\dot{+}m)$ are disjoint, or one of them, say $I\cup(I\dot{+}m)$, is contained in a dyadic subcube of J or $J\dot{+}m$.

We can hence decompose $\mathcal{D}_{\text{good}}$ into collections of pairwise m-compatible cubes by setting

$$\mathcal{D}^m_{k,v} := \{I\in\mathcal{D}_{\text{good}} : a^s(I) = k, b(I) = v\}, k = 0, \cdots, M(m), v = 0,1.$$

The total number of these collections is $2(1+M(m)) \lesssim (1+\log^+ |m|)$.

Note that for fixed $k, v, \{h^0_{I+m} - h^0_I\}_{I\in\mathcal{D}^m_{k,v}}$ form a martingale difference sequence. Thus

$$|B^0| \leq \sum_{\theta\in\{0,1\}^n\setminus\{0\}} \sum_{m\in\mathbb{Z}^n} \sum_{k,v} \sum_{s} \left|\left\langle\left\langle g_s, \sum_{I\in\mathcal{D}^m_{k,v}} \langle h^0_{I\dot{+}m}, Th^\theta_I\rangle\langle h^\theta_I, f_s\rangle(h^0_{I\dot{+}m} - h^0_I)\right\rangle\right\rangle\right|$$

$$\lesssim \sum_{s} \|g_s\|_{L_2(\mathcal{A})}\|f_s\|_{L_2(\mathcal{A})} \sum_{m\in\mathbb{Z}^n} (1+|m|)^{-n-\alpha}(1+\log^+ |m|) \lesssim \sum_{s} \|g_s\|_{L_2(\mathcal{A})}\|f_s\|_{L_2(\mathcal{A})}$$

where we used again the fact (3.3)Estimate of C^0_{good} . This term can be dealt with similarly as B^0_{good} since we can rewrite

$$C^0 = \sum_{\theta\in\{0,1\}^n\setminus\{0\}} \sum_{m\in\mathbb{Z}^n} \sum_{I\in\mathcal{D}} \sum_{s} \left\langle\left\langle f^*_s, \langle h^\theta_I, Th^0_{I\dot{m}m}\rangle\langle h^\theta_I, g^*_s\rangle(h^0_{I\dot{m}} - h^0_I)\right\rangle\right\rangle,$$

in the same form with B^0_{good} .

Estimate of $P+Q$ [63]. The estimate of $P+Q$ is completed through a similar argument used for Haar multiplier in Proposition 2.2

Lemma 3.1 (see [63]). We have

$$|P+Q| \lesssim \sum_{s} \|f_s\|_{L_2(\mathcal{A})}\|g_s\|_{L_2(\mathcal{A})}\|T1\|_{\text{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})} \lesssim \sum_{s} \|f_s\|_2\|g_s\|_2\|T1\|_{\text{BMO}(\mathbb{R}^n;\mathcal{M})} \tag{3.4}$$

with some constant independent of β. Here $\Sigma_{\mathcal{A}}$ is the filtration associated to the dyadic system $\mathcal{D}^\beta$.

Proof. We first rewrite P, Q as follows:

$$P = \sum_{\theta\in\{0,1\}^n\setminus\{0\}} \sum_{I\in\mathcal{D}} \sum_{s} \left\langle\left\langle T^*1, \langle h_I^\theta, f_s\rangle\langle 1_I/|I|, g_s^*\rangle h_I^\theta\right\rangle\right\rangle$$

$$= \left\langle T^*1, \sum_{k\in\mathbb{Z}} \sum_{s} \mathbb{D}_k(f_s)\mathbb{E}_{k-1}(g_s^*)\right\rangle$$

and

$$Q = \sum_{\theta\in\{0,1\}^n\setminus\{0\}} \sum_{I\in\mathcal{D}} \sum_{s} \left\langle\langle (T1)^*, \langle 1_I/|I|, f_s\rangle\langle h_I^\theta, g_s^*\rangle h_I^\theta\right\rangle\right\rangle$$

$$= \left\langle (T1)^*, \sum_{k\in\mathbb{Z}} \sum_{s} \mathbb{E}_{k-1}(f_s)\mathbb{D}_k(g_s^*)\right\rangle$$

Then by the symmetry condition (1.13), we get

$$|P+Q| \lesssim \|T^*1\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})} \sum_{s} \left\|\sum_{k\in\mathbb{Z}} \mathbb{D}_k(f_s)\mathbb{E}_{k-1}(g_s^*) + \sum_{k\in\mathbb{Z}} \mathbb{E}_{k-1}(f_s)\mathbb{D}_k(g_s^*)\right\|_{\mathrm{H}_1^m(\mathcal{A},\Sigma_{\mathcal{A}})},$$

which is controlled by

$$\|f_s\|_{L_2(\mathcal{A})}\|g_s\|_{L_2(\mathcal{A})}\|T^*1\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})}$$

by the same arguments in the proof of Proposition 2.2. Noting that for $b = T^*1$

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})} := \sup_J \left(\frac{1}{|J|}\int_J \|b - b_J\|_{\mathcal{M}}^2 dx\right)^{\frac{1}{2}}$$

where the supremum is taken over all the cubes J, while in the definition of $\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})$-norm, J runs over all the elements in $\mathcal{D}^\beta$. We get

$$\|T^*1\|_{\mathrm{BMO}(\mathcal{A},\Sigma_{\mathcal{A}})} \le \|T^*1\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})} < \infty$$

by the assumption (1.14), and thus finish the proof.

Remark 3.2 [63]. (i). Let $0 < \epsilon < \infty$ and $(1+\epsilon)$ be the conjugate index of $(1+\epsilon)$. Then using the arguments in the proof of Proposition 2.2, actually we are able to show

$$|P+Q| \lesssim \begin{cases} \|f_s\|_{L_{1+\epsilon}(\mathcal{A})}\|g_s\|_{\mathrm{H}_{1+\epsilon}^c(\mathcal{A},\Sigma_{\mathcal{A}})}\|T1\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})}, & \text{whenever } 0 \le \epsilon < \infty \\ \|f_s\|_{\mathrm{H}_{1+\epsilon}^c(\mathcal{A},\Sigma_{\mathcal{A}})}\|g_s\|_{L_{1+\epsilon}(\mathcal{A})}\|T1\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})}, & \text{whenever } 0 < \epsilon \le 1 \end{cases}$$

(ii). Here it is worthy to point out that in the argument above we did not use directly the boundedness of higher-dimensional Haar multiplier but the proof for one-dimensional Haar multiplier. We will see in the appendix that higherdimensional Haar multiplier is more difficult to handle.

3.3. Proof of Theorem 1.2 (see [63]). Combine the estimates of the four parts A_{good}, B_{good}^0, C_{good}^0 and $P+Q$, we proved

$$\|\sum_{s} Tf_s\|_{L_2(\mathcal{A})} \lesssim \sum_{s} \|f_s\|_{L_2(\mathcal{A})} \tag{3.5}$$

To prove other cases $1 < \epsilon \ne 1 < \infty$. We will use the atomic characterization of $\mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})$, which was first introduced by one of us [42]. Let us first recall the definition. Let $0 \le \epsilon < \infty$. The Hardy space $H_{1+\epsilon}^c(\mathbb{R}^n;\mathcal{M})$ is defined to be the space of functions $f_s \in L_1(\mathcal{A})$ for which we have

$$\|f_s\|_{\mathrm{H}_{1+\epsilon}^c(\mathbb{R}^n;\mathcal{M})} = \sum_{s} \left\|\left(\int_\Gamma \left[\frac{\partial \hat{f}_s^*}{\partial t}\frac{\partial \hat{f}_s}{\partial t} + \sum_j \frac{\partial \hat{f}_s^*}{\partial x_j}\frac{\partial \hat{f}_s}{\partial x_j}\right](x+\cdot,t)\frac{dxdt}{t^{n-1}}\right)^{\frac{1}{2}}\right\|_{L_{1+\epsilon}(\mathcal{A})} < \infty$$

with $\Gamma = \{(x,t) \in \mathbb{R}_+^{n+1} | |x| < t\}$ and $\hat{f}_s(x,t) = P_t f_s(x)$ for the Poisson semigroup $(P_t)_{t\ge 0}$.

According to [42], these Hardy spaces have nice duality and interpolation behavior. Observing that the adjoint operator T^* and its kernel have same properties as T and K, thus to finish the proof, it suffices to show

$$T: \mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M}) \to L_1(\mathcal{A}).$$

On the other hand, $\mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})$ has an atomic characterization. We say that $a^s \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^n))$ is an atom if there exists a cube I so that

- $\mathrm{supp}\, a^s \subseteq I$,
- $\int_I \sum_s a^s(x+\epsilon)d(x+\epsilon) = 0$,
- $\|a^s\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^n))} = \tau\left[\left(\int_I \sum_s |a^s(x+\epsilon)|^2 d(x+\epsilon)\right)^{\frac{1}{2}}\right] \le \frac{1}{\sqrt{|I|}}$.

By [42, Theorem 2.8], we have

$$\|f_s\|_{\mathrm{H}_1^c(\mathbb{R}^n;\mathcal{M})} \sim \inf\left\{\sum_k |\lambda_k| \mid f_s = \sum_k \sum_s \lambda_k a_k^s \text{ with } a_k^s \text{ atoms}\right\}.$$

Therefore, we only need to find a uniform upper estimate for the L_1 norm of $T(a^s)$ valid for an arbitrary atom

$$\|\sum_s T(a^s)\|_{L_1(\mathcal{A})} \leq \sum_s \|T(a^s)1_{2I}\|_{L_1(\mathcal{A})} + \sum_s \left\|T(a^s)1_{\mathbb{R}^n\setminus 2I}\right\|_{L_1(\mathcal{A})}$$

The second term is dominated by

$$\begin{aligned}\left\|T(a^s)1_{\mathbb{R}^n\setminus 2I}\right\|_{L_1(\mathcal{A})} &= \tau\int_{\mathbb{R}^n\setminus 2I}\sum_s \left|\int_I K(x,x+\epsilon)a^s(x+\epsilon)d(x+\epsilon)\right|dx\\ &\leq \int_I\left(\int_{\mathbb{R}^n\setminus 2I}\sum_s \|K(x,x+\epsilon)-K(x,c_I)\|_{\mathcal{M}}dx\right)\tau|a^s(x+\epsilon)|d(x+\epsilon)\\ &\lesssim \sum_s \tau\left(\int_I |a^s(x+\epsilon)|d(x+\epsilon)\right) \leq \sum_s \sqrt{|I|}\tau\left[\left(\int_I |a^s(x+\epsilon)|^2 d(x+\epsilon)\right)^{\frac{1}{2}}\right] \leq 1\end{aligned}$$

where we have used Kadison-Schwarz inequality in the third inequality. As for the first term, it suffices to show that $T\colon L_1(\mathcal{M}; L_2^c(\mathbb{R}^n)) \to L_1(\mathcal{M}; L_2^c(\mathbb{R}^n))$, since then we find again

$$\begin{aligned}\|T(a^s)1_{2I}\|_{L_1(\mathcal{A})} &= \tau\left(\int_{2I}\sum_s |T(a^s)(x)|dx\right)\\ &\leq \sqrt{|2I|}\tau\left[\left(\int_{2I}\sum_s |T(a^s)(x)|^2 dx\right)^{\frac{1}{2}}\right]\\ &\lesssim \sqrt{|2I|}\tau\left[\left(\int_I \sum_s |a^s(x)|^2 dx\right)^{\frac{1}{2}}\right] \lesssim 1\end{aligned}$$

The $L_1(\mathcal{M}; L_2^c(\mathbb{R}^n))$-boundedness of T follows from the duality

$$\|T(f_s)\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^n))} \leq \left(\sup_{\|g_s\|_{L_\infty(L_2^c)}\leq 1}\sum_s \|T^*(g_s)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}^n))}\right)\|f_s\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^n))}$$

Recall that the adjoint T^* has the same properties as T, and thus is bounded on $L_2(\mathcal{A})$. This gives rise to

$$\begin{aligned}\|T^*(g_s)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}^n))} &= \left\|\left(\int_{\mathbb{R}^n}\sum_s |T^*(g_s)(x)|^2 dx\right)^{\frac{1}{2}}\right\|_{\mathcal{M}}\\ &= \sup_{\|u\|_{L_2(\mathcal{M})}\leq 1}\left(\int_{\mathbb{R}^n}\sum_s \left\langle |T^*(g_s)(x)|^2 u, u\right\rangle_{L_2(\mathcal{M})} dx\right)^{\frac{1}{2}}\\ &= \sup_{\|u\|_{L_2(\mathcal{M})}\leq 1}\left(\int_{\mathbb{R}^n}\sum_s \|T^*(g_s u)(x)\|_{L_2(\mathcal{M})}^2 dx\right)^{\frac{1}{2}}\\ &\lesssim \sup_{\|u\|_{L_2(\mathcal{M})}\leq 1}\sum_s \left(\int_{\mathbb{R}^n}\|g_s(x)u\|_{L_2(\mathcal{M})}^2 dx\right)^{\frac{1}{2}}\\ &= \sum_s \left\|\left(\int_{\mathbb{R}^n}|g_s(x)|^2 dx\right)^{\frac{1}{2}}\right\|_{\mathcal{M}}\end{aligned}$$

The third identity above uses the right $\mathcal{M}$-module nature of T.

Remark 3.3 [63]. It is a quite interesting question to give a direct proof of Theorem 1.2 in the case $0 < \epsilon \neq 1 < \infty$ without using atomic decomposition, interpolation and duality like the one for perfect dyadic CZOs.

IV. Appendix

We show the following commutator estimate.

Theorem 4.1 (see [63]). If $b \in \mathrm{BMO}(\mathbb{R}^n; \mathcal{M})$, then the commutator $[R_j, b]$ is bounded on $L_2(\mathcal{A})$. Moreover we have the estimate

$$\left\| \sum_s [R_j, b] f_s \right\|_{L_2(\mathcal{A})} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})} \sum_s \|f_s\|_{L_2(\mathcal{A})} \tag{4.1}$$

When $n = 1$, the Riesz transforms reduce to the Hilbert transform. By noting the boundedness of the Haar multiplier-Proposition 2.2, the result has been essentially proven by Petermichl, see Section 2.3 of [49]. When $n > 1$, the situation becomes a little bit more complicated. Firstly, reviewing the proof of the boundedness of one-dimensional Haar multiplier, the higher-dimensional case is not trivial since $\mathbb{D}_k b \mathbb{D}_k f_s$ is not $k-1$-th measurable; Secondly, the higher-dimensional Haar systems are also more complicated.

[50] showed that the Riesz transforms also lie in the closed convex hull of some dyadic shifts. We write down explicitly the form of this class of dyadic shifts: Fix a dyadic system $\mathcal{D}$, let $\theta_0 \in \{0,1\}^n$ be the element with first coordinate 1 and others 0 ,

$$Sf_s = \sum_{I \in \mathcal{D}} \sum_{\theta \in \{0,1\}^n \setminus \{0\}} \sum_s \varepsilon_I^\theta \left\langle h_I^{\theta_0}, f_s \right\rangle h_{\hat{I}}^\theta \tag{4.2}$$

where $\hat{I}$ is the dyadic father of I and $\varepsilon_I^\theta = \pm 1$.

Associated to this fixed dyadic system $\mathcal{D}$, the Haar multiplier with noncommmuting symbol b is defined as

$$\Lambda_b(f_s) = \sum_{k \in \mathbb{Z}} \sum_s \mathbb{D}_k(b) \mathbb{E}_k(f_s)$$

As in the one-dimensional case-Proposition 2.2 one also gets

Proposition 4.2 (see [63]). If $b \in \mathrm{BMO}(\mathbb{R}^n; \mathcal{M})$, then we have

- Λ_b is bounded from $L_{2+\epsilon}(\mathcal{A})$ to $\mathrm{H}_{2+\epsilon}^c(\mathcal{A}, \Sigma_{\mathcal{A}})$ whenever $0 \le \epsilon < \infty$;
- Λ_b is bounded from $\mathrm{H}_{1+\epsilon}^c(\mathcal{A}, \Sigma_{\mathcal{A}})$ to $L_{1+\epsilon}(\mathcal{A})$ whenever $0 < \epsilon \le 1$.

Proof. It suffices to show the case $0 \le \epsilon < \infty$, since another case can be shown similarly. Let $(1+\epsilon)$ be the conjugate index of $(1+\epsilon)$. Let $f_s \in L_{1+\epsilon}(\mathcal{A})$, and $g_s \in \mathrm{H}_{1+\epsilon}^c(\mathcal{A}, \Sigma_{\mathcal{A}})$. By approximation, we can assume b, f_s and g_s are "nice", so that we do not to justify the infinite sum in the following calculations. By duality, it suffices to show

$$\left| \sum_s \varphi(\Lambda_b(f_s) g_s^*) \right| \lesssim \sum_s \|f_s\|_{L_{1+\epsilon}(\mathcal{A})} \|g_s\|_{\mathrm{H}_{1+\epsilon}^c(\mathcal{A},\Sigma_{\mathcal{A}})}$$

Noting that $\Lambda_b f_s = b f_s - \sum_k \mathbb{E}_{k-1}(b) \mathbb{D}_k(f_s)$, we have

$$\begin{aligned}
\left| \sum_s \varphi(\Lambda_b(f_s) g_s^*) \right| &= \sum_s \left(\left| \varphi\left(\left(b f_s - \sum_k \mathbb{E}_{k-1}(b) \mathbb{D}_k(f_s) \right) g_s^* \right) \right| \right) \\
&= \sum_s \left(\left| \varphi(b f_s g_s^*) - \varphi\left(b \sum_k \mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s) \mathbb{D}_k(g_s^*)\big) \right) \right| \right) \\
&= \sum_s \left(\left| \varphi\left(b \left(f_s g_s^* - \sum_k \mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s) \mathbb{D}_k(g_s^*)\big) \right) \right) \right| \right)
\end{aligned}$$

Now decompose $f_s g_s^*$, one gets

$$\begin{aligned}
\sum_s f_s g_s^* - \sum_k \sum_s \mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s) \mathbb{D}_k(g_s^*)\big) &= \sum_k \sum_s \mathbb{D}_k\big(\mathbb{D}_k(f_s) \mathbb{D}_k(g_s^*)\big) \\
&+ \sum_s \left(\sum_{k \in \mathbb{Z}} \mathbb{E}_{k-1}(f_s) \mathbb{D}_k(g_s^*) + \sum_{k \in \mathbb{Z}} \mathbb{D}_k(f_s) \mathbb{E}_{k-1}(g_s^*) \right)
\end{aligned}$$

The second term can be estimated as in the one-dimensional case. For the first term, note that $\mathbb{D}_k\big(\mathbb{D}_k(f_s) \mathbb{D}_k(g_s^*)\big)$ is a martingale difference, by duality and the fact that the dyadic BMO-norm is controlled by usual BMO-norm, we have

$$\left|\sum_s \varphi\left(b\sum_k \mathbb{D}_k\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right)\right| \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})}\sum_s \left\|\sum_k \mathbb{D}_k\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right\|_{\mathrm{H}_1^m(\mathcal{A},\Sigma_{\mathcal{A}})}$$
$$= \|b\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})}\int_{\mathbb{R}^n}\sum_s \sup_{\ell\in\mathbb{Z}}\left\|\sum_{k=-\infty}^{\ell} \mathbb{D}_k\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right\|_{L_1(\mathcal{M})} dx$$

Splitting

$$\sum_{k=-\infty}^{\ell}\sum_s \mathbb{D}_k\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big) = \sum_{k=-\infty}^{\ell}\sum_s \mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*) - \sum_{k=-\infty}^{\ell}\sum_s \mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)$$

noting that the first term can be dealt with as in Proposition 2.2, and we are reduced to show

$$\int_{\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}}\left\|\sum_{k=-\infty}^{\ell}\sum_s \mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right\|_{L_1(\mathcal{M})} dx \lesssim \sum_s \|f_s\|_{L_{1+\epsilon}(\mathcal{A})}\|g_s\|_{\mathrm{H}^c_{1+\epsilon}(\mathcal{A},\Sigma_{\mathcal{A}})} \tag{4.3}$$

Using twice the Hölder inequalities and vector-valued Doob's inequality (2.2), it suffices to show for any ℓ

$$\left\|\sum_{k=-\infty}^{\ell}\sum_s \mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right\|_{L_1(\mathcal{M})}$$
$$\le \sum_s \left\|\left(\sum_{k\in\mathbb{Z}}|\mathbb{D}_k(f_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})}\left\|\left(\sum_{k\in\mathbb{Z}}|\mathbb{D}_k(g_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})}$$

By duality and the Hölder inequality, using the trace-preserving property of conditional expectation,

$$\left\|\sum_{k=-\infty}^{\ell}\sum_s \mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right\|_{L_1(\mathcal{M})}$$
$$= \sup_{u,\|u\|_{\mathcal{M}}\le 1}\sum_s \left|\tau\left(u\sum_{k=-\infty}^{\ell}\mathbb{E}_{k-1}\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right)\right|$$
$$= \sup_{u,\|u\|_{\mathcal{M}}\le 1}\sum_s \left|\tau\left(\sum_{k=-\infty}^{\ell}\mathbb{E}_{k-1}(u)\big(\mathbb{D}_k(f_s)\mathbb{D}_k(g_s^*)\big)\right)\right|$$
$$= \sup_{u,\|u\|_{\mathcal{M}}\le 1}\sum_s \left|\tau\left(\sum_{k=-\infty}^{\ell}\mathbb{D}_k(g_s^*)(\mathbb{E}_{k-1}(u)\mathbb{D}_k(f_s))\right)\right|$$
$$= \sup_{u,\|u\|_{\mathcal{M}}\le 1}\sum_s \left|\tau\otimes\mathrm{tr}\left(\left(\sum_{k=-\infty}^{\ell}\mathbb{D}_k(g_s^*)\otimes e_{1k}\right)\left(\sum_{k=-\infty}^{\ell}\mathbb{E}_{k-1}(u)\mathbb{D}_k(f_s)\otimes e_{k1}\right)\right)\right|$$
$$\le \sup_{u,\|u\|_{\mathcal{M}}\le 1}\sum_s \left\|\sum_{k=-\infty}^{\ell}\mathbb{D}_k(g_s^*)\otimes e_{1k}\right\|_{L_{1+\epsilon}\left(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\right)}\left\|\sum_{k=-\infty}^{\ell}\mathbb{E}_{k-1}(u)\mathbb{D}_k(f_s)\otimes e_{k1}\right\|_{L_{1+\epsilon}\left(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\right)}$$
$$\le \sum_s \left\|\left(\sum_{k\in\mathbb{Z}}|\mathbb{D}_k(f_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})}\left\|\left(\sum_{k\in\mathbb{Z}}|\mathbb{D}_k(g_s)|^2\right)^{1/2}\right\|_{L_{1+\epsilon}(\mathcal{M})}.$$

This finishes the proof by noncommutative Burkholder-Gundy inequality.

Now we prove Theorem 4.1 (see [63])

Proof. Since the Riesz transforms [50] are shown to be in the convex hull of the dyadic shift operators such as (4.2), it suffices to estimate $[S,b]$ for one fixed dyadic shift operator S. Without loss of generality, we can assume $b=b^*$. Let $f_s\in L_2(\mathcal{A})$. By approximation, we can assume b and f_s are "nice" so that we can decompose $bf_s=\Lambda_b f_s+R_b f_s$, where

$$R_b f_s=\sum_k\sum_s \mathbb{E}_{k-1}(b)\mathbb{D}_k(f_s)=\sum_{\theta\in\{0,1\}^n\setminus\{0\}}\sum_{I\in\mathcal{D}}\sum_s \langle b\rangle_I\langle h_I^\theta,f_s\rangle h_I^\theta,$$

with $\langle b\rangle_I=\frac{1}{|I|}\int_I b$. Thus

$$[S,b]f_s=[S,\Lambda_b]f_s+[S,R_b]f_s$$

Observe that from the $L_2(\mathcal{A})$-boundedness of S and Λ_b we have

$$\left\|\sum_s [S,\Lambda_b]f_s\right\|_{L_2(\mathcal{A})}\leq 2\sum_s \|S\|\|\Lambda_b\|\|f_s\|_{L_2(\mathcal{A})}\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n,\mathcal{M})}\sum_s \|f_s\|_{L_2(\mathcal{A})}$$

For another term, we claim that

$$[S,R_b]f_s=\sum_{\theta,\eta\in\{0,1\}^n\setminus\{0\}}\sum_{I,J\in\mathcal{D}}\sum_s \langle h_J^\eta,Sh_I^\theta\rangle(\langle b\rangle_I-\langle b\rangle_J)\langle h_I^\theta,f_s\rangle h_J^\eta, \tag{4.4}$$

from which, we can conclude the proof. Indeed, by the orthogonality of the Haar basis h_I^θ 's,

$$[S,R_b]f_s=\sum_{\theta',\theta,\eta\in\{0,1\}^n\setminus\{0\}}\sum_{I\in\mathcal{D}}\sum_s (a^s)_I^{\theta',\theta,\eta}(\langle b\rangle_I-\langle b\rangle_{\hat{I}})\langle h_I^\theta,f_s\rangle h_{\hat{I}}^\eta$$

where

$$(a^s)_I^{\theta',\theta,\eta}=\varepsilon_I^{\theta'}\left\langle h_I^{\theta_0},h_I^\theta\right\rangle\left\langle h_{\hat{I}}^\eta,h_{\hat{I}}^{\theta'}\right\rangle,$$

which equals ±1 or 0 . Then the fact for any $e\in L_2(\mathcal{M})$ with norm 1 ,

$$\|(\langle b\rangle_I-\langle b\rangle_{\hat{I}})e\|_{L_2(\mathcal{M})}^2\lesssim\|b\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})}^2$$

yields

$$\left\|\left[\sum_s S,R_b\right]f_s\right\|_{L_2(\mathcal{A})}\lesssim\|b\|_{\mathrm{BMO}(\mathbb{R}^n;\mathcal{M})}\sum_s \|f_s\|_{L_2(\mathcal{A})}$$

Now let us show the formula (4.4). Note that $[S,R_b]f_s=SR_bf_s-R_bSf_s$. It is straightforward to compute

$$\begin{aligned}R_b(Sf_s)&=\sum_{\eta\in\{0,1\}^n\setminus\{0\}}\sum_{J\in\mathcal{D}}\sum_s \langle b\rangle_J\langle h_J^\eta,Sf_s\rangle h_J^\eta\\&=\sum_{\theta,\eta\in\{0,1\}^n\setminus\{0\}}\sum_{I,J\in\mathcal{D}}\sum_s \langle h_J^\eta,Sh_I^\theta\rangle\langle b\rangle_J\langle h_I^\theta,f_s\rangle h_J^\eta.\end{aligned}$$

For another term, we test it on $g_s=\sum_{\eta\in\{0,1\}^n\setminus\{0\}}\sum_{J\in\mathcal{D}}\sum_s \langle h_J^\eta,g_s\rangle h_J^\eta\in L_2(\mathcal{A})$, and obtain

$$\begin{aligned}\langle\langle g_s,SR_bf_s\rangle\rangle&=\sum_{\theta,\eta\in\{0,1\}^n\setminus\{0\}}\sum_{I,J\in\mathcal{D}}\sum_s \left\langle\left\langle\langle h_J^\eta,g_s\rangle h_J^\eta,\langle b\rangle_I\langle h_I^\theta,f_s\rangle Sh_I^\theta\right\rangle\right\rangle\\&=\sum_{\theta,\eta\in\{0,1\}^n\setminus\{0\}}\sum_{I,J\in\mathcal{D}}\sum_s \left\langle\left\langle g_s,\langle h_J^\eta,Sh_I^\theta\rangle\langle b\rangle_I\langle h_I^\theta,f_s\rangle h_J^\eta\right\rangle\right\rangle,\end{aligned}$$

which yields

$$SR_bf_s=\sum_{\theta,\eta\in\{0,1\}^n\setminus\{0\}}\sum_{I,J\in\mathcal{D}}\sum_s \langle h_J^\eta,Sh_I^\theta\rangle\langle b\rangle_I\langle h_I^\theta,f_s\rangle h_J^\eta.$$

From the above two identities, we get (4.4).

Remark 4.3 [63]. (i). The above argument works also for general dyadic shifts such as those introduced in [37]. But at the time of writing, the authors have no idea how to show similar results for general Calderón-Zygmund singular integral operators.

(ii). In the framework of noncommutative harmonic analysis, it would be also interesting to show the result for $\epsilon\neq1$. But now the proof is not trivial at all.

References

[1] P. Auscher, S. Hoffman, C. Muscalu, T. Tao and C. Thiele, Carleson measures trees, extrapolation, and T(b) theorems, Pub. Mat. 46 (2002), 257-325.

[2] O. Blasco, S. Pott, Embeddings between operator-valued dyadic BMO spaces, Illinois J. Math. 52 (3) (2008), 799-814.

[3] O. Blasco, S. Pott, Operator-valued dyadic BMO spaces, (English summary) J. Oper. Theory. 63 (2) (2010), 333-347.

[4] J. Bourgain, Vector-valued singular integrals and the H^1-BMO duality, In J. A. Chao, W. Woyczyński (eds.), Probability Theory and Harmonic Analysis, Marcel Dekker, New York, 1986, 1-19.

[5] D. L. Burkholder, Martingales and Fourier analysis in Banach spaces. In G. Letta, M. Pratelli (eds.), Probability and Analysis, Lecture Notes in Math. 1206, Springer-Verlag, 1986.

[6] T. Figiel, Singular integral operators: a martingale approach. In P. F. X. Müller, W. Schachermayer (eds.), Geometry of Banach Spaces. Proceedings of the conference held in Strobl, Austria, 1989. London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press, 1990.

[7] A. P. Calderón, Cauchy integrals on Lipschitz curves and related operators, Proc. Nat. Acad. Sci. USA. 74 (1977), 1324-1327.
[8] Z. Chen, Q. Xu and Z. Yin, Harmonic analysis on quantum tori, Comm. Math. Phys. 322 (2013), 755-805.
[9] M.Christ, Lectures on singular integral operators, CBMS Regional Conference Series in Mathematics, Vol. 77, Amer. Math. Soc., Providence, RI, 1990.
[10] G. David, J.-L. Journé. A boundedness criterion for generalized Calderón-Zygmund operators, Ann. Math. 120 (1984), 371-397.
[11] T. Ferguson, T. Mei, B. Simanek, H^∞-calculus for semigroup generators on BMO. Adv. Math. 347 (2019), 408-441.
[12] T. A. Gillespie, S. Pott, S. Treil, A. Volberg, The transfer method in estimates of vector Hankel operators, Algebra iAnaliz. 12 (6) (2000), 178-193; (English version to appear in St. Petersburg Math. J.)
[13] A. Gonzalez-Pérez, M. Junge, J. Parcet, Singular integrals in quantum Euclidean spaces, Ann. Sci. Éc. Norm. Supér. 50 (4) (2017), 879-925.
[14] A. Harcharras, Fourier analysis, Schur multipliers on S_p and non-commutative $\Lambda(p)$-sets, Studia Math. 137 (3) (1999), 203-260.
[15] G. Hong, M. Junge, J. Parcet, Algebraic Davis decomposition and Asymmetric Doobindqualities, Commun. Math. Phys. 346 (2016), 995-1019.
[16] G. Hong, M. Junge, J. Parcet, Asymmetric Doobindqualities in continuous time, J. Funct. Anal. 273 (4) (2017), 1479-1503.
[17] G. Hong, L.D. López-Sánchez, J. Martell, J. Parcet, Calderón-Zygmund operators associated to matrix-valued kernels, Int. Math. Res. Not. 5 (2014), 1221-1252.
[18] G. Hong, T. Ma, Vector valued q-variation for ergodic averages and analytic semigroups, J. Math. Anal. Appl. 437 (2016), 1084-1100.
[19] G. Hong, T. Ma, Vector valued q-variation for differential operators and semigroups I, Math. Z. 286 (2017), 89-127.
[20] G. Hong, T. Mei, John-Nirenberg inequality and atomic decomposition for noncommutative martingales, J. Funct. Anal. 263 (4) (2012), 1064-1097.
[21] G. Hong and Z. Yin, Wavelet approach to operator-valued Hardy spaces, Rev. Mate. Iberoa. 29 (1) (2013), 293-313.
[22] T. Hytönen, An operator-valued Tb theorem, J. Funct. Anal. 234 (2) (2006), 420-463.
[23] T. Hytönen, Vector-valued singular integrals revisited-with random dyadic cubes, Bull. Pol. Acad. Sci. Math. 60 (3) (2010), 269-283.
[24] T. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, Ann. of Math. 175 (3) (2012), 1473-1506.
[25] T. Hytönen, L. Weis, A $T1$ theorem for integral transformations with operator-valued kernel, J. Reine Angew. Math. 599 (2006), 155-200.
[26] M. Junge, Doob's inequality for non-commutative martingales, Journal für die Reine und AngewandteMathematik. 549 (2002), 149-190.
[27] M. Junge, C. Le Merdy, and Q. Xu, H^∞ functional calculus and square functions on noncommutative L_p-spaces, Astérisque. 305 (2006).
[28] M. Junge and T. Mei, Noncommutative Riesz transforms-A probabilistic approach, Amer. J. Math. 132 (2010), 611-681.
[29] M. Junge and T. Mei, BMO spaces associated with semigroups of operators, Math. Ann. 352 (3) (2012), 691-743.
[30] M. Junge, T. Mei, and J. Parcet, Smooth Fourier multipliers on group von Neumann algebras, Geom. Funct. Anal. 24 (6) (2014), 1913-1980.
[31] M. Junge, T. Mei, and J. Parcet, Noncommutative Riesz transforms-Dimension free bounds and Fourier multipliers, J. Eur. Math. Soc. 20 (3) (2018), 529-595.
[32] M. Junge, C. Palazuelos, J. Parcet and M. Perrin, Hypercontractivity in group von Neumann algebras, Mem. Amer. Math. Soc. 249 (1183) (2017).
[33] M. Junge, C. Palazuelos, J. Parcet, M. Perrin and E. Ricard, Hypercontractivity for free products, Ann. Sci. Éc. Norm. Supér. 48 (4) (2015), 861-889.
[34] M. Junge, M. Perrin, Theory of $\mathcal{H}_p$ space for continuous filtrations in von Neumann algebras, Astérisque. 362 (2014).
[35] M. Junge, Q. Xu, Non-commutative Burkholder/Rosenthal Inequalities, Ann. Prob. 31 (2) (2003), 948-995.
[36] M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities II: applications, Israel J. Math. 167 (2008), 227-282.
[37] M. Lacey, S. Petermichl, M. Reguera, Sharp A_2 inequality for Haar shift operators, Math. Ann. 348 (2010), 127-141.
[38] T. Martínez, J. L. Torrea, Q. Xu, Vector-valued Littlewood-Paley-Stein theory for semigroups, Adv. Math. 203 (2) (2006), 430-475.
[39] E. McDonald, F. Sukochev and X. Xiong, Quantum differentiability on quantum tori. Commun. Math. Phys. (2019).
[40] J. Mateu, J. Orobitg, J. Verdera, Estimates for the maximal singular integral in terms of the singular integral: the case of even kernels, Ann. of Math. 174 (3) (2011), 1429-1483.
[41] T. Mei, Notes on matrix valued paraproducts, Indiana Univ. Math. J. 55 (2) (2006), 747-760.
[42] T. Mei, Operator valued Hardy spaces, Mem. Amer. Math. Soc. 188 (2007).
[43] T. Mei, An extrapolation of operator-valued dyadic paraproducts, J. Lond. Math. Soc. 81 (3) (2007), 650-662.
[44] M. Musat, Interpolation Between Non-commutative BMO and Non-commutative L_p-spaces, J. Funct. Analysis. 202 (1) (2003), 195-225.
[45] F. Nazarov, G. Pisier, S. Treil, A. Volberg, Sharp estimates in vector Carleson imbedding theorem and for vector paraproducts, J. Reine Angew. Math. 542 (2002), 147-171.
[46] F. Nazarov, S. Treil, A. Volberg, Counterexample to the infinite dimensional Carleson embedding theorem, C.R. Acad. Sci. Paris Ser. I Math. 325 (1997), 383-389.

[47] F. Nazarov, S. Treil, A. Volberg, The Tb-theorem on non-homogeneous spaces, Acta. Math, 190 (2) (2003), 151-239.
[48] J. Parcet, Pseudo-localization of singular integrals and noncommutative Calderón-Zygmund theory, J. Funct. Anal. 256 (2009), 509-593.
[49] S. Petermichl, Dyadic shifts and a logarithmic bound for Hankel operators with matrix symbols, C. R. Acad. Sci. Paris Ser. I Math. 330 (2000), 455-460.
[50] S. Petermichl, S. Treil and A. Volberg, Why the Riesz transforms are averages of the dyadic shifts? Proc. of the 6th Inter. Conf. on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000) (2002) 209-228.
[51] G. Pisier, Non-commutative Vector Valued L_p-Spaces and Completely p-Summing Maps, Astérisque 247, 1998.
[52] G. Pisier, Q. Xu, Non-commutative martingale inequalities, Comm. Math. Phys. 189 (1997), 667-698.
[53] G. Pisier, Q. Xu, Non-commutative L_p-spaces, Handbook of the Geometry of Banach Spaces II. In: Johnson, W.B., Lindenstrauss, J.(eds.) pp. 1459-1517. North-Holland (2003).
[54] S. Pott, A. Stoica, Linear bounds for Calderón-Zygmund operators with even kernel on UMD spaces. J. Funct. Anal. 266 (5) (2014), 3303-3319.
[55] N. Randrianantoanina, Non-commutative martingale transform, J. Funct. Analysis, 194 (1) (2002), 181-212.
[56] N. Randrianantoanina. Conditioned square functions for noncommutative martingales, Ann. Probab. 35 (3) (2007), 1039-1070.
[57] E. Ricard and Q. Xu, A noncommutative martingale convexity inequality, Ann. Probab, 44 (2016), 867-882.
[58] F. Sukochev and D. Zanin, Connes integration formula for the noncommutative plane, Commun. Math. Phys. 359 (2018), 449-466.
[59] S. Treil, A. Volberg, Wavelets and the angle between past and future, J. Funct. Anal. 143 (1997), 269-308.
[60] R. Xia, X. Xiong, Q. Xu, Characterizations of operator-valued Hardy spaces and applications to harmonic analysis on quantum tori, Adv. Math. 291 (2016), 183-227.
[61] X. Xiong, Q. Xu, Z. Yin, Sobolev, Besov and Triebel-Lizorkin spaces on quantum tori, Mem. Amer. Math. Soc. 252 (2018).
[62] Q. Xu, Littlewood-Paley theory for functions with values in uniformly convex spaces, J. Reine. Angew. Math. 504 (1998), 195-226.
[63] Guixiang Hong, Honghai Liu, Tao Mei, An Operator-Valued $T1$ Theory for Symmetric Czos, v1 Math.Ca (2019) 1-22.