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Research Paper

On Order Simple Labeled Graph $C*$-Algebras Associated to Disagreeable Order Labeled Spaces:

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Abstract
By an order labeled graph C^*-algebra we mean a C^*-algebra associated to an order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ consisting of an order labeled graph$(E_q, \mathcal{L}_q)$ and the smallest normal accommodating set $\mathcal{E}$ of vertex subsets. Every graph C^*-algebra $C^*(E_q)$ is an order labeled graph C^*-algebra and it is well known that $C^*(E_q)$ is simple if and only if the graph E_q is cofinal and satisfies Condition (L). Bates and Pask extend these conditions of graphs E_q to the order labeled spaces, and show that if a set-finite and receiver set-finite order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ is cofinal and disagreeable, then its C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple. J. AJeonga, G. H. Park [23] show that the converse is also true. On the same way as [23] we introduce a specific order of definiteness.
Keywords: Labeled graph C^*-algebra, Simple C^*-algebra.

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1. Introduction

A class of C^*-algebras associated to directed graphs including the Cuntz-Krieger algebras [9] was seen in [18],[19], and since then its generalizations have attracted much attention. The C^*-algebras associated to ultragraphs, infinite matrices, higher-rank graphs, subshifts, Boolean dynamical systems, and labeled spaces are examples (see [1],[3],[4],[7],[10],[11],20],[22]).

In the study of these generalized Cuntz-Krieger algebras is to describe the ideal structure of a C^*-algebra in question in terms of structural properties of the object to which the C^*-algebra is associated. The ideal structure of a graph C^*-algebra is now well understood, and if we recall it for a row-finite order graph E_q with no singular vertices, it says that there exists a one to one correspondence between the gauge-invariant ideals of the graph C^*-algebra $C^*(E_q)$ and the hereditary saturated vertex subsets of the order graph E_q ([2],[5],[10]), and moreover $C^*(E_q)$ is simple if and only if E_q is cofinal and satisfies Condition (L) ([10],[18]). Here the gauge action is the action of the unit circle on a graph C^*-algebra which always exists because of the universal property of a graph C^*-algebra. Many put a great deal of effort to extend this result to the classes of generalized Cuntz-Krieger algebras, and we will look at the order labeled graph C^*-algebras and focus on the question of when these algebras are simple.

If $(E_q, \mathcal{L}_q)$ is an order labeled graph, that is, $\mathcal{L}_q: E_q^1 \to \mathcal{A}$ is an order labeling map of the edges E_q^1 onto an alphabet $\mathcal{A}$, then we consider a collection $\mathcal{B}$ consisting of certain vertex subsets so that a universal family of order projections $\{p_{A_q}: A_q \in \mathcal{B}\}$ and order partial isometries $\{s_{a_q}: a_q \in \mathcal{A}\}$ satisfying the relations imposed by the triple $(E_q, \mathcal{L}_q, \mathcal{B})$ exists and thus one can form the C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{B})$ generated by this universal family of operators $\{p_{A_q}, s_{a_q}\}$. We call $C^*(E_q, \mathcal{L}_q, \mathcal{B})$ the C^*-algebra of an order labeled space $(E_q, \mathcal{L}_q, \mathcal{B})$. So, if $\mathcal{E}$ is the smallest normal accommodating set, we will simply call $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ the order labeled graph C^*-algebra of $(E_q, \mathcal{L}_q)$ for convenience. We will be mostly interested in these order labeled graph C^*-algebras $C^*(E_q, \mathcal{L}_q, \mathcal{E})$.

Every graph C^*-algebra is a labeled graph C^*-algebra ([1]) and the class of Morita equivalence classes of C^*-algebras of labeled spaces strictly contains the class of Morita equivalence classes of graph C^*-algebras (see [1] and [13]). By the universal property of an order labeled graph C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{B})$, there exists a gauge action of the unit circle on $C^*(E_q, \mathcal{L}_q, \mathcal{B})$, and it is known [16] that if E_q has no sinks and $(E_q, \mathcal{L}_q, \mathcal{B})$ is a set-finite and receiver set-finite order normal labeled space, there is a one to one correspondence between the gauge-invariant ideals of $C^*(E_q, \mathcal{L}_q, \mathcal{B})$ and the hereditary saturated subsets of $\mathcal{B}$. (A gauge invariant uniqueness theorem [3] used in [16] was turned out to be incorrect, but was corrected in [6] for normal order labeled spaces and in [1] for general order labeled spaces.)

[4] considered the question of when a C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ of a set-finite and receiver set-finite order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ is simple, and proved that $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple if $(E_q, \mathcal{L}_q, \mathcal{E})$ is cofinal and disagreeable. The notion of a disagreeable order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ introduced in [4] is an analogue of Condition (L) of usual directed graphs. The cofinal condition for $(E_q, \mathcal{L}_q, \mathcal{E})$ used in [4] needs

to be modified to obtain the simplicity result for $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ as noted in [15] where a condition called strongly cofinal was used instead. The definition of a strongly cofinal order labeled space given in [15] is weaker than the one here (see Definition 2.10 or [14]), and throughout if we mention strong cofinality. It then follows from [15] that $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple whenever $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable and strongly cofinal.

As for the converse of Bates and Pask's simplicity result, the strong cofinality of $(E_q, \mathcal{L}_q, \mathcal{E})$ can be derived in [15] by slightly modifying the proof there. On the other hand, it is not clear whether the order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ has to be disagreeable when its C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple, and this is the question we will consider here. From a [8],[16] on simplicity of a C^*-algebra associated to a Boolean dynamical system, we know that for an order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ whose Boolean dynamical system satisfies a sort of domain condition, the C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple if and only if $(E_q, \mathcal{L}_q, \mathcal{E})$ has no cycles without an exit and there are no nonempty hereditary saturated subsets of $\mathcal{E}$. The first condition of having no cycles without an exit is always satisfied whenever the order labeled space is disagreeable while the converse does not hold in general, and the second condition of having no nonempty saturated hereditary subsets is equivalent to the absence of gauge-invariant proper ideals in $C^*(E_q, \mathcal{L}_q, \mathcal{E})$. Thus the question of whether the converse of Bates and Pask's simplicity result holds true is not answered directly from [8] while it is known [15] that the converse holds if $\mathcal{E}$ contains $\{v_q\}$ for every vertex v_q in E_q.

We figure out whether the converse of Bates and Pask's simplicity result holds and it is proved that the order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ is always disagreeable if $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple. This establishes the following: the order labeled graph C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple if and only if $(E_q, \mathcal{L}_q, \mathcal{E})$ is strongly cofinal and disagreeable.

2. Preliminaries

We state notation, review definitions and basic results. See [1] or [14].

A directed graph $E_q = (E_q^0, E_q^1, r, s)$ consists of the vertex set E_q^0 and the edge set E_q^1 together with the range, source maps $r, s: E_q^1 \to E_q^0$. We call a vertex $v_q \in E_q^0$ a sink (a source, respectively) if $s^{-1}(v_q) = \emptyset$ $(r^{-1}(v_q) = \emptyset$, respectively). If every vertex in E_q emits only finitely many edges, E_q is called row-finite.

For each $n \geq 1, E_q^n$ denotes the set of all paths of length n, and the vertices in E_q^0 are regarded as finite paths of length zero. The maps r, s naturally extend to the set $E_q^* = \cup_{n\geq 0} E_q^n$ of all finite paths, especially with $r(v_q) = s(v_q) = v_q$ for $v_q \in E_q^0$. By E_q^∞ we denote the set of all infinite paths $x = \lambda_1\lambda_2 \cdots$, where we define $s(x) := s(\lambda_1)$. For $A_q, B_q \subset E_q^0$ and $n \geq 0$, we use the following notation

$$A_q E_q^n := \{\lambda \in E_q^n : s(\lambda) \in A_q\}, E_q^n B_q := \{\lambda \in E_q^n : r(\lambda) \in B_q\},$$

and $A_q E_q^n B_q := A_q E_q^n \cap E_q^n B_q$ with $E_q^n v_q := E_q^n\{v_q\}, v_q E_q^n := \{v_q\}E_q^n$. Also the sets of paths like $E_q^{\geq k}, A_q E_q^{\geq k}$, and $A_q E_q^\infty$ which have their obvious meaning will be used. A loop is a finite path $\lambda \in E_q^{\geq 1}$ such that $r(\lambda) = s(\lambda)$, and an exit of a loop λ is a path $\delta \in E_q^{\geq 1}$ such that $|\delta| \leq |\lambda|, s(\delta) = s(\lambda)$, and $\delta \neq \lambda_1 \cdots \lambda_{|\delta|}$. A graph E_q is said to satisfy Condition (L) if every loop has an exit.

For $\mathcal{A}$ be a countable alphabet and $\mathcal{A}^*$ ($\mathcal{A}^\infty$, respectively) denote the set of all finite words (infinite words, respectively) in symbols of $\mathcal{A}$. An order labeled graph $(E_q, \mathcal{L}_q)$ over $\mathcal{A}$ consists of a directed graph E_q and an ordering labeling map $\mathcal{L}_q: E_q^1 \to \mathcal{A}$ which is always assumed to be onto. Given a graph E_q, one can define a so-called trivial labeling map $(\mathcal{L}_q)_{id} := id: E_q^1 \to E_q^1$ which is the identity map on E_q^1 with the alphabet E_q^1. The labeling map naturally extends to any finite and infinite labeled paths, namely if $\lambda = \lambda_1 \cdots \lambda_n \in E_q^n$, then $\mathcal{L}_q(\lambda) := \mathcal{L}_q(\lambda_1) \cdots \mathcal{L}_q(\lambda_n) \in \mathcal{L}_q(E_q^n) \subset \mathcal{A}^*$, and similarly to infinite paths. We often call these labeled paths just paths for convenience if there is no risk of confusion, and use notation $\mathcal{L}_q^*(E_q) := \mathcal{L}_q(E_q^{\geq 1})$. For a vertex $v_q \in E_q^0$ and a vertex subset $A_q \subset E_q^0$, we set $\mathcal{L}_q(v_q) := v_q$ and $\mathcal{L}_q(A_q) := A_q$, respectively. A subpath $\alpha_i \cdots \alpha_j$ of $\alpha = \alpha_1\alpha_2 \cdots \alpha_{|\alpha|} \in \mathcal{L}_q^*(E_q)$ is denoted by $\alpha_{[i,j]}$ for $1 \leq i \leq j \leq |\alpha|$, and each $\alpha_{[1,j]}, 1 \leq j \leq |\alpha|$, is called an initial path of α. The range and source of a path $\alpha \in \mathcal{L}_q^*(E_q)$ are defined to be the following sets of vertices

$$r(\alpha) = \{r(\lambda) \in E_q^0 : \lambda \in E_q^{\geq 1}, \mathcal{L}_q(\lambda) = \alpha\},$$

$$s(\alpha) = \{s(\lambda) \in E_q^0 : \lambda \in E_q^{\geq 1}, \mathcal{L}_q(\lambda) = \alpha\},$$

and the relative range of $\alpha \in \mathcal{L}_q^*(E_q)$ with respect to $A_q \subset E_q^0$ is defined by

$$r(A_q, \alpha) = \{r(\lambda) : \lambda \in A_q E_q^{\geq 1}, \mathcal{L}_q(\lambda) = \alpha\}.$$

A collection $\mathcal{B}$ of subsets of E_q^0 is said to be closed under relative ranges for $(E_q,\mathcal{L}_q)$ if $r(A_q,\alpha)\in\mathcal{B}$ whenever $A_q\in\mathcal{B}$ and $\alpha\in\mathcal{L}_q^*(E_q)$. We call $\mathcal{B}$ an accommodating set for $(E_q,\mathcal{L}_q)$ if it is closed under relative ranges, finite intersections and unions and contains the ranges $r(\alpha)$ of all paths $\alpha\in\mathcal{L}_q^*(E_q)$. A set $A_q\in\mathcal{B}$ is called minimal (in $\mathcal{B}$) if $A_q\cap B_q$ is either A_q or $\emptyset$ for all $B_q\in\mathcal{B}$.

If $\mathcal{B}$ is accommodating for $(E_q,\mathcal{L}_q)$, the triple $(E_q,\mathcal{L}_q,\mathcal{B})$ is called an order labeled space. We say that an order labeled space $(E_q,\mathcal{L}_q,\mathcal{B})$ is set-finite (receiver set-finite, respectively) if for every $A_q\in\mathcal{B}$ and $k\geq 1$ the set $\mathcal{L}_q(A_qE_q^k)$ $(\mathcal{L}_q(E_q^kA_q)$, respectively) is finite. The order labeled space $(E_q,\mathcal{L}_q,\mathcal{B})$ is said to be weakly left-resolving if

$$r(A_q,\alpha)\cap r(B_q,\alpha)=r(A_q\cap B_q,\alpha)$$

holds for all $A_q,B_q\in\mathcal{B}$ and $\alpha\in\mathcal{L}_q^*(E_q)$. If $\mathcal{B}$ is closed under relative complements, we call $(E_q,\mathcal{L}_q,\mathcal{B}\ )$ a normal labeled space.

Notation 2.1. For $A_q\in\mathcal{B}$, we will use the following notation

$$A_q\sqcap\mathcal{B}:=\{B_q\in\mathcal{B}:B_q\subset A_q\}.$$

Assumptions. Throughout, we assume that graphs E_q have no sinks and sources, and the order labeled spaces $(E_q,\mathcal{L}_q,\mathcal{B})$ are weakly left-resolving, set-finite, receiver set-finite, and normal.

Definition 2.2 (see [23]). A representation of the order labeled space $(E_q,\mathcal{L}_q,\mathcal{B})$ is a family of order projections $\left\{p_{A_q}:A_q\in\mathcal{B}\right\}$ and order partial isometries $\left\{s_{a_q}:a_q\in\mathcal{A}\right\}$ such that for $A_q,B_q\in\mathcal{B}$ and $a_q,b_q\in\mathcal{A}$,

(i) $p_\emptyset=0, p_{A_q\cap B_q}=p_{A_q}p_{B_q}$, and $p_{A_q\cup B_q}=p_{A_q}+p_{B_q}-p_{A_q\cap B_q}$,

(ii) $p_{A_q}s_{a_q}=s_{a_q}p_{r(A_q,a_q)}$,

(iii) $s_{a_q}^*s_{a_q}=p_{r(a_q)}$ and $s_{a_q}^*s_{b_q}=0$ unless $a_q=b_q$,

(iv) $p_{A_q}=\sum_{a_q\in\mathcal{L}_q(A_qE_q^1)} s_{a_q}p_{r(A_q,a_q)}s_{a_q}^*$.

It is known [1],[3] that given the order labeled space $(E_q,\mathcal{L}_q,\mathcal{B})$, there exists a C^*-algebra $C^*(E_q,\mathcal{L}_q,\mathcal{B})$ generated by a universal representation $\left\{s_{a_q},p_{A_q}\right\}$ of $(E_q,\mathcal{L}_q,\mathcal{B})$, so that if $\left\{t_{a_q},\tilde{q}_{A_q}\right\}$ is a representation of $(E_q,\mathcal{L}_q,\mathcal{B})$ in a C^*-algebra B_q, there exists a $*$-homomorphism

$$\phi: C^*(E_q,\mathcal{L}_q,\mathcal{B})\to B_q$$

such that $\phi\left(s_{a_q}\right)=t_{a_q}$ and $\phi\left(p_{A_q}\right)=\tilde{q}_{A_q}$ for all $a_q\in\mathcal{A}$ and $A_q\in\mathcal{B}$.

Definition 2.3 (see [23]). We call the C^*-algebra $C^*(E_q,\mathcal{L}_q,\mathcal{B})$ generated by a universal representation of $(E_q,\mathcal{L}_q,\mathcal{B})$ the C^*-algebra of an order labeled space $(E_q,\mathcal{L}_q,\mathcal{B})$.

The C^*-algebra $C^*(E_q,\mathcal{L}_q,\mathcal{B})$ is unique up to isomorphism, and we simply write

$$C^*(E_q,\mathcal{L}_q,\mathcal{B})=C^*\left(s_{a_q},p_{A_q}\right)$$

to indicate the generators s_{a_q},p_{A_q} that are nonzero for all $a_q\in\mathcal{A}$ and $A_q\in\mathcal{B},A_q\neq\emptyset$.

Remark 2.4 [23]. Let $(E_q,\mathcal{L}_q,\mathcal{B})$ be an order labeled space with $C^*(E_q,\mathcal{L}_q,\mathcal{B})=C^*\left(s_{a_q},p_{A_q}\right)$. By ϵ, we denote a symbol (not in $\mathcal{L}_q^*(E_q)$) such that $a_q\epsilon=\epsilon a_q, r(\epsilon)=E_q^0$, and $r(A_q,\epsilon)=A_q$ for all $a_q\in\mathcal{A}$ and $A_q\subset E_q^0$. We write $\mathcal{L}_q^\#(E_q)$ for the union $\mathcal{L}_q^*(E_q)\cup\{\epsilon\}$. Let s_ϵ denote the unit of the multiplier algebra of $C^*(E_q,\mathcal{L}_q,\mathcal{B})$. Then one can easily check the following

$$\left(s_\alpha p_{A_q}s_\beta^*\right)\left(s_\gamma p_{B_q}s_\delta^*\right)=\begin{cases} s_{\alpha\gamma'}p_{r(A_q,\gamma')\cap B_q}s_\delta^*, & \text{if } \gamma=\beta\gamma' \\ s_\alpha p_{A_q\cap r(B_q,\beta')}^{*}s_{\delta\beta'}, & \text{if } \beta=\gamma\beta' \\ s_\alpha p_{A_q\cap B_q}s_\delta^*, & \text{if } \beta=\gamma \\ 0, & \text{otherwise}\end{cases}$$

for $\alpha,\beta,\gamma,\delta\in\mathcal{L}_q^\#(E_q)$ and $A_q,B_q\in\mathcal{B}$ (see [3, Lemma 4.4]). Since $s_\alpha p_{A_q}s_\beta^*\neq 0$ if and only if $A_q\cap r(\alpha)\cap r(\beta)\neq\emptyset$, we have

$$C^*(E_q,\mathcal{L}_q,\mathcal{B})=\overline{\operatorname{span}}\left\{s_\alpha p_{A_q}s_\beta^*:\alpha,\beta\in\mathcal{L}_q^\#(E_q)\text{ and }A_q\subseteq r(\alpha)\cap r(\beta)\right\}$$

For the order labeled graph $(E_q,\mathcal{L}_q)$, there are many accommodating sets to be considered to form an order labeled space, and the C^*-algebras $C^*(E_q,\mathcal{L}_q,\mathcal{B})$ are not necessarily isomorphic to each other, in general. By $\mathcal{E}$ we denote the smallest accommodating set for which $(E_q,\mathcal{L}_q,\mathcal{E})$ is an order normal labeled space. We are mostly interested in the C^*-algebras of these order labeled spaces $(E_q,\mathcal{L}_q,\mathcal{E})$ throughout.

For each $l \geq 1$, the relation $\sim_l$ on E_q^0 given by $v_q \sim_l w_q$ if and only if $\mathcal{L}_q(E_q^{\leq l} v_q) = \mathcal{L}_q(E_q^{\leq l} w_q)$ is an equivalence relation, and the equivalence class $[v_q]_l$ of $v_q \in E_q^0$ is called a generalized vertex (or a vertex simply). If $k > l$, then $[v_q]_k \subset [v_q]_l$ is obvious and $[v_q]_l = \cup_{i=1}^m [(v_q)_i]_{l+1}$ for some vertices $(v_q)_1, \dots, (v_q)_m \in [v_q]_l$ ([4, Proposition 2.4]). We have

$$\mathcal{E} = \left\{ \cup_{i=1}^n [(v_q)_i]_l : (v_q)_i \in E_q^0, l \geq 1, n \geq 0 \right\}$$

with the convention $\cup_{i=1}^0 [(v_q)_i]_l := \emptyset$ by [12].

Recall that a Cuntz-Krieger E_q-family for a graph E_q is a representation of the order labeled space $(E_q, (\mathcal{L}_q)_{id}, \mathcal{E})$ with the trivial labeling, and the Cuntz-Krieger uniqueness theorem for graph C^*-algebras says that if E_q satisfies Condition (L), then every Cuntz-Krieger E_q-family of nonzero operators generates the same C^*-algebra $C^*(E_q)$ up to isomorphism (see [5], [10], and [18]). A condition of an order labeled space corresponding to Condition (L) of a directed graph was suggested in [4] as below, and it is shown there in [4] that for a graph E_q, the order labeled space $(E_q, (\mathcal{L}_q)_{id}, \mathcal{E})$ with the trivial labeling is disagreeable if and only if E_q satisfies Condition (L).

Definition 2.5. ([4]) A path $\alpha \in \mathcal{L}_q^*(E_q)$ with $s(\alpha) \cap [v_q]_l \neq \emptyset$ is called agreeable for $[v_q]_l$ if $\alpha = \beta\alpha' = \alpha'\gamma$ for some $\alpha', \beta, \gamma \in \mathcal{L}_q^*(E_q)$ with $|\beta| = |\gamma| \leq l$. Otherwise α is called disagreeable. We say that $[v_q]_l$ is disagreeable if there is an $N \geq 1$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}_q(E_q^{\geq n})$ which is disagreeable for $[v_q]_l$.

The order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ is said to be disagreeable if for every $v_q \in E_q^0$, there is an $L_{v_q} \geq 1$ such that every $[v_q]_l$ is disagreeable for all $l \geq L_{v_q}$.

It is then natural to ask whether every loop in a disagreeable order labeled space must have an exit, which first leads us to try to seek a right definition for a loop in the order labeled space and then to work on whether the important results known for graph C^*-algebras $C^*(E_q)$ which involve the loop structure of E_q can be generalized to the order labeled graph C^*-algebras. We take the following definition and will see that every loop in a disagreeable order labeled space has an exit.

Definition 2.6. ([12]) Let $(E_q, \mathcal{L}_q, \mathcal{E})$ be an order labeled space. For a path $\alpha \in \mathcal{L}_q^*(E_q)$ and a nonempty set $A_q \in \mathcal{E}$, we call (α, A_q) a loop if

$$A_q \subset r(A_q, \alpha)$$

We say that a loop (α, A_q) has an exit if one of the following holds:

(I) there exists a path $\beta \in \mathcal{L}_q(A_q E_q^{\geq 1})$ such that $|\beta| = |\alpha|, \beta \neq \alpha$,

(II) $A_q \subsetneq r(A_q, \alpha)$.

In [8], the notion of cycle was introduced to define Condition $(L_{\mathcal{B}})$ for an order labeled space $(E_q, \mathcal{L}_q, \mathcal{B})$ (more generally for Boolean dynamical systems) which can be regarded as another condition analogous to Condition (L) for usual directed graphs.

Definition 2.7. ([8]) For $\alpha \in \mathcal{L}_q^*(E_q)$ and a nonempty $A_q \in \mathcal{E}$, the pair (α, A_q) is called a cycle if

$$B_q = r(B_q, \alpha) \text{ for all } B_q \in A_q \sqcap \mathcal{E}$$

Clearly every cycle is a loop, and if (α, A_q) is a cycle with an exit, then the exit must be of type (I).

If $(E_q, \mathcal{L}_q, \mathcal{E})$ is an order labeled space satisfying our standing assumptions and if, in addition, for each path $\alpha \in \mathcal{L}_q^*(E_q)$,

$$r(D_\alpha, \alpha) = r(\alpha) \text{ for some } D_\alpha \in \mathcal{E} \quad (1)$$

(that is, every path has a domain in $\mathcal{E}$), then the order labeled graph C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ can be regarded as a C^*-algebra associated to a Boolean dynamical system as discussed in [8]. A Boolean dynamical system on a Boolean algebra $\mathcal{B}$ is said to satisfy Condition $(L_{\mathcal{B}})$ if it has no cycle without an exit: if this is the case for the Boolean dynamical system induced from an order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$, we will simply say that $(E_q, \mathcal{L}_q, \mathcal{E})$ satisfies Condition $(L_{\mathcal{E}})$.

Theorem 2.8 below is the Cuntz-Krieger uniqueness theorem for order labeled graph C^*-algebras: if $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable, it satisfies Condition $(L_{\mathcal{E}})$ (see Proposition 3.2 or [14]). We need to understand Condition $(L_{\mathcal{E}})$ and disagreeability of order labeled spaces not only to answer the simplicity question of order labeled graph C^*-algebras, but also to be able to apply this useful uniqueness theorem for order labeled graph C^*-algebras.

Theorem 2.8 (see [23]). ([4], [8]) Let $\{t_{a_q}, \tilde{q}_{A_q}\}$ be a representation of an order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ such that $\tilde{q}_{A_q} \neq 0$ for all nonempty $A_q \in \mathcal{E}$. If $(E_q, \mathcal{L}_q, \mathcal{E})$ satisfies Condition $(L_{\mathcal{E}})$, in particular if $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable, then the canonical homomorphism $\phi: C^*(E_q, \mathcal{L}_q, \mathcal{E}) = C^*\left(s_{a_q}, p_{A_q}\right) \to C^*\left(t_{a_q}, \tilde{q}_{A_q}\right)$ such that $\phi\left(s_{a_q}\right) = t_{a_q}$ and $\phi\left(p_{A_q}\right) = \tilde{q}_{A_q}$ is an isomorphism.

Definition 2.9. ([14]) A subset H of $\mathcal{E}$ is hereditary if it is closed under subsets, finite unions, and relative ranges. A hereditary set H is saturated if $A_q \in H$ whenever $A_q \in \mathcal{E}$ and $r(A_q, \alpha) \in H$ for all $\alpha \in \mathcal{L}_q^*(E_q)$.

Let $\overline{\mathcal{L}_q(E_q^\infty)}$ be the set of all infinite sequences $x \in \mathcal{A}^{\mathbf{N}}$ such that every finite words of x occurs as an order labeled path in $(E_q, \mathcal{L}_q)$, namely

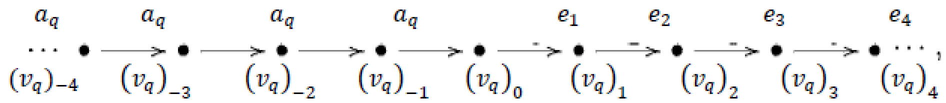
$$\overline{\mathcal{L}_q(E_q^\infty)} := \{x \in \mathcal{A}^{\mathbf{N}} \mid x_{[1,n]} \in \mathcal{L}_q(E_q^n) \text{ for all } n \geq 1\}.$$

Clearly $\mathcal{L}_q(E_q^\infty) \subset \overline{\mathcal{L}_q(E_q^\infty)}$, and the notation $\overline{\mathcal{L}_q(E_q^\infty)}$ comes from the fact that $\overline{\mathcal{L}_q(E_q^\infty)}$ is the closure of $\mathcal{L}_q(E_q^\infty)$ in the totally disconnected perfect space $\mathcal{A}^{\mathbf{N}}$ which is equipped with the topology that has a countable basis of open-closed cylinder sets $Z(\alpha) := \{x \in \mathcal{A}^{\mathbf{N}} : x_{[1,n]} = \alpha\}, \alpha \in \mathcal{A}^n, n \geq 1$ (see Section 7.2 of [17]).

Definition 2.10. ([14]) We say that the order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ is strongly cofinal if for each $x \in \overline{\mathcal{L}_q(E_q^\infty)}$ and $[v_q]_l \in \mathcal{E}$, there exist an $N \geq 1$ and a finite number of paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}_q^*(E_q)$ such that

$$r(x_{[1,N]}) \subset \cup_{i=1}^m r([v_q]_l, \lambda_i).$$

The above definition of a strongly cofinal order labeled space is stronger than the one given in [15]: for example, in the following order labeled space



if $x := a_q^\infty \in \overline{\mathcal{L}_q(E_q^\infty)} \setminus \mathcal{L}_q(E_q^\infty)$ and $N, n, l \geq 1$, then

$$r(x_{[1,N]}) = r(a_q^N) = r(a_q) = \{(v_q)_{-k} : k \geq 0\} \not\subset \cup_{i=1}^m r\left(\left[(v_q)_n\right]_l, \lambda_i\right)$$

for any paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}_q^*(E_q)$, namely the order labeled space is not strongly cofinal although it is in the sense of [15]. The result [15] can be improved as below with a slightly modified proof which we provide here.

Theorem 2.11 (see [23]). If $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple, then $(E_q, \mathcal{L}_q, \mathcal{E})$ is strongly cofinal.

Proof. Suppose to the contrary that there exist $[v_q]_l$ and $x \in \overline{\mathcal{L}_q(E_q^\infty)}$ such that

$$r(x_{[1,N]}) \not\subset \cup_{i=1}^m r([v_q]_l, \lambda_i) \tag{2}$$

for all $N \geq 1$ and any finite number of labeled paths $\lambda_1, \ldots, \lambda_m$. Let I be the ideal generated by the projection $p_{[v_q]_l}$ and let $p_{x_1} := p_{r(x_1)}$. Since $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple, we must have $p_{x_1} \in I$ and thus there is an element $\sum_{j=1}^m c_j \left(s_{\alpha_j} p_{(A_q)_j} s_{\beta_j}^*\right) p_{[v_q]_l} \left(s_{\gamma_j} p_{(B_q)_j} s_{\delta_j}^*\right)$ in I with $c_j \in \mathbb{C}$ such that

$$\left\| \sum_{j=1}^m c_j \left(s_{\alpha_j} p_{(A_q)_j} s_{\beta_j}^*\right) p_{[v_q]_l} \left(s_{\gamma_j} p_{(B_q)_j} s_{\delta_j}^*\right) - p_{x_1} \right\| < 1 \tag{3}$$

and the paths δ_i 's have the same length $|\delta_i| = N_0 \geq 1$. Then

$$\begin{aligned}
1 &> \left\| \sum_j c_j \left(s_{\alpha_j} p_{(A_q)_j} s_{\beta_j}^*\right) p_{[v_q]_l} \left(s_{\gamma_j} p_{(B_q)_j} s_{\delta_j}^*\right) - p_{x_1} \right\| \\
&\geq \left\| \sum_j c_j \left(s_{\alpha_j} p_{(A_q)_j} s_{\beta_j}^*\right) p_{[v_q]_l} \left(s_{\gamma_j} p_{(B_q)_j} s_{\delta_j}^*\right) p_{x_1} - p_{x_1} \right\| \\
&= \left\| \sum_j c_j \left(s_{\alpha_j} p_{(A_q)_j} s_{\beta_j}^*\right) p_{[v_q]_l} \left(s_{\gamma_j} p_{r([v_q]_l, \gamma_j) \cap (B_q)_j \cap r(x_1 \delta_j)} s_{\delta_j}^*\right) - p_{x_1} \right\|.
\end{aligned}$$

We first show that for each $j = 1, \ldots, m$,

$$r(x_1\delta_j) \subset \cup_{i=1}^m r([v_q]_l, \gamma_i) \quad (4)$$

If $r(x_1\delta_j) \not\subset \cup_{i=1}^m r([v_q]_l, \gamma_i)$ for some j, then $r(x_1\delta_j) \setminus \cup_{i=1}^m r([v_q]_l, \gamma_i) \neq \emptyset$ hence $p_j := p_{r(x_1\delta_j)\setminus\cup_{i=1}^m r([v_q]_l,\gamma_i)} \neq 0$. Then with $J := \{i \mid \delta_i = \delta_j\}$,

$$\begin{aligned}
1 &> \left\| \left(\sum_i c_i \left(s_{\alpha_i} p_{(A_q)_i} s_{\beta_i}^* \right) p_{[v_q]_l} \left(s_{\gamma_i} p_{r([v_q]_l,\gamma_i)\cap(B_q)_i\cap r(x_1\delta_i)} s_{\delta_i}^* \right) - p_{x_1} \right) s_{\delta_j} \right\| \\
&= \left\| \sum_{i\in J} c_i \left(s_{\alpha_i} p_{(A_q)_i} s_{\beta_i}^* \right) p_{[v_q]_l} s_{\gamma_i} p_{r([v_q]_l,\gamma_i)\cap(B_q)_i\cap r(x_1\delta_i)} - p_{x_1} s_{\delta_j} \right\| \\
&= \left\| \sum_{i\in J} c_i \left(s_{\alpha_i} p_{(A_q)_i} s_{\beta_i}^* \right) p_{[v_q]_l} s_{\gamma_i} p_{r([v_q]_l,\gamma_i)\cap(B_q)_i\cap r(x_1\delta_i)} - s_{\delta_j} p_{r(x_1\delta_j)} \right\| \\
&\geq \left\| \sum_{i\in J} c_i \left(s_{\alpha_i} p_{(A_q)_i} s_{\beta_i}^* \right) p_{[v_q]_l} s_{\gamma_i} p_{r([v_q]_l,\gamma_i)\cap(B_q)_i\cap r(x_1\delta_i)} p_j - s_{\delta_j} p_{r(x_1\delta_j)} p_j \right\| \\
&= \left\| s_{\delta_j} p_j \right\| = 1,
\end{aligned}$$

which is a contradiction and (4) follows. Also $\delta_i \neq x_{[2,N_0+1]}$ for each $1 \leq i \leq m$. In fact, if $\delta_i = x_{[2,N_0+1]}$ for some i, then by (4),

$$r(x_{[1,N_0+1]}) = r(x_1\delta_i) \subseteq \cup_{j=1}^m r([v_q]_l, \gamma_j),$$

which is not possible because of (2). Thus $s_{\delta_i}^* s_{x_{[2,N_0+1]}} = 0$ for $i = 1, \ldots, m$. Then the partial isometry $y := p_{x_1} s_{x_{[2,N_0+1]}} = s_{x_{[2,N_0+1]}} p_{r(x_{[1,N_0+1]})}$ is nonzero since $s_{x_{[2,N_0+1]}}^* y = p_{r(x_{[1,N_0+1]})} \neq 0$, and $s_{\delta_i}^* y = s_{\delta_i}^* p_{x_1} s_{x_{[2,N_0+1]}} = p_{r(x_1\delta_i)} s_{\delta_i}^* s_{x_{[2,N_0+1]}} = 0$ for all i. From (3), we have

$$1 > \left\| \left(\sum_{i=1}^m c_i \left(s_{\alpha_i} p_{(A_q)_i} s_{\beta_i}^* \right) p_{[]_l} \left(s_{\gamma_i} p_{(B_q)_i} \delta_{\delta_i}^* \right) \right) yy^* - p_{x_1} yy^* \right\| = \|yy^*\| = 1,$$

a contradiction, and we conclude that $(E_q, \mathcal{L}_q, \mathcal{E})$ is strongly cofinal.

3. Disagreeable order labeled spaces of simple order labeled graph C^*-algebras

We prove that simplicity of an order labeled graph C^*-algebra implies that the order labeled space is disagreeable. For this, we will use condition (c) of the following lemma which is equivalent to disagreeability of the order labeled space since the original definition of disagreeability seems a little complicated as recalled in Definition 2.5.

Lemma 3.1 (see [23]). ([14]) For an order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$, the following are equivalent:

(a) $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable.

(b) $[v_q]_l$ is disagreeable for all $v_q \in E_q^0$ and $l \geq 1$.

(c) For any nonempty set $A_q \in \mathcal{E}$ and a path $\beta \in \mathcal{L}_q^*(E_q)$, there is an $n \geq 1$ such that $\mathcal{L}_q\left(A_q E_q^{|\beta|n}\right) \neq \{\beta^n\}$.

If $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable, then it satisfies Condition $(L_\mathcal{E})$ as shown in [14] and [8]. But the converse is not true in general as we see from the following proposition.

Proposition 3.2 (see [23]). Consider the following conditions of an order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$.

(a) $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable.

(b) Every loop in $(E_q, \mathcal{L}_q, \mathcal{E})$ has an exit.

(c) $(E_q, \mathcal{L}_q, \mathcal{E})$ satisfies $(L_\mathcal{E})$, that is, every cycle has an exit.

Then $(a) \Rightarrow (b) \Rightarrow (c)$ hold. But the other implications are not true, in general.

Proof. $(b) \Rightarrow (c)$ is clear since every cycle is a loop.

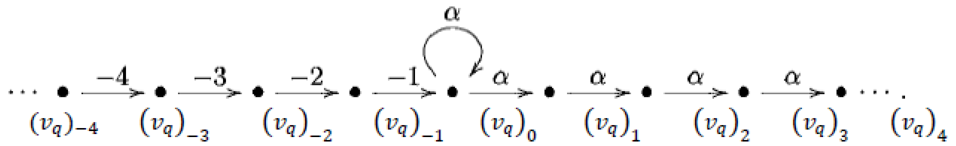
$(a) \Rightarrow (b)$ Let (A_q, α) be a loop so that $A_q \subset r(A_q, \alpha)$. If $A_q \subsetneq r(A_q, \alpha)$, then the loop has an exit of type (II). So we may assume that $A_q = r(A_q, \alpha)$. By Lemma 3.1, there is an $n \geq 1$ such that $\mathcal{L}_q(A_q E_q^{n|\alpha|}) \neq \{\alpha^n\}$. Choose $\beta \in \mathcal{L}_q(A_q E_q^{n|\alpha|})$ with $\beta \neq \alpha^n$. If $\beta_{[1,|\alpha|]} \neq \alpha$, then $\mathcal{L}_q(A_q E_q^{|\alpha|}) \neq \{\alpha\}$ and (A_q, α) has an exit of type (I). If $\beta := \alpha^k \delta$ for some $1 \leq k \leq n-1$ and $\delta \in \mathcal{L}_q^*(E_q)$ with $\delta_{[1,|\alpha|]} \neq \alpha$, then the loop (A_q, α) has an exit of type (I) since from $A_q = r(A_q, \alpha^k)$ we have

$$\alpha \neq \delta_{[1,|\alpha|]} \in \mathcal{L}_q(r(A_q, \alpha^k) E_q^{|\alpha|}) = \mathcal{L}_q(A_q E_q^{|\alpha|})$$

$(b) \nRightarrow (a)$ The order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ of the following order labeled graph is obviously not disagreeable while (b) is trivially satisfied since it has no loops.

$$\cdots \underset{(v_q)_{-4}}{\cdot} \xrightarrow{-4} (v_q)_{-3} \overset{-3}{\cdot} (v_q)_{-2} \overset{-2}{\cdot} (v_q)_{-1} \overset{-1}{\cdot} (v_q)_0 \xrightarrow{\alpha} \underset{(v_q)_1}{\cdot} \xrightarrow{\alpha} \underset{(v_q)_2}{\cdot} \xrightarrow{\alpha} \underset{(v_q)_3}{\cdot} \xrightarrow{\alpha} (v_q)_4 \cdots$$

$(c) \nRightarrow (b)$ and $(c) \nRightarrow (a)$ Note that the order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ of the following the order labeled graph, which is not disagreeable clearly, has loops $((A_q)_i, \alpha), i = 0,1$, where $(A_q)_0 = \{(v_q)_0\}$ and $(A_q)_1 := r(\alpha) = \{(v_q)_0, (v_q)_1, \ldots\}$. The loop $((A_q)_1, \alpha)$ has no exits while $((A_q)_0, \alpha)$ has an exit of type (II). Thus (b) is not satisfied for $(E_q, \mathcal{L}_q, \mathcal{E})$. But the order labeled space has no cycles and thus (c) is trivially satisfied.



Lemma 3.3 ([14]) (see [23]). Let the order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ have a loop (β, A_q) without an exit. If A_q is a minimal set in $\mathcal{E}$, then the C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ has a hereditary subalgebra which is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \geq 1$, in particular $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is not simple.

Lemma 3.4 (see [23]). For a nonempty set $A_q \in \mathcal{E}$, let

$$H_{A_q} := \{\cup_{i=1}^{k} C_i : k \geq 1, C_i \in r(A_q, \beta) \sqcap \mathcal{E}, \beta \in \mathcal{L}_q^{\#}(E_q)\}$$

Then H_{A_q} is a hereditary subset of $\mathcal{E}$ and

$$\bar{H}_{A_q} := \left\{B_q \in \mathcal{E} : \exists n \geq 1 \text{ such that } r(B_q, \alpha) \in H_{A_q} \text{ for all } \alpha \in \mathcal{L}_q(E_q^{\geq n})\right\}$$

is a hereditary saturated subset of $\mathcal{E}$ with $H_{A_q} \subset \bar{H}_{A_q}$.

Proof. It is easy to check that H_{A_q} is a hereditary set. For convenience, we write a number n in the definition of $\bar{H}_{A_q}$ for $B_q \in \mathcal{E}$ as n_{B_q} although it is not unique. Clearly $\bar{H}_{A_q}$ is closed under subsets. If $(A_q)_1, (A_q)_2 \in \bar{H}_{A_q}$, then $r((A_q)_1 \cup (A_q)_2, \alpha) = r((A_q)_1, \alpha) \cup r((A_q)_2, \alpha) \in H_{A_q}$ whenever $|\alpha| \geq \max\{n_{(A_q)_1}, n_{(A_q)_2}\}$. Hence $\bar{H}_{A_q}$ is closed under finite unions. Let $B_q \in \bar{H}_{A_q}$ and $|\sigma| \geq 1$. Then

$$r(r(B_q, \sigma), \alpha) = r(B_q, \sigma\alpha) \in H_{A_q}$$

whenever $|\alpha| \geq n_{B_q}$ because then $|\sigma\alpha| \geq n_{B_q}$. Thus $\bar{H}_{A_q}$ is also closed under relative ranges, which shows that $\bar{H}_{A_q}$ is a hereditary subset of $\mathcal{E}$. To see that $\bar{H}_{A_q}$ is saturated, let $B_q \in \mathcal{E}$ satisfy $r(B_q, \alpha) \in \bar{H}_{A_q}$ for all paths α with $|\alpha| \geq 1$. We have to show that $B_q \in \bar{H}_{A_q}$. Since our order labeled space is assumed to be set-finite, there are only finitely many order labeled edges, say $\delta_1, \ldots, \delta_k$, emitting out of B_q. Then $r(B_q, \delta_i) \in \bar{H}_{A_q}$ for each i, and thus there is an $n_i \geq 1$ such that $r(r(B_q, \delta_i), \alpha) \in H_{A_q}$ for all $\alpha \in \mathcal{L}_q(E_q^{\geq n_i})$. For $n := \max_{1 \leq i \leq k}\{n_i\}$, we then have $r(B_q, \delta_i\alpha) = r(r(B_q, \delta_i), \alpha) \in H_{A_q}$ whenever $|\alpha| \geq n$ and $1 \leq i \leq k$. This means that $r(B_q, \alpha) \in H_{A_q}$ for all α with $|\alpha| \geq n + 1$. Thus $B_q \in \bar{H}_{A_q}$ follows as desired.

Notation 3.5. For a path $\beta := \beta_1 \cdots \beta_{|\beta|} \in \mathcal{L}_q^*(E_q)$, let

$$\bar{\beta} := \beta\beta\beta \ldots$$

denote the infinite repetition of β, namely

$$\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 \cdots = \beta_1 \cdots \beta_{|\beta|}\beta_1 \cdots \beta_{|\beta|} \cdots$$

Then for each $j \geq 1$, we have $\bar{\beta}_j = \beta_k$ for some $1 \leq k \leq |\beta|$ with $k = j(\mathrm{mod}|\beta|)$. The initial path $\bar{\beta}_1 \cdots \bar{\beta}_j$ of $\bar{\beta}, j \geq 1$, is denoted by $\bar{\beta}_{[1,j]}$ as before.

We will call a path $\beta \in \mathcal{L}_q^*(E_q)$ irreducible if it is not a repetition of its proper initial path. The following Lemma 3.6 will be used to derive a contradiction in the proof of Theorem 3.7, but then we see from Theorem 3.7 that there does not exist an order labeled space $(E_q, \mathcal{L}_q, \mathcal{E})$ satisfying the assumptions of Lemma 3.6.

Lemma 3.6 (see [23]). Let $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ be a simple C^*-algebra and $(A_q)_0 \in \mathcal{E}$ be a nonempty set. If there exists an irreducible path β such that

$$\mathcal{L}_q((A_q)_0 E_q^{\geq 1}) = \{\bar{\beta}_{[1,n]} : n \geq 1\} = \{\beta^m\beta' : m \geq 0, \beta' \text{ is an initial path of } \beta\}, \quad (5)$$

or equivalently $\mathcal{L}_q\left((A_q)_0 E_q^{n|\beta|}\right) = \{\beta^n\}$ for all $n \geq 1$, then the following hold.

(i) There is an $N \geq 1$ such that for all $k \geq 1$,

$$r((A_q)_0, \bar{\beta}_{[1,N+k]}) \subset \cup_{j=1}^{N} r((A_q)_0, \bar{\beta}_{[1,j]}).$$

(ii) There is an $N_0 \geq 1$ such that for all $k \geq 1$,

$$r((A_q)_0, \beta^{N_0+k}) \subset \cup_{j=1}^{N_0} r((A_q)_0, \beta^j). \quad (6)$$

Moreover $A_q = r(A_q, \beta)$ for $A_q := \cup_{j=1}^{N_0} r((A_q)_0, \beta^j)$.

Proof. We will frequently use the following observation,

$$r((A_q)_0, \bar{\beta}_{[1,j]}) \cap r((A_q)_0, \bar{\beta}_{[1,k]}) \neq \emptyset \Rightarrow j = k(\mathrm{mod}|\beta|). \quad (7)$$

In fact, if $D := r((A_q)_0, \bar{\beta}_{[1,j]}) \cap r((A_q)_0, \bar{\beta}_{[1,k]}) \neq \emptyset$ for some $j \neq k(\mathrm{mod}|\beta|)$, then without loss of generality we can write

$$j := m|\beta| + j_0 \text{ and } k := n|\beta| + j_0 + r$$

for some $m, n \geq 0$ and $0 \leq j_0 < |\beta|, j_0 < j_0 + r < |\beta|$ (here we set $\bar{\beta}_{[1,0]} := \epsilon$), then from (5) we must have

$$\mathcal{L}_q\left(DE_q^{|\beta|}\right) = \{\beta_{[j_0+1,j_0+r]}\beta_{[j_0+r+1,|\beta|]}\beta_{[1,j_0]}\} = \{\beta_{[j_0+r+1,|\beta|]}\beta_{[1,j_0]}\beta_{[j_0+1,j_0+r]}\}.$$

But then the subpaths

$$\mu := \beta_{[j_0+1,j_0+r]} \text{ and } \nu := \beta_{[j_0+r+1,|\beta|]}\beta_{[1,j_0]}$$

of β satisfy $\mu\nu = \nu\mu$, which contradicts to irreducibility of β (see [14]).

(i) Since $\bar{H}_{(A_q)_0}$ is a nonempty hereditary saturated set by Lemma 3.4 and $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is simple, it follows from [16] that $\mathcal{E} = \bar{H}_{(A_q)_0}$. Suppose

$$r((A_q)_0, \bar{\beta}_{[1,n]}) \setminus \cup_{j=1}^{n-1} r((A_q)_0, \bar{\beta}_{[1,j]}) \neq \emptyset \quad (8)$$

for infinitely many $n \geq 1$. Then by (7), $r\left((A_q)_0, \beta^n\right) \not\subset \cup_{j=1}^{n-1} r\left((A_q)_0, \beta^j\right)$ for infinitely many n, which implies that $r(\beta^r) \notin H_{(A_q)_0}$ for all $r \geq 1$. In fact, if $r(\beta^r) = \cup_{i=1}^{k} C_i \in H_{(A_q)_0}$ with some $C_i \in r\left((A_q)_0, \bar{\beta}_{[1,m_i]}\right) \sqcap \mathcal{E}$, then each m_i must be a multiple of $|\beta|$ by (7) and thus for $m := \max_i\{m_i\}/|\beta|$ we have $r(\beta^r) \subset \cup_{i=1}^{m} r\left((A_q)_0, \beta^i\right)$. But then for all sufficiently large number $n > m|\beta|$,

$$r(\beta^n) \subset r(\beta^r) \subset \cup_{i=1}^{m} r\left((A_q)_0, \beta^i\right) \subset \cup_{j=1}^{n-1} r\left((A_q)_0, \bar{\beta}_{[1,j]}\right),$$

which is not possible by (8). Hence $r(\beta^r) \notin H_{(A_q)_0}$ for all $r \geq 1$, which then easily implies that $r(\beta^r) \notin \bar{H}_{(A_q)_0}$ for all $r \geq 1$. But this contradicts to $\bar{H}_{(A_q)_0} = \mathcal{E}$, and thus the left hand side of (8) must be empty for all but finitely many n 's. Therefore we see from (5) that there exists an $N \geq 1$ such that

$$r\left((A_q)_0, \bar{\beta}_{[1,n]}\right) \subset \cup_{j=1}^{n-1} r\left((A_q)_0, \bar{\beta}_{[1,j]}\right) \text{ for all } n \geq N.$$

Then $r\left((A_q)_0, \bar{\beta}_{[1,N+2]}\right) \subset \cup_{j=1}^{N+1} r\left((A_q)_0, \bar{\beta}_{[1,j]}\right) \subset \cup_{j=1}^{N} r\left((A_q)_0, \bar{\beta}_{[1,j]}\right)$ because $r\left((A_q)_0, \bar{\beta}_{[1,N+1]}\right) \subset \cup_{j=1}^{N} r\left((A_q)_0, \bar{\beta}_{[1,j]}\right)$, and actually an induction gives

$$r\left((A_q)_0, \bar{\beta}_{[1,N+k]}\right) \subset \cup_{j=1}^{N} r\left((A_q)_0, \bar{\beta}_{[1,j]}\right)$$

for all $k \geq 1$, which proves (i).

(ii) We can take $N = |\beta|N_0$, a multiple of $|\beta|$ in (i). Then N_0 satisfies (6) by (i) and (7) since (7) implies that for each $k \geq 1$,

$$r\left((A_q)_0, \bar{\beta}_{[1,N+k]}\right) \subset \bigcup_{\substack{1 \leq j \leq N \\ j = k(\mathrm{mod}|\beta|)}} r\left((A_q)_0, \bar{\beta}_{[1,j]}\right)$$

To show $A_q = r(A_q, \beta)$ for $A_q := \cup_{j=1}^{N_0} r((A_q)_0, \beta^j)$, first note from (6) that

$$A_q \supset r(A_q, \beta) \supset r(A_q, \beta^2) \supset \cdots$$

Suppose $B_q := A_q \setminus r(A_q, \beta) \neq \emptyset$. Then for $l > k \geq 1$,

$$r(B_q, \beta^k) \cap r(B_q, \beta^l) \subset r(B_q, \beta^k) \cap r(A_q, \beta^{k+1}) = r(B_q \cap r(A_q, \beta), \beta^k) = \emptyset.$$

Thus $r(B_q, \beta^n) \setminus \cup_{j=1}^{n-1} r(B_q, \beta^j) \neq \emptyset$ for infinitely many n. But this contradicts to (6) with B_q in place of $(A_q)_0$ since $\mathcal{L}_q\left(B_q E_q^{n|\beta|}\right) = \{\beta^n\}$ for all $n \geq 1$, and we conclude that $B_q = \emptyset$.

In the following Theorem 3.7, $(a) \Leftrightarrow (c)$ is known in [8] for the Boolean dynamical system induced from an order labeled space with the domain property (1).

Theorem 3.7 (see [23]). Let $(E_q, \mathcal{L}_q, \mathcal{E})$ be an order labeled space. Then the following are equivalent:

(a) $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ is a simple C^*-algebra.
(b) $(E_q, \mathcal{L}_q, \mathcal{E})$ is strongly cofinal and disagreeable.

Also these conditions imply the following.

(c) $(E_q, \mathcal{L}_q, \mathcal{E})$ has no cycles without exits and there is no proper hereditary saturated subsets in $\mathcal{E}$.

If $(E_q, \mathcal{L}_q, \mathcal{E})$ satisfies the domain condition (1), then (c) is equivalent to (a) and (b).

Proof. We only need to show that (a) implies that $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable.

Suppose $(E_q, \mathcal{L}_q, \mathcal{E})$ is not disagreeable. Then by Lemma 3.1, there exists a nonempty set $(A_q)_0 \in \mathcal{E}$ and a path $\beta \in \mathcal{L}_q^*(E_q)$ such that for all $n \geq 1$,

$$\mathcal{L}_q\left((A_q)_0 E_q^{|\beta|n}\right) = \{\beta^n\}$$

where we assume β to be irreducible. Choose an integer $N_0 \geq 1$ such that

$$r\big((A_q)_0, \beta^{N_0+k}\big) \subset \cup_{j=1}^{N_0} r\big((A_q)_0, \beta^j\big)$$

for all $k \geq 1$, which exists by Lemma 3.6(ii). Then for

$$A_q := \cup_{j=1}^{N_0} r\big((A_q)_0, \beta^j\big),$$

we have $A_q = r(A_q, \beta)$ by the same lemma. A simple computation shows that the hereditary subalgebra $p_{A_q} C^*(E_q, \mathcal{L}_q, \mathcal{E}) p_{A_q}$ of $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ generated by p_{A_q} is equal to

$$\mathrm{Her}\left(p_{A_q}\right) := \overline{\mathrm{span}}\left\{s_\mu p_{B_q} s_\nu^* : B_q \in r(A_q, \mu) \sqcap \mathcal{E}, \mu, \nu \in \beta_{[1,j]}^*, 0 \leq j \leq |\beta|\right\}$$

where we use notation

$$\beta_{[1,j]}^* := \{\beta^r \beta_{[1,j]} : r, j \geq 0\} \text{ with } \beta^0 := \epsilon =: \beta_{[1,0]}.$$

Let $(A_q)_1 \in A_q \sqcap \mathcal{E}$ be a nonempty subset. Then $\cup_{j=1}^{N} r\big((A_q)_1, \beta^j\big) \subset A_q$ for all $N \geq 1$ since $A_q = r(A_q, \beta)$, but one can actually show that there exists an $N_1 \geq 1$ such that

$$A_q = \cup_{j=1}^{N_1} r\big((A_q)_1, \beta^j\big). \tag{9}$$

In fact, an integer $N_1 \geq 1$ for which

$$\cup_{j=1}^{N_1} r\big((A_q)_1, \beta^j\big) = \cup_{j=1}^{\infty} r\big((A_q)_1, \beta^j\big)$$

holds (N_1 exists again by Lemma 3.6(ii)) satisfies (9) because otherwise one can easily show that

$$\emptyset \neq A_q \setminus \cup_{j=1}^{N_1} r\big((A_q)_1, \beta^j\big) \notin \bar{H}_{(A_q)_1}$$

which is a contradiction to simplicity of $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ (or to $\bar{H}_{(A_q)_1} = \mathcal{E}$).

Now we claim that $\mathrm{Her}\left(p_{(A_q)_1}\right) = \mathrm{Her}\left(p_{A_q}\right)$ for any nonempty subset $(A_q)_1 \in A_q \sqcap \mathcal{E}$. The hereditary subalgebra generated by $p_{(A_q)_1}$ is also equal to

$$\mathrm{Her}\left(p_{(A_q)_1}\right) = \overline{\mathrm{span}}\left\{s_\mu p_{B_q} s_\nu^* : B_q \in r\big((A_q)_1, \mu\big) \sqcap \mathcal{E}, \mu, \nu \in \beta_{[1,j]}^*, 0 \leq j \leq |\beta|\right\},$$

and for each positive element of the form $s_\mu p_{B_q} s_\mu^* \in \mathrm{Her}\left(p_{A_q}\right)$ with $B_q \in r(A_q, \mu) \sqcap \mathcal{E}$, the following computation

$$s_\mu p_{B_q} s_\mu^* \leq s_\mu p_{r(A_q,\mu)} s_\mu^* = s_\mu p_{r\left(\cup_{i=1}^{N_1} r((A_q)_1, \beta^i), \mu\right)} s_\mu^* \leq \sum_{i=1}^{N_1} s_\mu p_{r((A_q)_1, \beta^i \mu)} s_\mu^*$$

where we apply (9) for the second equality shows that $s_\mu p_{B_q} s_\mu^* \in \mathrm{Her}\left(p_{(A_q)_1}\right)$. Then for each $s_\mu p_{B_q} s_\nu^* \in \mathrm{Her}\left(p_{A_q}\right)$, the identity

$$s_\mu p_{B_q} s_\nu^* = \left(s_\mu p_{B_q} s_\mu^*\right) s_\mu p_{B_q} s_\nu^* \left(s_\nu p_{B_q} s_\nu^*\right)$$

proves that $s_\mu p_{B_q} s_\nu^* \in \mathrm{Her}\left(p_{(A_q)_1}\right)$ (for this, see [21]). Thus $\mathrm{Her}\left(p_{(A_q)_1}\right) = \mathrm{Her}\left(p_{A_q}\right)$ follows for any nonempty subset $(A_q)_1 \in A_q \sqcap \mathcal{E}$. However, this is not possible if A_q has a proper subset $(A_q)_1 \in A_q \sqcap \mathcal{E}$ since $p_{A_q} = p_{(A_q)_1} + p_{A_q \setminus (A_q)_1} \nsim p_{(A_q)_1}$. Hence A_q must be a minimal set. But then, by Lemma 3.3 the C^*-algebra $C^*(E_q, \mathcal{L}_q, \mathcal{E})$ contains a nonsimple hereditary subalgebra (isomorphic to (T)), and from this contradiction to simplicity of $C^*(E_q, \mathcal{L}_q, \mathcal{E})$, we finally conclude that $(E_q, \mathcal{L}_q, \mathcal{E})$ is disagreeable.

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