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Research Paper

On Orthogonality over Locally Compact Abelian Groups via Pair of Frames

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Abstract
Following the statements of the pioneers authors in [42], applying an application, who provide necessary and sufficient conditions for the orthogonality of two Bessel families when such families have the form of generalized translation invariant (GTI) systems over a second countable locally compact abelian (LCA) groupsG_r.A recent raise notion given by Jakobsen and Lemvig on GTI systems in $L^2(G_r)$, and the concept of the orthogonality of a pair of frames studied by Balan, Han, and Larson are considered.
Furthermore, [42] deduce similar results for several function systems including the case of TI systems, GTI systems on compact abelian groups,and results to the Bessel families having wave-packet structure (asort of combination of wavelet, Gabor structure), and hence obtained a characterization for pairwise orthogonal wave-packet frame systems over LCA groups. Hence, they relate the well-established theory from known literature with their results by observing several deductions of wavelet and Gabor systems over LCA groups with G_r equal to more respective spaces.

Keywords: *Generalized translation invariant system, locally compact abelian group, orthogonal frames,wave-packet system, wavelet system, Gabor system.*

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I. Introduction

The frame theory on locally compact abelian (LCA) groups has become the focus, both in theory and in applications, due to its potential to unify the continuous and discrete theories. Several researchers establishing the theory required to analyse frame properties on LCA groups (see [5], [6], [8], [11],[15], [17], [18], [28], [29], [31], [32], [41]). Among these properties, the "orthogonality or strongly disjointness" of a pair of frames in Hilbert spaces, and in multiple access communications, hiding data, and in synthesizing superframes and frames, etc. (see [19,20,22,23,36,41]). Moreover, [21] have used this property of frame and Bessel sequences as the essential ingredient to examine the density and connectedness results of the wavelet frames.

The orthogonality of frames was initially introduced and investigated by [22], and [3,4]. The property of a pair of frames says that the associated mixed dual Gramian operator is zero. The corresponding frames are termed as pairwise orthogonal (simply, orthogonal). Inspired by the wide applications of pairwise orthogonal frames, a lot of mathematicians and engineering specialists have contributed in developing different aspects of such frames over various function spaces (see $[18-20,22,23,31,36,41]$).

We study pairwise orthogonal frames for Hilbert spaces associated with LCA groups. Note that in the Euclidean case, [41] studied orthogonalframes of translates with various applications, and later, [31] discussed such frames in a shift-invariant (SI) space for $L^2(\mathbb{R}^d)$, while [36] described the orthogonality of a pair of discrete Gabor frames in $\ell^2(\mathbb{Z}^d)$.[42] give a characterization of pairwise orthogonal frames which arise from translations of generating functions by a countable family of closed, cocompact subgroups of a second countable LCA group G_r. They call such frames as pairwise orthogonal GTI frame systems. As an application of their characterization result, they get characterizations of the orthogonality of various structured frame systems such as wave-packet, wavelet, and Gabor frame systems over LCA-group setting.

We consider orthogonality of GTI frame systems, which is a class of systems introduced recently by [29]. The notion of GTI systems connects the well-established discrete frame theory of generalized shift-invariant (GSI) systems and its continuous version. Therefore, the characterization result on pairwise orthogonal

GTI frame systems represents a unified way to deduce similar results for several other function systems including the case of GSI systems studied by [32], and the translation invariant (TI) systems considered by [5]. In association with this, [32] presented a combined theory for many of the known function systems (e.g., Gabor systems and GSI systems on $\mathbb{R}^d$) by introducing the notion of GSI systems in the LCA-group setting. This approach is an extension of the theory of [24], and [39] on GSI systems in $L^2(\mathbb{R}^d)$.

Another significant fact which we wish to remark here is that among all function systems mentioned above, SI and GSI systems are based on translation along uniform lattices while TI and GTI systems, respectively, generalize the concept of SI and GSI systems for the continuous case by considering translation along co-compact subgroups of an LCA group. The motivation behind the consideration of co-compact subgroups for TI systems in [5] and GTI systems in [29] is related to the necessity of overcoming the limitation on the existence of uniform lattices for an LCA group, which says there exist LCA groups that do not contain any uniform lattices, for example, the $(1+\epsilon)$-adic numbers $\mathbb{Q}_{1+\epsilon}$, whose only discrete subgroup is the neutral element which is not a uniform lattice. Another example is the $(1+\epsilon)$-adic integers which have only trivial examples of uniform lattices but have a lot of non-trivial cocompact subgroups. Hence, the concept of co-compact subgroups in [5] and [29] generalizes the work on function systems with translation along uniform lattices considered in [6] and [32], respectively. On the study of above discussed general function systems, the investigation of frame properties for structured function systems (e.g., Gabor, wavelet. and shearlet systems) in different settings have got special attention due to their interesting theory and enormous applications in pure mathematics as well as in engineering areas such as signal processing, image processing, etc. (see [2,8,11,12,15,36]).[42] apply the characterization result on Bessel families with wave-packet, Gabor, and wavelet structure to deduce necessary and sufficient conditions for pairwise orthogonal wave-packet frames, Gabor frames, and wavelet frames over LCA groups.

We discussing basic properties about continuous frames for Hilbert spaces. Such frames were introduced independently by [1] and [30]. For a brief and self-sufficient introduction to continuous frames, see [14,38]

Definition 1.1 (see [42]). Let $\mathcal{H}$ be a complex Hilbert space, and let $(M,\sum_M,\mu_M)$ be a measure space, where $\sum_M$ denotes the σ -algebra and μ_M the non-negative measure. Then, a family of functions $\{f_m^r\}_{r,m\in M}$ in $\mathcal{H}$, is called a continuous frame for $\mathcal{H}$ with respect to $(M,\sum_M,\mu_M)$ if
(1) $m\mapsto f_m^r$ is weakly measurable; that is, for all $h_r\in\mathcal{H}$, the mapping $M\to\mathbb{C}, m\mapsto\langle h_r,f_m^r\rangle$ is measurable, and
(2) there exist constants $\epsilon\geq 0$ (called continuous frame bounds) such that

$$(1+\epsilon)\parallel h_r\parallel^2\leq\int_M|\langle h_r,f_m^r\rangle|^2 d\mu_M(m)\leq(1+2\epsilon)\parallel h_r\parallel^2\text{, for all } h_r\in\mathcal{H}\tag{1.1}$$

A continuous frame $\{f_m^r\}_{r,m\in M}$ is called tight if we can choose $\epsilon=0$, and Parseval if $\epsilon=0$. The family $\{f_m^r\}_{r,m\in M}$ is called Bessel with constant $(1+2\epsilon)$ as its Bessel constant if the right side of inequality in (1.1) holds. In this case, we say that the family $\{f_m^r\}_{r,m\in M}$ satisfies the Bessel condition. Since we deal with only separable Hilbert spaces, we can use Petti's theorem to replace weak measurability of $m\mapsto f_m^r$ with (strong) measurability with respect to the Borel algebra in $\mathcal{H}$. If μ_M is a counting measure and $M=\mathbb{N}$, then $\{f_m^r\}_{r,m\in M}$ reduces to a discrete frame. So continuous frames can be realized as the generalization of discrete frames. Now, we call continuous frames as just frames.
Given the family of functions $\mathbb{F}_r:=\{f_m^r\}_{r,m\in M}$, which is Bessel with respect to a measure space $(M,\sum_M,\mu_M)$, define the synthesis operator if $\Theta_{\mathbb{F}_r}: L^2(M,\mu_M)\to\mathcal{H}$ by

$$\langle\Theta_{\mathbb{F}_r}\varphi^r,h_r\rangle=\int_M\langle f_m^r,h_r\rangle\varphi_m^r d\mu_M(m),\varphi^r=\{\varphi_m^r\}_{r,m\in M}\in L^2(M,\mu_M),h_r\in\mathcal{H}$$

which is a well-defined, linear and bounded operator [38, Theorem 2.6]. Further, the adjoint of the synthesis operator, known as the analysis operator of $\mathbb{F}_r$, is defined by $\Theta_{\mathbb{F}_r}^*:\mathcal{H}\to L^2(M,\mu_M)$ with

$$(\Theta_{\mathbb{F}_r}^* h_r)(m)=\langle h_r,f_m^r\rangle, r,m\in M$$

Given two Bessel families $\mathbb{F}_r$ and $\mathbb{G}_r:=\{g_m^r\}_{r,m\in M}$ with respect to the measure space $(M,\sum_M,\mu_M)$ for $\mathcal{H}$, define the mixed dual Gramian operator corresponding to $\mathbb{F}_r$ and $\mathbb{G}_r$ as

$$\Theta_{G_r}\Theta_{F_r}^*:\mathcal{H}\to\mathcal{H}; h_r\mapsto\int_M\langle h_r,f_m^r\rangle g_m^r d\mu_M(m)\tag{1.2}$$

We define the orthogonality of a pair of Bessel families (frames):
Definition 1.2 (see [42]). Let $\mathbb{F}_r$ and $\mathbb{G}_r$ be Bessel families (frames) with respect to $(M,\sum_M,\mu_M)$ for $\mathcal{H}$. Then. if the mixed dual Gramian operator of $\mathbb{F}_r$ and $\mathbb{G}_r$ (as defined in (1.2)) is zero, that is, $\Theta_{G_r}\Theta_{\mathbb{F}_r}^*=0$, the Bessel families (frames) are said to be pairwise orthogonal (simply, orthogonal). In other words, we say that $\mathbb{F}_r$ and $\mathbb{G}_r$ satisfy the orthogonality property in this case.

We state some basic, notation and definitions on LCA groups. We characterize pairwise orthogonal GTI frame systems in $L^2(G_r)$ and deduce similar results for several function systems including the case of TI systems, GSI systems and GTI systems on compact abelian groups. We discuss applications of our characterization result on the Bessel families with wave-packet, Gabor, and wavelet structure over LCA groups.

We review some basic results from Fourier analysis on locally compact abelian (LCA) groups.

Here and throughout, let G_r denote a second countable locally compact abelian (LCA) group, with the additive group composition, denoted by the symbol " $+$ ", and the neutral element 0 . Note that the second countable property of G_r is equivalent to saying that G_r is metrizable and σ -compact. It is wellknown that on every LCA group G_r, there exists a Haar measure, that is, a non-negative, regular Borel measure, denoted as μ_{G_r} (not identically zero) which is translation invariant, i.e., $\mu_{G_r}(E+x)=\mu_{G_r}(E)$ for every element $x \in G_r$ and every Borel set $E \subseteq G_r$. This measure on any LCA group is unique up to a positive constant.

Denote by $\widehat{G}_r$, the set of all continuous characters, that is, all continuous homomorphisms from G_r into the torus $\mathbb{T} \cong \{z \in \mathbb{C}: |z|=1\}$. Then, under the pointwise multiplication $\widehat{G}_r$ forms an LCA group with unit element 1, that is called the dual group associated to G_r, when equipped with the compact convergence topology and the composition $(\gamma+\gamma')(x):=\gamma(x)\gamma'(x), \gamma, \gamma' \in \widehat{G}_r, x \in G_r$, and thus possesses a Haar measure that we denote by $\mu_{\widehat{G}_r}$ · It turns out that there exists a topological group isomorphism mapping the group$\widehat{G}_r$, that is, the dual group of $\widehat{G}_r$, onto G_r. More precisely, $\widehat{G}_{\mathrm{r}} \cong G_r$[13, Pontryagin duality theorem]. Note that if an LCA group G_r is discrete then $\widehat{G}_r$ is compact, and vice yersa.

Given an LCA group G_r with Haar measure μ_{G_r}, the integral over G_r is translation invariant in the sense that,

$$\int_{G_r} f^r(x+y)\, d\mu_{G_r}(x)=\int_{G_r} f^r(x)\, d\mu_{G_r}(x)$$

for each element $y \in G_r$ and for each Borel-measurable function f^r on G_r. For $0 \leq \epsilon<\infty$, we define the space $L^{1+\epsilon}(G_r, \mu_{G_r})$ (simply, $L^{1+\epsilon}(G_r)$) as follows:

$$L^{1+\epsilon}(G_r):=\left\{f^r: G_r \rightarrow \mathbb{C} \text{ is a measurable function and } \int_{G_r} |f^r(x)|^{1+\epsilon} d\mu_{G_r}(x)<\infty\right\}.$$

Since G_r is a second countable LCA group, $L^{1+\epsilon}(G_r)$ is separable, for all $0 \leq \epsilon<\infty$. In this article, we will focus only on $\epsilon=1$ case. Here, note that $L^2(G_r)$ is a Hilbert space with inner product given by

$$\langle f^r, g^r\rangle=\int_{G_r} f^r(x)\overline{g^r(x)}\, d\mu_{G_r}(x), \text{ for all } f^r, g^r \in L^2(G_r).$$

Let the Fourier transform $\widehat{\ }: L^1(G_r) \rightarrow C_0(\widehat{G}_r), f^r \mapsto \hat{f}^r$, be defined by the operator where $C_0(\widehat{G}_r)$ denotes the functions on $\widehat{G}_r$ vanishing at infinity. If $f^r \in L^1(G_r), \hat{f}^r \in L^1(\widehat{G}_r)$, and the measures on G_r and $\widehat{G}_r$ are normalized appropriately so that the Plancherel theorem holds, then the inverse Fourier transform can be defined by

$$f^r(x)=\mathcal{F}^{-1}\hat{f}^r(x)=\int_{\widehat{G}_r} \hat{f}^r(\xi^r)\xi^r(x)\, d\mu_{\widehat{G}_r}(\xi^r), x \in G_r$$

Note that the Fourier transform $\mathcal{F}$ can be extended from $L^1(G_r) \cap L^2(G_r)$ to a surjective isometry between $L^2(G_r)$ and $L^2(\widehat{G}_r)$[13, Plancherel theorem]. Thus, the Parseval formula holds and is given by

$$\langle f^r, g^r\rangle=\int_{G_r} f^r(x)\overline{g^r(x)}\, d\mu_{G_r}(x)=\int_{\widehat{G}_r} \hat{f}^r(\xi^r)\overline{g^r(\xi^r)}\, d\mu_{\widehat{G}_r}(\xi^r)=\langle\hat{f}^r, \widehat{g^r}\rangle, \text{ for all } f^r, g^r \in L^2(G_r)$$

The following definitions will be used in the sequel: Given G_r an LCA group, we call a subgroup Γ in G_r as co-compact if the quotient group G_r/Γ is compact, whereas Γ in G_r is said to be a uniform lattice if in addition, Γ is discrete. Let $\Gamma \subseteq G_r$ be a closed subgroup of an LCA group G_r. Then, the quotient G_r/Γ is a regular topological group. Further, we note that it is a second countable LCA group under the quotient topology by using the fact that G_r is second countable. For a subgroup Γ of an LCA group G_r, the annihilator $\Gamma^{\perp}$ of Γ is defined by

$$\Gamma^{\perp}:=\{\xi^r \in \widehat{G}_r: \xi^r(x)=1, \text{ for all } x \in \Gamma\}$$

It follows from the definition of the topology on $\widehat{G}_r$ that the annihilator $\Gamma^{\perp}$ is a closed subgroup in $\widehat{G}_r$. Moreover, if Γ is closed, then $(\Gamma^{\perp})^{\perp}=\Gamma$ and the following hold: there exists a topological group isomorphism mapping $\widehat{G_r/\Gamma}$ onto $\Gamma^{\perp}$, that is, $\widehat{G_r/\Gamma} \cong \Gamma^{\perp}$; there exists a topological group isomorphism mapping $\widehat{G}_r/\Gamma^{\perp}$ onto Γ, that is, $\widehat{G}_r/\Gamma^{\perp} \cong \Gamma$. For more information on harmonic analysis over locally compact abelian groups, we refer the reader to the classical books [13,26,27].

We study and characterize pairwise orthogonal generalized translation invariant (GTI) frame systems over locally compact abelian (LCA) groups and deduce similar results for the function systems related to GTI systems. The statement of our first main characterization result, that is, Theorem 3.5 (along with several deductions in Remark 3.6) is stated, while the proof for the characterization is discussed. Theorem 4.1 provides a characterization for pairwise orthogonal wave-packet frame systems over LCA groups. We shall discuss the definition of wave-packet systems over LCA groups along with the characterization result. We begin by considering GTI systems introduced by [29] along with the following definition and the standing hypotheses on it. Note that such systems model various discrete and continuous systems, e.g., the wavelet, shearlet and Gabor systems, etc.

Definition 3.1 (see [42]). Let $J \subset \mathbb{Z}$ be a countable index set. For each $j \in J$, let P_j be a countable or an uncountable index set, let $g^r_{j,1+\epsilon} \in L^2(G_r)$ for $(1+\epsilon) \in P_j$, and let Γ_j be a closed, co-compact subgroup in G_r. Then, the generalized translation invariant (GTI) system generated by $\{g^r_{j,1+\epsilon}\}_{(1+\epsilon)\in P_j, j\in J}$ with translation along closed, co-compact subgroups $\{\Gamma_j\}_{j\in J}$ is the family of functions given by $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$, where for $y \in G_r$, the operator T_y, called the translation by y, is defined as

$$T_y: L^2(G_r) \to L^2(G_r), (T_y f^r)(x) = f^r(x-y), x \in G_r.$$

Note that the GTI system in Definition 3.1 reduces to the translation invariant (TI) system discussed in [5], if $\Gamma_j = \Gamma$ for each $j \in J$. However, if each P_j is countable and each Γ_j is a uniform lattice in Definition 3.1, we arrive at the generalized shift-invariant (GSI) system considered in [24,32].

Standing Hypotheses:

For GTI systems considered in **Definition 3.1,** we assume that these systems satisfy the following criterion. Before proceeding, we introduce some notation. Let $\left(P_j, \sum_{P_j}, \mu_{P_j}\right)$ be a measure space for each $j \in J$, where $J \subset \mathbb{Z}$ is a countable index set. For a topological space X, by B_X, we denote the Borel algebra of X. By the symbol $P_j \times G_r$, we represent the product measure space formed by the Cartesian product of G_r with the measure space $P_j, \sum_{P_j} \otimes B_{G_r}$ denotes the tensor-product σ -algebra on $P_j \times G_r$ which is formed by the tensor-product of B_{G_r} with the σ -algebra $\sum_{P_j}$ on P_j, and the notation $\mu_{P_j} \otimes \mu_{G_r}$ specifies the product measure on $P_j \times G_r$. By assuming above notation, we state the conditions as follows. For each $j \in J$:

(1) $\left(P_j, \sum_{P_j}, \mu_{P_j}\right)$ is a σ -finite measure space,

(2) the mapping $(1+\epsilon) \mapsto g^r_{j,1+\epsilon}, \left(P_j, \sum_{(1+\epsilon)_j}\right) \to \left(L^2(G_r), B_{L^2(G_r)}\right)$ is measurable,

(3) the mapping $(1+\epsilon, x) \mapsto g^r_{j,1+\epsilon}(x)$, that is, $\left(P_j \times G_r, \sum_{P_j} \otimes B_{G_r}\right) \to (\mathbb{C}, B_{\mathbb{C}})$ is measurable.

Further, it is relevant to note that in order to investigate frame properties for GTI systems considered in Definition 3.1, we need to view the family of functions $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ in the set-up of continuous g-frames. Recall that these frames are a generalized version of continuous frames, more precisely, for a countable index set $J \subset \mathbb{Z}$, a family of functions $\cup_{j\in J}\{f^r_{j,m}\}_{r,m\in M_j}$ is a continuous generalized frame (continuous g-frame) for a complex Hilbert space $\mathcal{H}$ with respect to a collection of measure spaces $\left\{\left(M_j, \sum_{M_j}, \mu_j\right) : j \in J\right\}$ if$(\mathcal{C}_1) m \mapsto f^r_{j,m}, M_j \to \mathcal{H}$ is measurable for each $j \in J$, and $(\mathcal{C}_2)$ there exist constants $0 < 1+\epsilon \le 1+2\epsilon$ such that

$$(1+\epsilon) \parallel h_r \parallel^2 \le \sum_{j\in J} \int_{M_j} \left|\langle h_r, f^r_{j,m}\rangle\right|^2 d\mu_{Mj}(m) \le (1+2\epsilon)||h_r \parallel^2 \text{, for all } h_r \in \mathcal{H}$$

Note that all the definitions and operators associated to Definition 1.1 can be easily visualized for the case of continuous g-frames. For more details, see[20,40].

GTI System as a Continuous g-frame: In order to study pairwise orthogonal GTI frame systems, first we need to define GTI frame systems. The GTI system as a continuous g-frame is well-explained in [29], but we include the details here for the sake of completion. Hence, our next motive is to compare the GTI system $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ with the family of functions $\cup_{j\in J}\{f^r_{j,m}\}_{r,m\in M_j}$ considered in the above definition of continuous g-frame. Then, it follows that we can view the GTI system as a family of functions in $L^2(G_r)$ with respect to the collection of measure spaces $\left\{\left(M_j, \sum_{M_j}, \mu_j\right) := \left(P_j \times \Gamma_j, \sum_{P_j} \otimes B_{\Gamma_j}, \mu_{P_j} \otimes \mu_{\Gamma_j}\right) : j \in J\right\}$.

Next, to realize GTI system as a continuous g-frame, we first verify the condition $(\mathcal{C}_1)$. Let $j \in J$. Consider a function $F: P_j \times \Gamma_j \to L^2(G_r); (1+\epsilon, \gamma) \mapsto T_\gamma g^r_{j,1+\epsilon}$. The function F is continuous in γ and measurable in $(1+\epsilon)$, and hence represents a Carathéodory function $\tilde{F}$ which is defined on P_j by $\tilde{F}(1+\epsilon)(\gamma) = F(1+\epsilon, \gamma)$. Since $\Gamma_j \subset G_r$ is second countable and locally compact, and $L^2(G_r)$ is separable, it follows that $\tilde{F}$, and hence the function F, is jointly measurable on $\left(M_j, \sum_{M_j}\right) = \left(P_j \times \Gamma_j, \sum_{P_j} \otimes B_{\Gamma_j}\right)$ (for more details, see [37]). Thus, the condition $(\mathcal{C}_1)$ holds, and the GTI system $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ is automatically weakly measurable.

In addition, if the GTI system satisfies the condition $(\mathcal{C}_2)$ with respect to the set of measure spaces $\{(P_j \times$

$\Gamma_j, \sum_{P_j} \otimes B_{\Gamma_j}, \mu_{P_j} \otimes \mu_{\Gamma_j}\big) : j \in J\big\}$, we call $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ as the genemlized translation invariant frame system (GTI frame system) for $L^2(G_r)$. But, in case only the right side of inequality in the condition $(\mathcal{C}_2)$ holds, the system $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ is termed as a GTI Bessel system in $L^2(G_r)$. Similarly, we can define TI and GSI Bessel (frame) systems in $L^2(G_r)$ by replacing GTI system in the definition of GTI Bessel (frame) system with TI and GSI sytems, respectively.

It is known that pairwise orthogonal frames for a Hilbert space play a key role in constructing superframes and frames $[18-20,22,36,41]$, and also in developing the theory of frames and its applications (e.g., see [18,21,22,31]). We define such frames as GTI systems satisfying a special case of Definition 1.2 as follows:

Definition 3.2 (see [42]).Orthogonal GTI Bessel (frame) systems:
Let the systems $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ and $\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ be GTI Bessel (frame) systems in $L^2(G_r)$. Then, we term these systems as pairwise orthogonal GTI Bessel (frame) systems, or simply, orthogonal GTI Bessel (frame) systems in $L^2(G_r)$ if they satisfy the orthogonality property in the sense of Definition 1.2. In particular, by replacing GTI systems with TI systems and GSI systems, this definition corresponds to orthogonal TI Bessel (frame) systems and orthogonal GSI Bessel (frame) systems, respectively.

3.1. Statement of the characterization result.
We state the first main result, which provides a characterization for orthogonal GTI Bessel (frame) systems defined in Definition 3.2 over LCA group set-up.

We mention that for stating our main characterization result, we require some technical assumption in the form of a local integrability condition as follows. For the case of GSI systems, such condition was originally introduced by [24] for $L^2(\mathbb{R}^n)$, and later generalized by [32] for $L^2(G_r)$. This condition was further proposed in a more generalized form by [29] for GTI systems in $L^2(G_r)$. We state these conditions as follows:

Definition 3.3 (see [42]). Consider two GTI systems $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ and$\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ in $L^2(G_r)$.

(i) We say that $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ satisfies the local integrability condition (LIC) if

$$\sum_{j\in J}\int_{P_j}\sum_{\alpha_r\in\Gamma_j^\dagger}\int_{\operatorname{supp}\hat{f}^r}\left|\hat{f}^r(\xi^r+\alpha_r)\hat{g}^r_{j,1+\epsilon}(\xi^r)\right|^2 d\mu_{\hat{G}_r}(\xi^r)d\mu_{P_j}(1+\epsilon)<\infty,\text{ for all } f^r\in\mathfrak{D}\quad(3.1)$$

where for a Borel set B in $\hat{G}_r$ with $\mu_{\hat{G}_r}(\bar{B})=0$, we define the subset $\mathfrak{D}$ in $L^2(G_r)$ as follows:

$$\mathfrak{D}:=\{f^r\in L^2(G_r):\hat{f}^r\in L^\infty(\hat{G}_r)\text{ and supp }\hat{f}^r\text{ is compact in }\hat{G}_r\setminus B\}.$$

(ii) $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ and $\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ satisfy the dual α_r-local integmbility condition
(dual α_r-LIC) if

$$\sum_{j\in J}\int_{P_j}\sum_{\alpha_r\in\Gamma_j^\dagger}\int_{\hat{G}_r}\left|\hat{f}^r(\xi^r)\hat{f}^r(\xi^r+\alpha_r)\hat{g}^r_{j,1+\epsilon}(\xi^r)\widehat{(h_r)}_{j,1+\epsilon}(\xi^r+\alpha_r)\right|d\mu_{\hat{G}_r}(\xi^r)d\mu_{P_j}(1+\epsilon)<\infty,\text{ for }\in\mathfrak{D}\ (3.2)$$

In case $g^r_{j,1+\epsilon}=(h_r)_{j,1+\epsilon}$ for each j and $(1+\epsilon)$, we refer to (3.2) as the α_r-local integrability condition (α_r-LIC) for the GTI system $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$.

Note that the integrands in (3.1) and (3.2) are measurable on $P_j\times\hat{G}_r$, therefore, we are allowed to reorder sums and integrals in the local integrability conditions.

Further, note that the subset $\mathfrak{D}$ used in Definition 3.3 is dense in $L^2(G_r)$, and since it is sufficient to prove the various frame properties on a dense subset of the Hilbert space, we may verify our results for $\mathfrak{D}$ and then extend on $L^2(G_r)$ by a density argument.

Remark 3.4 [42]. In view of [29, Lemma 3.9], it is clear that
(i) LIC implies the α_r-LIC while the converse need not be true (e.g., see [29, Example 1]).
(ii) if two GTI systems satisfy the LIC, then they satisfy the dual α_r-LIC.

Next, we provide the statement of our first main characterization result on orthogonal frames. Here, we would like to add that the characterization results for orthogonal frames with the form of GTI systems, TI systems, and GSI systems discussed here are new. The results obtained here are also helpful in studying the orthogonality of special structured systems which lead to our second main result on orthogonal wave-packet Bessel (frame) systems, and hence for the case of wavelet and Gabor systems over LCA groups. Moreover, we can easily deduce the orthogonality conditions for a pair of frames in case of LCA group $G_r=\mathbb{R}^d,\mathbb{Z}^d$, etc. That means, our orthogonality results on GTI systems generalize the existing similar work done for the classical case (e.g., see [31,36,41]) as well.

Theorem 3.5 (see [42]). Let $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ and $\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ be GTI Bessel (frame) systems in $L^2(G_r)$ which satisfy the dual α_r-LIC. Then, the following assertions are equivalent:
(i) $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P,}$ and $\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ are orthogonal GTI Bessel (frame) systems in $L^2(G_r)$ in the sense of Definition 1.2,
(ii) for each $\alpha_r\in\cup_{j\in J}\Gamma_j^\perp\setminus\{0\}$, we have

$$\sum_{j\in J:\alpha_r\in\Gamma_j^\dagger}\int_{P_j}^{(\bar{h_r})_{j,1+\epsilon}(\xi^r)}\hat{g}^r_{j,1+\epsilon}(\xi^r+\alpha_r)d\mu_{P_j}(1+\epsilon)=0,\text{ for a.e. }\xi^r\in\hat{G}_r \tag{3.3}$$

and

$$\sum_{j\in J}\int_{P_j}\overline{\widehat{(h_r)}_{J,1+\epsilon}(\xi^r)}\hat{g}^r_{j,1+\epsilon}(\xi^r)d\mu_{P_j}(1+\epsilon)=0,\text{ for a.e. }\xi^r\in\hat{G}_r$$

We point out that Theorem 3.5 can be used to deduce similar results for several function systems since in these cases some of the assumptions trivially hold. We have the following observation:

Remark 3.6 [42]. We remark that Theorem 3.5 leads to the characterization results on the orthogonality of TI Bessel (frame) systems, GSI Bessel (frame) systems, and GTI Bessel (frame) systems (over a compact abelian group). For this, observe that
(i) in the case for TI Bessel systems, and for GTI Bessel systems over compact abelian groups, the dual α_r -LIC will be satisfied automatically in view of [29] and Remark 3.4.
(ii) for each j in J if we take P_j as countable and Γ_j as a uniform lattice in Theorem 3.5, then the orthogonality result for the case of GSI Bessel (frame) systems is obtained.

3.2. Proof of the characterization result.

We obtain a proof for Theorem 3.5, that gives necessary and sufficient conditions for two GTI systems to form orthogonal frames for $L^2(G_r)$ by following the Definition 1.2. For this, we have the next result.

Proposition 3.7 (see [42]). Suppose $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ and $\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ are GTI Bessel systems in $L^2(G_r)$ satisfying the dual α_r-LIC. Then, the following statements are equivalent:
(i) the mixed dual Gramian operator corresponding to the systems $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ and $\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ commutes with the family of translations $\{T_x\}_{x\in G_r}$,
(ii) for each $\alpha_r\in\cup_{j\in J}\Gamma_j^\perp\setminus\{0\}$,

$$t_{\alpha_r}(\xi^r):=\sum_{j\in J:a\in\Gamma_j^\dagger}\int_{P_j}\widehat{\overline{(h_r)}_{j,1+\epsilon}}(\xi^r)\hat{g}^r_{j,1+\epsilon}(\xi^r+\alpha_r)d\mu_{P_j}(1+\epsilon)=0,\text{ for a.e. }\xi^r\in\hat{G}_r \tag{3.4}$$

Moreover, if (i) or (ii) holds, then the mixed dual Gramian operator is a Fourier multiplier whose symbol is

$$t_{\alpha_r}(\xi^r)=\sum_{j\in J}\int_{P_j}\overline{(\hat{h}_r)_{J,1+\epsilon}(\xi^r)}\hat{g}^r_{j,1+\epsilon}(\xi^r)d\mu_P(1+\epsilon),\text{ for a.e. }\xi^r\in\hat{G}_r \tag{3.5}$$

We first remark that the equations (3.4) and (3.5) are well-defined which can be easily verified by using Cauchy-Schwartz inequality in the following computation:

$$\begin{aligned}\sum_{j\in J:\alpha_r\in\Gamma_j^\dagger}\int_{P_j}\left|(h_r)\widehat{_{j,1+\epsilon}}(\xi^r)\hat{g}^r_{j,1+\epsilon}(\xi^r+\alpha_r)\right|d\mu_{P_j}(1+\epsilon) &\le\sum_{j\in J}\int_{P_j}\left|\widehat{(h_r)}_{j,1+\epsilon}(\xi^r)\parallel\hat{g}^r_{j,1+\epsilon}(\xi^r+\alpha_r)\right|d\mu_{P_j}(1+\epsilon)\\ &\le\sum_{j\in J}\left(\int_{P_j}\left|\widehat{(h_r)}_{j,1+\epsilon}(\xi^r)\right|^2d\mu_{P_j}(1+\epsilon)\right)^{1/2}\left(\int_{P_j}\left|\hat{g}^r_{j,1+\epsilon}(\xi^r+\alpha_r)\right|^2d\mu_{P_j}(1+\epsilon)\right)^{1/2}\\ &\le\left(\sum_{j\in J}\int_{P_j}\left|\widehat{(h_r)}_{j,1+\epsilon}(\xi^r)\right|^2d\mu_{P_j}(1+\epsilon)\right)^{1/2}\left(\sum_{j\in J}\int_{P_j}\left|\hat{g}^r_{j,1+\epsilon}(\xi^r+\alpha_r)\right|^2d\mu_{P_j}(1+\epsilon)\right)^{1/2}\end{aligned}$$

and hence we can write

$$\sum_{j\in J:\alpha_r\in\Gamma_j^\dagger}\int_{P_j}\left|\overline{\widehat{(h_r)}_{J,1+\epsilon}(\xi^r)}\hat{g}^r_{j,1+\epsilon}(\xi^r+\alpha_r)\right|d\mu_{P_j}(1+\epsilon)\le\beta,\text{ for a.e. }\xi^r\in\hat{G}_r \tag{3.6}$$

Lemma 3.8 (see [42]). Suppose $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ and $\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_{j,}(1+\epsilon)\in P_j}$ satisfy the assumptions ofProposition 3.7, and assume that the symbol Θ is their corresponding mixed dual Gramian operator. For $f^r\in\mathfrak{D}$, define the function $w_{f^r}: G_r\to\mathbb{C}, x\mapsto\langle\Theta T_x f^r, T_x f^r\rangle$. Then, the following hold true: (i) The operator Θ commutes with all translations T_x for $x\in G_r$, if, and only if, w_{f^r} is constant for all $f^r\in\mathfrak{D}$, that means, $w_{f^r}(x)=w_{f^r}(0)=\langle\Theta f^r, f^r\rangle$ for all $x\in G_r$, where 0 denotes the neutral element of the LCA group G.

(ii) Let $f^r\in\mathfrak{D}$. Then, w_{f^r} is a continuous function that coincides pointwise with its absolutely convergent (almost periodic) Fourier series

$$w_{f^r}(x)=\sum_{\alpha_r\in U_{j\in J}\Gamma_j^\perp}\alpha_r(x)\hat{w}_{f^r}(\alpha_r) \tag{3.7}$$

where

$$\widehat{w}_{f^r}(\alpha_r):=\int_{G_r}\hat{f^r}(\xi^r)\bar{\hat{f^r}}(\xi^r+\alpha_r)t_{\alpha_r}(\xi^r)d\mu_{\bar{G}_r}(\xi^r) \tag{3.8}$$

and the last integral converges absolutely.

(iii) w_{f^r} is constant for all $f^r\in\mathfrak{D}$ if, and only if, for all $\alpha_r\in\cup_{j\in J}\Gamma_j^\perp\setminus\{0\}, t_{\alpha_r}(\xi^r)=0$ a.e. $\xi^r\in\hat{G}_r$.

Proof. (i) Let $\Theta T_x=T_x\Theta$, for all $x\in G_r$. Then, the direct part of (i) can be concluded by observing that

$$\begin{aligned} w_{f^r}(x) &=\langle\Theta T_x f^r, T_x f^r\rangle=\langle T_x\Theta f^r, T_x f^r\rangle\\ &=\langle\Theta f^r, T_x^*T_x f^r\rangle=\langle\Theta f^r, f^r\rangle,\end{aligned}$$

for all $x\in G_r$ and $f^r\in\mathfrak{D}$, since for each x, T_x is an unitary operator. Conversely, let w_{f^r} be constant for all $f^r\in\mathfrak{D}$. Then, for all $x\in G_r$,

$$w_{f^r}(x)=\langle\Theta T_x f^r, T_x f^r\rangle=\langle T_{-x}\Theta T_x f^r, f^r\rangle=\langle\Theta f^r, f^r\rangle,$$

which by using unitary nature of T_x for each x and polarization identity, leads to $T_{-x}\Theta T_x=\Theta$, and hence we get $\Theta T_x=T_x\Theta$.

(ii) For each $f^r\in\mathfrak{D}$ and $x\in G_r$, we can write the function

$$\begin{aligned} w_{f^r}(x)=\langle\Theta T_x f^r, T_x f^r\rangle &=\left\langle\sum_{j\in J}\int_{(1+\epsilon)\in P_j}\int_{\Gamma_j}\langle T_x f^r, T_\gamma(h_r)_{j,1+\epsilon}\rangle T_\gamma g^r_{j,1+\epsilon}d\mu_{\Gamma_j}(\gamma)d\mu_{P_j}(1+\epsilon), T_x f^r\right\rangle \qquad (*)\\ &=\sum_{j\in J}\int_{(1+\epsilon)\in P,\Gamma_j}\int\langle T_x f^r, T_\gamma(h_r)_{j,1+\epsilon}\rangle\langle T_\gamma g^r_{j,1+\epsilon}, T_x f^r\rangle d\mu_{\Gamma_j}(\gamma)d\mu_{P_j}(1+\epsilon).\end{aligned}$$

Now, by proceeding in the same way as in the proof of [29, Theorem 3.4], the result follows.

(iii) Given that dual α_r -LIC holds for all $f^r\in\mathfrak{D}$. From (3.7) and (*), it follows that

$$\begin{aligned} w_{f^r}(x) &=\sum_{j\in J}\int_{(1+\epsilon)\in P_j}\int_{\Gamma_j}\langle T_x f^r, T_\gamma(h_r)_{j,1+\epsilon}\rangle\langle T_\gamma g^r_{j,1+\epsilon}, T_x f^r\rangle d\mu_{\Gamma_j}(\gamma)d\mu_{P_j}(1+\epsilon)\\ &=\sum_{\alpha_r\in U_{j\in J}\Gamma_{j+}}\alpha_r(x)\widehat{w}_{f^r}(\alpha_r). \end{aligned}\tag{3.9}$$

Consider now the function $z_{f^r}(x):=w_{f^r}(x)-\langle\Theta f^r, f^r\rangle$ which is continuous in view of continuity of the function w_{f^r}

Now, for the direct part, assume that the function w_{f^r} is constant for all $f^r\in\mathfrak{D}$. We claim that $t_{\alpha_r}(\xi^r)=0$, for all $\alpha_r\in\cup\Gamma_j^\perp\setminus\{0\}$ and $a.e.\,\xi^r\in\hat{G}_r$. Here, note that by the construction, z_{f^r} is identical to the zero function. Additionally, sincew_{f^r} equals an absolute convergent generalized Fourier series, also z_{f^r} can be expressed as an absolute convergent generalized Fourier series $z_{f^r}(x)=\sum_{\alpha_r\in U\in J_j^\perp}\alpha_r(x)\hat{z}_{f^r}(\alpha_r)$, with

$$\hat{z}_{f^r}(\alpha_r)=\begin{cases}\widehat{w}_{f^r}(0)-\langle\Theta f^r, f^r\rangle, & \text{if } \alpha_r=0,\\ \widehat{w}_{f^r}(\alpha_r), & \text{if } \alpha_r\neq 0.\end{cases}$$

By the uniqueness theorem for generalized Fourier series [10, Theorem 7.12], the function $z_{f^r}(x)$ is identical to zero if, and only, if $\hat{z}_{f^r}(\alpha_r)=0$ for all $\alpha_r\in\cup_{j\in J}\Gamma_j^\perp$. Thus, for $\alpha_r\in\cup_{j\in J}\Gamma_j^\perp$ and $f^r\in\mathfrak{D}$, we have

$$\widehat{w}_{f^r}(\alpha_r)=\delta_{\alpha_r,0}\langle\Theta f^r, f^r\rangle. \tag{3.10}$$

Let $\alpha_r\neq 0$. Then, for all $f^r\in\mathfrak{D}$, (3.10) reduces to $\widehat{w}_{f^r}(\alpha_r)=0$, and hence we get

$$\int_\sigma\hat{f^r}(\xi^r)\bar{\hat{f^r}}(\xi^r+\alpha_r)t_{\alpha_r}(\xi^r)d\mu_{\hat{G}_r}(\xi^r)=0, \text{ for a.e. } \xi^r\in\hat{G}_r \tag{3.11}$$

Now, define the multiplication operator $M_{\bar{t}_a}: L^2(\hat{G}_r) \to L^2(\hat{G}_r)$ by $M_{\tau_a}\hat{f}^r(\xi^r) = \overline{t_{\alpha_r}(\xi^r)}\hat{f}^r(\xi^r)$ which is a bounded linear operator in view of the fact that $t_{\alpha_r}(\xi^r) \in L^\infty(\hat{G}_r)$ (for details, see (3.6)). For all $f^r \in \mathfrak{D}$ and a.e. $\xi^r \in \hat{G}_r$, we can now rewrite the term in left hand side of (3.11) as

$$\begin{aligned}\int_{G_r} \hat{f}^r(\xi^r)\overline{\overline{t_{\alpha_r}(\xi^r)}T_{\alpha_r}\hat{f}^r(\xi^r)}d\mu_{\widehat{G_r}}(\xi^r) &= \int_{\hat{G}_r} \hat{f}^r(\xi^r)\overline{M_{\bar{t}_{\alpha_r}}(T_{\alpha_r}\hat{f}^r)(\xi^r)}d\mu_{\widehat{G_r}}(\xi^r) \\ &= \int_{\hat{G}_r} \hat{f}^r(\xi^r)\overline{(M_{\overline{\tau_o}}T_{\alpha_r})\hat{f}^r(\xi^r)}d\mu_{\hat{G}_r}(\xi^r) \\ &= \left\langle \widehat{f^r}, M_{\bar{t}_{\alpha_r}}T_{\alpha_r}\widehat{f^r}\right\rangle_{L^2(\hat{G}_r)}\end{aligned}$$

which is equal to zero in view of (3.11). From the above equality and the fact that $\mathfrak{D}$ is dense in the complex Hilbert space $L^2(G_r)$, it follows that $M_{\bar{z}_a}T_{\alpha_r}\widehat{f^r} = 0$, which is if, and only if, $M_{\bar{a}}T_{\alpha_r} = 0$ means, $M_{\tau_a}T_{\alpha_r}(\hat{g}^r) = 0$ for all $\hat{g}^r \in L^2(\hat{G}_r)$, and hence $M_{\bar{t}_{\alpha_r}}T_{\alpha_r}\hat{g}^r(\xi^r) = \overline{t_{\alpha_r}(\xi^r)}T_{\alpha_r}\hat{g}^r(\xi^r) = 0$ for all $\hat{g}^r \in L^2(\hat{G}_r)$ and a.e. $\xi^r \in \widehat{G_r}$. Thus, (3.11) holds if, and only if, for a.e. $\xi^r \in \hat{G}_r$, we have $t_{\alpha_r}(\xi^r) = 0$, for all $\alpha_r \in \cup_{j\in J}\Gamma_j^\perp \setminus \{0\}$. Conversely, for each $\alpha_r \in \cup_{j\in J}\Gamma_j^\perp \setminus \{0\}$, let $t_{\alpha_r}(\xi^r) = 0$ for a.e. $\xi^r \in \widehat{G_r}$, which implies that $\widehat{w}_{f^r}(\alpha_r) = 0$, and by using this in (3.9) along with the fact from (3.10) that for $\alpha_r = 0$, we have $\hat{w}_{f^r}(\alpha_r) = \langle \Theta f^r, f^r\rangle$, and hence

$$w_{f^r}(x) = \sum_{\alpha_r \in U_\Gamma^\perp \setminus \{\{0\}} \alpha_r(x)\widehat{w}_{f^r}(\alpha_r) + \sum_{\alpha_r \in \{0\}} \alpha_r(x)\widehat{w}_{f^r}(\alpha_r) = 0 + \widehat{w}_{f^r}(0) = \langle \Theta f^r, f^r\rangle,$$

for a.e. $x \in G_r$. Therefore, w_{f^r} is constant for all $f^r \in \mathfrak{D}$.

Proof of Proposition 3.7 (see [42]). Clearly, part (i) is true if, and only if, (3.4) holds in view of Lemma 3.8. Further, it is well-known that if the mixed dual Gramian operator, say Θ, commutes with T_x for all $x \in G_r$, then it is a Fourier multiplier (see [33, Theorem 4.1.1]), and hence there exists a unique $s \in L^\infty(\hat{G}_r)$ such that $\widehat{\Theta f}^r(\xi^r) = s(\xi^r)\widehat{f^r}(\xi^r)$, where $s(\xi^r)$ represents the symbol corresponding to Θ. Now, for a.e. $\xi^r \in \hat{G}_r$, we are interested in finding the expression for $s(\xi^r)$. For this, observe that

$$\langle \Theta f^r, f^r\rangle = \langle \widehat{\Theta f}^r, \hat{f}^r\rangle_{L^2(\hat{G}_r)} = \int_{\tilde{G}_r} \widehat{\Theta f}^r(\xi^r)\overline{f^r(\xi^r)}d\mu_{\hat{G}_r}(\xi^r) = \int_{\tilde{G}_r} s(\xi^r)\hat{f}^r(\xi^r)\overline{f^r(\xi^r)}d\mu_{\hat{G}_r}(\xi^r). \quad (3.12)$$

Moreover, for $\alpha_r = 0$, it follows from (3.8) and (3.10) that for all $f^r \in \mathfrak{D}$,

$$\langle \Theta f^r, f^r\rangle = \widehat{w}_{f^r}(0) = \int_{G_r} \hat{f}^r(\xi^r)\bar{f}^r(\xi^r)\sum_{j\in J}\int_{P_j} \widehat{(h_r)_{J,1+\epsilon}}(\xi^r)\hat{g}^r_{j,1+\epsilon}(\xi^r)d\mu_{P_j}(1+\epsilon)d\mu_{\hat{G}_r}(\xi^r) \quad (3.13)$$

Since (3.12) and (3.13) are valid for all $f^r \in \mathfrak{D}$ and s is unique, it is clear that the symbol of Θ_i that is, $s(\xi^r) = \sum_{j\in JP_j}\int \overline{\widehat{(h_r)}_{J,1+\epsilon}(\xi^r)}\hat{g}^r_{j,1+\epsilon}(\xi^r)d\mu_{P_j}(1+\epsilon)$.

Now, we prove our first main result, Theorem 3.5:

Proof of Theorem 3.5. By Definition 1.2, the part (i) is equivalent to saying that the mixed dual Gramian operator corresponding to the GTI systems $\cup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_{j,(1+\epsilon)\in P_j}}$ and$\cup_{j\in J}\{T_\gamma (h_r)_{j,1+\epsilon}\}_{\gamma\in\Gamma_{j,(1+\epsilon)\in P_j}}$, say Θ, is equal to zero. Next, we claim that $\Theta = 0$ if, and only if, Θ commutes with the translations T_x for all $x \in G_r$, and, act as a Fourier multiplier with symbol

$$s(\xi^r) = \sum_{j\in J}\int_{P_j} \overline{\widehat{(h_r)}_{J,1+\epsilon}(\xi^r)}\hat{g}^r_{j,1+\epsilon}(\xi^r)d\mu_{P_j}(1+\epsilon) = 0, \text{ for a.e. } \xi^r \in \hat{G}_r$$

For proving the above claim, let $\Theta = 0$. Then, $\Theta T_x(f^r) = 0$, for all $x \in G_r$ and $f^r \in L^2(G_r)$. Since for each x, the translation T_x is a linear operator, therefore $T_x(0) =$ zero function in $L^2(G_r) = 0$, and hence $T_x\Theta f^r = T_x(0) = 0$, which implies that $\Theta T_x = T_x\Theta$ for all $x \in G_r$. Thus by [33, Theorem 4.1.1], Θ is a Fourier multiplier. So for all $f^r \in L^2(G_r)$ we have $0 = \widehat{\Theta f}^r(\xi^r) = s(\xi^r)\hat{f}^r(\xi^r), \xi^r \in \hat{G}_r$ a.e., where $s(\xi^r)$, the symbol of Θ as a Fourier multiplier, is given by (3.5). Conversely, if Θ is a Fourier multiplier with symbol $s(\xi^r) = 0$, then $\widehat{\Theta}f^r(\xi^r) = 0$, which implies that $\Theta f^r = 0$ for all $f^r \in L^2(G_r)$, and hence $\Theta = 0$. Now, the result follows by considering the above claim along with Proposition 3.7.

We discuss applications of our first main result to the Bessel families having wave-packet structure, which are obtained by applying certain collections of dilations, modulations, and translations to a countable family of functions in $L^2(G_r)$. As a consequence, we obtain results for wavelet and Gabor systems. Along with this, we connect the already existing results with the theory discussed by providing various examples in case of $G_r = \mathbb{R}^d, \mathbb{Z}^d$, etc.

4.1. Wave-Packet Systems.

For a given second countable LCA group G_r, let Epi (G_r), Epick (G_r), and Aut (G_r) denote respectively, the semigroup of continuous group homomorphisms α_r from G_r onto G_r, the semigroup of $\alpha_r \in$ Epi (G_r) having compact kernel ker α_r, and the group of topological automorphismsα_r of G_r onto itself. Note that Aut $(G_r) \subset$

Epick $(G_r) \subset$ Epi (G_r). For $\alpha_r \in$ Epick (G_r), we define the isometric dilation operator D_{α_r} by

$$D_{\alpha_r}: L^2(G_r) \to L^2(G_r); D_{\alpha_r} f^r(x) = (\Delta(\alpha_r))^{-1/2} f^r(\alpha_r(x)), \text{ for all } x \in G_r$$

where the modular function Δ: Epick $(G_r) \to (0,\infty)$ is a semigroup homomorphism such that

$$\int_{G_r} (g^r \circ \alpha_r)(x) d\mu_{G_r}(x) = \Delta(\alpha_r) \int_{G_r} g^r(x) d\mu_{G_r}(x)$$

for all integrable functions g^r on G_r with respect to the Haar measure μ_{G_r} (see [5, Theorem 6.2]). For a character χ in $\widehat{G}_r$, we define the modulation opemtor M_X on $L^2(G_r)$ as

$$M_\chi(f^r)(x) = \chi(x) f^r(x), \text{ for all } x \in G_r$$

and observe that for each $\chi \in \widehat{G}_r$, it is associated with the translation operator on $L^2(\widehat{G}_r)$ by the relation

$$\left(\widehat{M_\chi f^r}\right)(\xi^r) = \int_{G_r} \chi(x) f^r(x) \overline{\xi^r(x)} d\mu_G(x)$$

$$= \int_{G_r} f^r(x) \overline{(\xi^r - \chi)(x)} d\mu_{G_r}(x) = \hat{f}^r(\xi^r - \chi) = T_\chi \hat{f}^r(\xi^r), \quad (4.1)$$

for all $f^r \in L^2(G_r)$ and a.e. $\xi^r \in \widehat{G}_r$. Further, note that for each $\alpha_r \in$ Epick (G_r), the dilation operator on $L^2(G_r)$ satisfies the following relation (see [5, Lemma 6.6]) :

$$\left(\widehat{D_{\alpha_r} f^r}\right)(\chi) = \begin{cases} (\Delta(\alpha_r))^{\frac{1}{2}} \hat{f}^r(\beta^{-1}(\chi)) & \text{for } \chi \in \beta(\widehat{G}_r) = (\ker\ \alpha_r)^\perp \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

for all $f^r \in L^2(G_r)$, where by $\beta := \alpha_r^*$, we denote the adjoint of $\alpha_r \in$ Epick (G_r) which is a topological isomorphism $\beta: \widehat{G}_r \to (\ker\ \alpha_r)^\perp; \chi \mapsto \chi \circ \alpha_r$ in view of [5, Proposition 6.5].

Let $\mathcal{A}$ be a subset of Epick (G_r), let Γ and Λ be co-compact subgroups of G_r and $\widehat{G}_r$, respectively, and for some index set $J \subset \mathbb{Z}$, let $\Psi := \{\psi_j : j \in J\}$ be a subset of $L^2(G_r)$. Then, we define the wave-packet system generated by Ψ as:

$$\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda) := \{D_{\alpha_r} T_\gamma M_\chi \psi_j : \alpha_r \in \mathcal{A}, \gamma \in \Gamma, \chi \in \Lambda, j \in J\}. \quad (4.3)$$

In the case of $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^d)$, the systems of the above form have been studied by several authors, see[7,25,34]. The wave-packet systems were originally introduced by [9], and the collection defined in (4.3) generalizes the notion of such systems of LCA groups. In particular, the wavelet and Gabor systems can be seen as special cases of (4.3) which we shall discuss.

The following commutator relation helps in representing the collection (4.3) in the form of a GTI system. Now for each $\alpha_r \in \mathcal{A}, \gamma \in \Gamma, \chi \in \Lambda$, and $j \in J$, we have:

$$\begin{aligned} D_{\alpha_r} T_\gamma M_\chi \psi_j(x) &= (\Delta(\alpha_r))^{-1/2} T_\gamma M_\chi \psi_j(\alpha_r(x)) = (\Delta(\alpha_r))^{-1/2} M_\chi \psi_j(\alpha_r(x) - \gamma) \\ &= (\Delta(\alpha_r))^{-1/2} M_\chi \psi_j(\alpha_r(x - \gamma_1)) = D_{\alpha_r} M_\chi \psi_j(x - \gamma_1) = T_{\gamma_1} D_{\alpha_r} M_\chi \psi_j(x) \end{aligned}$$

for all $x \in G_r$, and for some $\gamma_1 \in \alpha_r^{-1}\Gamma$ such that $\alpha_r(\gamma_1) = \gamma$. Let $\mathcal{A}$ be a countable subset of Epick (G_r). Then, by using the above commutator relation, the wave-packet system $\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda)$ will represent a GTI system of the form $\cup \{T_\gamma g^r_{\alpha_r,1+\epsilon}\}_{\gamma \in \Gamma_{a,1+\epsilon}(1+\epsilon) \in P_a}$ for $\Gamma_{\alpha_r} := \alpha_r^{-1}\Gamma$ with $\alpha_r \in \mathcal{A}, g^r_{\alpha_r,1+\epsilon} = g^r_{\alpha_r,(j,x)} = D_{\alpha_r} M_x \psi_j$ for $(\alpha_r, 1+\epsilon) = (\alpha_r, (j,\chi))$in $\mathcal{A} \times (J \times \Lambda)$. In this case, for each $\alpha_r \in \mathcal{A}$, the measure space $P_{\alpha_r} := \{(j,\chi): j \in J, \chi \in \Lambda\}$ isequipped with the measure $\mu_{P_a} := \mu_{J \times \Lambda} = (\Delta(\alpha_r))^{-1}(\mu_J \otimes \mu_\Lambda)$, where the quantity $(\Delta(\alpha_r))^{-1}$ helps inavoiding the scaling factor in the calculations and μ_J represents the counting measure on J. Clearly, the measure μ_{P_a} is σ -finite. Here, note that $\Gamma_{\alpha_r} = \alpha_r^{-1}\Gamma$ is a closed co-compact subgroup of G_r for each $\alpha_r \in \mathcal{A}$, in view of [5, Proposition 6.4] and the fact that α_r is a continuous group homomorphism from G_r onto G_r along with Γ as a closed subgroup of G_r.

Next, we apply Theorem 3.5 to the wave-packet systems $\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda)$ and $\mathcal{W}(\Phi, \mathcal{A}, \Gamma, \Lambda)$, where for any index set $J \subset \mathbb{Z}, \Psi := \{\psi_j\}_{j \in J}$ and $\Phi := \{\varphi_j^r\}_{j \in J}$ are subsets in $L^2(G_r)$. Further, we simplify (3.3) by considering $\mathcal{W}(\Psi, \mathcal{A}, \Gamma, \Lambda)$ and $\mathcal{W}(\Phi, \mathcal{A}, \Gamma, \Lambda)$ as GTI systems $\cup \{T_\gamma g^r_{\alpha_r,1+\epsilon}\}_{\gamma \in \Gamma_{a},(1+\epsilon) \in P_a}$ and $\cup_{\alpha_r,A} \{T_\gamma (h_r)_{\alpha_r,1+\epsilon}\}_{\gamma \in \Gamma_a,(1+\epsilon) \in P_a}$, respectively, where $g^r_{\alpha_r,1+\epsilon} = g^r_{\alpha_r,(j,x)} = D_{\alpha_r} M_x \psi_j$ and $(h_r)_{\alpha_r,1+\epsilon} = (h_r)_{o,(j,x)} = D_{\alpha_r} M_\chi \varphi_j^r$ for $(\alpha_r, 1+\epsilon) = (\alpha_r, (j,\chi)) \in \mathcal{A} \times P_{\alpha_r} = \mathcal{A} \times (J \times \Lambda)$. Hence, for each $\tilde{\alpha}_r \in \cup_{a \in \mathcal{A}} \Gamma_{\alpha_r}^\perp$ and for a.e. $\xi^r \in \cup_{a \in \mathcal{A}} (\ker\ \alpha_r)^\perp$, the expression (3.3) takes the following form in view of (4.1) and (4.2) along with $\beta = \alpha_r^*$:

$$\begin{aligned}\mathcal{T}_{\tilde{\alpha}_r}(\xi^r) &:= \sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha}_r\in\Gamma_{\alpha_r}}\int_{P_{\alpha_r}}\overline{\widehat{(h_r)}_{\alpha_r,1+\epsilon}(\xi^r)}\hat{g}^{\mathrm{r}}_{\alpha_r,1+\epsilon}(\xi^r+\tilde{\alpha}_{\mathrm{r}})d\mu_{P_{\alpha_r}}(1+\epsilon)\\ &= \sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha}_r\in\Gamma^{\perp}_{\tilde{\alpha}_r}}\int_{J\times\Lambda}\overline{\widehat{(h_r)}_{\alpha_r,(j,\chi)}(\xi^r)}\hat{g}^{\mathrm{r}}_{\alpha_r,(j,\chi)}(\xi^r+\tilde{\alpha}_{\mathrm{r}})d\mu_{J\times\Lambda}((j,\chi))\\ &= \sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha}_r\in(\alpha_{\mathrm{r}}^{-1}\Gamma)}\sum_{j\in J}\int_{\Lambda}\overline{\left(\widehat{D_{\alpha_r}M_\chi\varphi_j^r}\right)(\xi^r)}\left(\widehat{D_{\alpha_r}M_\chi\psi_j}\right)(\xi^r+\tilde{\alpha}_{\mathrm{r}})\frac{1}{\Delta(\alpha_r)}d\mu_\Lambda(\chi)\\ &= \sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha}_r\in\beta\Gamma^{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{\widehat{M_\chi\varphi_j^r}(\beta^{-1}\xi^r)}\widehat{M_\chi\psi_j}(\beta^{-1}(\xi^r+\tilde{\alpha}_{\mathrm{r}}))d\mu_\Lambda(\chi)\\ &= \sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha}_r\in\beta\Gamma^{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{T_\chi\hat{\varphi}_j^r(\beta^{-1}\xi^r)}T_\chi\hat{\psi}_j(\beta^{-1}(\xi^r+\tilde{\alpha}_{\mathrm{r}}))d\mu_\Lambda(\chi)\\ &= \sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha}_r\in\beta\Gamma^{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{\hat{\varphi}_j^r(\beta^{-1}\xi^r-\chi)}\hat{\psi}_j(\beta^{-1}(\xi^r+\tilde{\alpha}_{\mathrm{r}})-\chi)d\mu_\Lambda(\chi) =: \tilde{\mathcal{T}}_{\tilde{\alpha}_r}(\xi^r) \text{ (say)},\end{aligned}$$

whereas for the case of $\xi^r\in\hat{G}_{\mathrm{r}}\setminus\bigcup_{\alpha_r\in\mathcal{A}}(\ker\ \alpha_r)^{\perp}$ a.e., we get $\mathcal{T}_{\tilde{\alpha}_r}(\xi^r)=0$ by proceeding in the similar way as above. Hence, we can write

$$\mathcal{T}_{\tilde{\alpha}_r}(\xi^r)=\begin{cases}\tilde{\mathcal{T}}_{\tilde{\alpha}_r}(\xi^r) & \text{for a.e. } \xi^r\in\bigcup\limits_{\alpha_r\in\mathcal{A}}(\ker\ \alpha_r)^{\perp}\\ 0 & \text{otherwise.}\end{cases}\qquad(4.4)$$

Now, to apply Theorem 3.5 on the wave-packet systems, we require that for a.e. $\xi^r\in G_r, \mathcal{T}_{\tilde{\alpha}_r}(\xi^r)$ in (4.4) should be equal to 0 for all $\tilde{\alpha}_{\mathrm{r}}\in\bigcup_{\alpha_r\in\mathcal{A}}\Gamma^{\perp}_{\alpha_r}$.

The above discussion leads to our second main result, that is, Theorem 4.1 which provides the conditions on Ψ and Φ such that the wave-packet systems generated by Ψ and Φ form pairwise orthogonal Bessel families (frames) which we call as pairwise orthogonal (simply, orthogonal) wavepacket Bessel (frame) systems in $L^2(G_r)$. Note that the general LCA group approach applies to all groups of the form $G_r=\mathbb{R}^s\times\mathbb{Z}^{1+\epsilon}\times\mathbb{T}^q\times\mathbb{Z}_m$. Therefore, the following characterization result on wavepacket systems can be easily used to verify the concrete conditions for any choice of G_r specified as above, while the direct derivation would be rather complex.

Theorem 4.1 (see [42]). Let the wave-packet systems $\mathcal{W}(\Psi,\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{W}(\Phi,\mathcal{A},\Gamma,\Lambda)$ be Bessel families (frames) in $L^2(G_r)$ satisfying the corresponding dual α_r-LIC, where $\mathcal{A}$ is a countable subset of Epick (G_r). Then, the following assertions are equivalent:

(i) $\mathcal{W}(\Psi,\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{W}(\Phi,\mathcal{A},\Gamma,\Lambda)$ form orthogonal wave-packet Bessel (frame) systems in $L^2(G_r)$,

(ii) for a.e. $\xi^r\in\hat{G}_r$ and $\tilde{\alpha}_{\mathrm{r}}\in\bigcup_{\alpha_r\in\mathcal{A}}\Gamma^{\perp}_{\alpha_r}$, the following holds:

$$\sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha}_r\in\Gamma^{\perp}_{\perp}}\sum_{j\in J}\int_{\Lambda}\overline{\hat{\varphi}_j^r(\beta^{-1}\xi^r-\chi)}\hat{\psi}_j(\beta^{-1}(\xi^r+\tilde{\alpha}_{\mathrm{r}})-\chi)d\mu_\Lambda(\chi)=0$$

where for $\beta=\alpha_r^*, \Gamma^{\perp}_{\alpha_r}$ is given by $\beta\Gamma^{\perp}$.

Proof. The proof can be concluded by observing that if we consider $\mathcal{W}(\Psi,\mathcal{A},\Gamma,\Lambda)$ and $\mathcal{W}(\Phi,\mathcal{A},\Gamma,\Lambda)$ as Bessel families (frames) satisfying the corresponding dual α_r -LIC, then, the wave-packet systems generated by Ψ and Φ form pairwise orthogonal Bessel families (frames) in $L^2(G_r)$ if, and only if, in view of (4.4), $\mathcal{T}_{\tilde{\alpha}_r}(\xi^r)$ is equal to 0 for all $\tilde{\alpha}_{\mathrm{r}}\in\bigcup_{\alpha_r\in\mathcal{A}}\Gamma^{\perp}_{\alpha_r}$ and a.e. $\xi^r\in\hat{G}_r$.

In the following, by applying Theorem 4.1 to the case $G_r=\mathbb{R}^d$, we get a characterization result for the orthogonality of a pair of wave-packet systems in $L^2(\mathbb{R}^d)$. Hence, the wave-packet systems within $L^2(\mathbb{R}^d)$ are easily covered within our framework.

Example 4.2 (see [42]). Let $G_r=\mathbb{R}^d$ (equipped with Lebesgue measure), $\Gamma=\mathbb{Z}^d$ and $\Lambda=\mathbb{R}^d$. Then, $\hat{G}_{\mathrm{r}}=\mathbb{R}^d$, with Euclidean metric, we have $\Gamma^{\perp}=\mathbb{Z}^d$ and $\Lambda^{\perp}=\{0\}$. Further, by assuming the matrix A in $\mathrm{GL}(d,\mathbb{R})$, set $\mathcal{A}=\{x\mapsto A^kx: k\in\mathbb{Z}\}$. Under these assumptions, from (4.3), the wave-packet system generated by $\Psi=\{\psi_l\}_{l=1}^L\subset L^2(\mathbb{R}^d)$ can be written as

$$\begin{aligned}\mathcal{W}(\Psi,\mathcal{A},\mathbb{Z}^d,\mathbb{R}^d) &:= \left\{D_{A^k}T_\gamma M_\chi\psi_l(\cdot): l=1,\dots,L, k\in\mathbb{Z},\gamma\in\mathbb{Z}^d,\chi\in\mathbb{R}^d\right\}\\ &= \left\{|\det\ A|^{-k/2}\chi(A^k\cdot-\gamma)\psi_l(A^k\cdot-\gamma): l=1,\dots,L, k\in\mathbb{Z},\gamma\in\mathbb{Z}^d,\chi\in\mathbb{R}^d\right\}\end{aligned}$$

By letting $\mathcal{W}(\Psi,\mathcal{A},\mathbb{Z}^d,\mathbb{R}^d)$ as a wave-packet system in $L^2(\mathbb{R}^d)$ which satisfies the Bessel condition (frame inequality), we conclude from Theorem 4.1 that the wave-packet systems generated by Ψ and Φ form pairwise orthogonal Bessel families (frames) in $L^2(\mathbb{R}^d)$ if, and only if, the following holds:

$$\sum_{k\in\mathbb{Z}:\tilde{\alpha}_{\mathrm{r}}\in B^k\mathbb{Z}^d}\sum_{l=1}^{L}\int_{\mathbb{R}^d}\overline{\hat{\varphi}_l^r(B^{-k}\xi^r-\chi)}\hat{\psi}_l(B^{-k}(\xi^r+\tilde{\alpha}_{\mathrm{r}})-\chi)d(\chi)=0$$

for a.e. $\xi^r\in\mathbb{R}^d$ and for each α_r along with $B=A^*$.

In the next, by applying Theorem 4.1 we deduce the orthogonality conditions for the case of Gabor and wavelet systems over LCA groups. First we discuss the case of Gabor systems over LCA-group setting:

4.2. Special cases of Wave-Packet Systems.

4.2.1. Gabor Systems. In (4.3), by assuming $\mathcal{A}=\{I_{G_r}\}$, where I_{G_r} denotes the identity group homo-morphism on G_r, we consider the following system as a special case of wave-packet system defined in (4.3) which we call the Gabor system generated by Ψ :

$$\mathcal{G}(\Psi,\Gamma,\Lambda):=\{T_\gamma M_\chi \psi_j:\gamma\in\Gamma,\chi\in\Lambda,j\in J\}$$

At this juncture, it is relevant to note that the system $\mathcal{G}(\Psi,\Gamma,\Lambda)$ is a frame for $L^2(G_r)$ if, and only if, $\{M_\chi T_\gamma \psi_j:\gamma\in\Gamma,\chi\in\Lambda,j\in J\}$ is a frame for $L^2(G_r)$ (see [29, Lemma 2.4]), where the later system is termed as a co-compact Gabor system in [28]. Further, observe that $\mathcal{G}(\Psi,\Gamma,\Lambda)$ is a TI system of the form $\bigcup_{j\in J}\{T_\gamma g^r_{j,1+\epsilon}\}_{\gamma\in\Gamma_j,(1+\epsilon)\in P_j}$ with $\Gamma_j=\Gamma$ for $j\in J\subset\mathbb{Z}$ and $g^r_{j,1+\epsilon}=g^r_{j,\chi}=M_\chi\psi_j$, where $(j,1+\epsilon)=(j,\chi)\in J\times\Lambda$. In this case, for each $j\in J, P_j=\{\chi:\chi\in\Lambda\}$ is equipped with the measure $\mu_{P_j}:=(\Delta(\alpha_r))^{-1}\mu_\Lambda$ that satisfies the standing hypothesis. Since for TI systems the dual α_r -LIC is automatically satisfied, Theorem 4.1 leads to the following result on the orthogonality of Gabor systems. Here, note that the Bessel families (frames) with co-compact Gabor structure which satisfy the orthogonality property are termed as pairwise orthogonal (simply, orthogonal) co-compact Gabor Bessel (frame) systems in $L^2(G_r)$:

Proposition 4.3 (see [42]). Let the Gabor systems $\mathcal{G}(\Psi,\Gamma,\Lambda)$ and $\mathcal{G}(\Phi,\Gamma,\Lambda)$ be Bessel families (frames) in $L^2(G_r)$. Then, they form orthogonal co-compact Gabor Bessel (frame) systems in $L^2(G_r)$ if, and only if, for each $\tilde{\alpha}_r\in\Gamma^\perp$ and for a.e. $\xi^r\in\hat{G}_r$, the following assertion holds:

$$\sum_{j\in J}\int_\Lambda \overline{\hat{\varphi}^r_j(\xi^r-\chi)}\hat{\psi}_j((\xi^r+\tilde{\alpha}_r)-\chi)d\mu_\Lambda(\chi)=0$$

Using above result, we can deduce a characterization for the orthogonality of Gabor Bessel families (frames) in $\ell^2(\mathbb{Z}^d)$ given by [36]. This property of Gabor frames has found its significance in developing frame theory and its applications including the construction of Gabor superframes in various set-ups. For more details, see [2,16,19,20,22,35,36]. In the next part, we provide a characterization for pairwise orthogonal Bessel families (frames) with wavelet structure which we call as pairwise orthogonal (simply, orthogonal) wavelet Bessel (frame) systems in $L^2(G_r)$.

4.2.2. Wavelet Systems. By letting $\Lambda=\{\chi_0\}\subset\hat{G}_r$ in (4.3), where χ_0 being the neutral element of $\hat{G}_r$. we define the collection $\mathcal{U}(\Psi,\mathcal{A},\Gamma)$ as the wavelet system generated by Ψ :

$$\mathcal{U}(\Psi,\mathcal{A},\Gamma):=\{D_{\alpha_r}T_\gamma\psi_j:\alpha_r\in\mathcal{A},\gamma\in\Gamma,j\in J\}$$

as a special case of wave-packet system defined in (4.3). For a countable subset $\mathcal{A}$ in **Epick**(G_r), the system (4.5) is a GTI system of the form $\bigcup_{\alpha_r\in\mathcal{A}}\{T_\gamma g^r_{\alpha_r,1+\epsilon}\}_{\gamma\in\Gamma_{\alpha_r},(1+\epsilon)\in P_{\alpha_r}}$ for $\Gamma_{\alpha_r}=\alpha_r^{-1}\Gamma$ with $\alpha_r\in\mathcal{A}$, $g^r_{\alpha_r,1+\epsilon}=g^r_{\alpha_r,j}=D_{\alpha_r}\psi_j$ for $(\alpha_r,1+\epsilon)=(\alpha_r,j)$ in $\mathcal{A}\times J$. In this case, for each $\alpha_r\in\mathcal{A}$, the measure space $P_{\alpha_r}:=\{j:j\in J\}$ is equipped with a counting measure $\mu_{P_{\alpha_r}}:=(\Delta(\alpha_r))^{-1}(\mu_J)$ which is clearly σ -finite.

Thus, Theorem 4.1 for the case of wave-packet systems now reduces to the following result on wavelet systems which generalizes similar results discussed in [31,41] along with various applications:

Proposition 4.4 (see [42]). Let $\mathcal{U}(\Psi,\mathcal{A},\Gamma)$ and $\mathcal{U}(\Phi,\mathcal{A},\Gamma)$ be Bessel families (frames) in $L^2(G_r)$ which satisfy the corresponding dual α_r - LIC, where $\mathcal{A}$ is a countable subset of Epick (G_r). Then, they form orthogonal wavelet Bessel (frame systems in $L^2(G_r)$ if, and only if, for a.e. $\xi^r\in\hat{G}_r$ and $\tilde{\alpha}_r\in\bigcup_{\alpha_r\in\mathcal{A}}\Gamma^\perp_{\alpha_r}$, we have

$$\sum_{\alpha_r\in\mathcal{A}:\tilde{\alpha_r}\in\Gamma^\perp_{\alpha_r}}\sum_{j\in J}\bar{\hat{\varphi}}^r_j(\beta^{-1}\xi^r)\hat{\psi}_j(\beta^{-1}(\xi^r+\tilde{\alpha}_r))=0$$

where for $\beta=\alpha_r^*,\Gamma^\perp_{\alpha_r}$ is given by $\beta\Gamma^\perp$.

Example 4.5 (see [42]). By assuming $\Lambda=\{\chi_0\}\subset\hat{G}_r$ in Example 4.2, where χ_0 being the neutral element of $\hat{G}_r$, we obtain a wavelet system generated by Ψ :

$$\begin{aligned}\mathcal{U}(\Psi,\mathcal{A},\mathbb{Z}^d) &=\{D_{A^k}T_\gamma\psi_l(\cdot):l=1,\dots,L,k\in\mathbb{Z},\gamma\in\mathbb{Z}^d\}\\ &=\{|\det A|^{-k/2}\psi_l(A^k\cdot-\gamma):l=1,\dots,L,k\in\mathbb{Z},\gamma\in\mathbb{Z}^d\}\end{aligned}$$

which is a special case of wave-packet system $\mathcal{W}(\Psi,\mathcal{A},\mathbb{Z}^d,\mathbb{R}^d)$. It follows that two Bessel families (frames) $\mathcal{U}(\Psi,\mathcal{A},\mathbb{Z}^d)$ and $\mathcal{U}(\Phi,\mathcal{A},\mathbb{Z}^d)$ are pairwise orthogonal Bessel families (frames) if, and only if, we have

$$\sum_{k\in\mathbb{Z}:\tilde{\alpha_r}\in B^k\mathbb{Z}^d}\sum_{l=1}^{L}\overline{\hat{\varphi}^r_l(B^{-k}\xi^r)}\hat{\psi}_l(B^{-k}(\xi^r+\tilde{\alpha}_r))=0,\text{ for a.e. }\xi^r\in\mathbb{R}^d$$

and for all $\tilde{\alpha}_r\in\bigcup_{k\in\mathbb{Z}}B^k\mathbb{Z}^d$. Note that the above result coincides with the characterization of two wavelet systems to be pairwise orthogonal frames in $L^2(\mathbb{R}^d)$ which is given by [41].

References

[1] S. T. Ali, J. P. Antoine and J. P. Gazeau, Continuous frames in Hilbert space, Ann. Physics, 222(1) (1993), 1–37.
[2] R. Balan, A study of Weyl-Heisenberg and wavelet frames, Ph.D. Thesis, Princeton University, (1998).
[3] R. Balan, Density and redundancy of the noncoherentWeyl-Heisenberg superframes, The Functional and Harmonic Analysis of Wavelets and Frames, Contemp. Math., 247, Amer. Math. Soc., Providence, RI, (1999), 29–41.
[4] R. Balan, Multiplexing of signals using superframes, Wavelet Applications in Signal and Image Processing VIII, A. Aldroubi, A. Laine, and M. Unser, eds., SPIE, Bellingham, WA, (2000), 118–129.
[5] M. Bownik and K. A. Ross, The structure of translation-invariant spaces on locally compact abelian groups, J. Fourier Anal. Appl., 21(4) (2015), 849–884.
[6] C. Cabrelli and V. Paternostro, Shift-invariant spaces on LCA groups, J. Funct. Anal., 258(6) (2010), 2034–2059.
[7] O. Christensen and A. Rahimi, Frame properties of wave packet systems in L2(Rd), Adv. Comput. Math., 29(2) (2008), 101–111.
[8] O. Christensen, An introduction to frames and Riesz bases, Second edition, Applied and Numerical Harmonic Analysis, Birkh¨auser/Springer, [Cham], (2016).
[9] A. C´ordoba and C. Fefferman, Wave packets and Fourier integral operators, Comm. Partial Differential Equations, 3(11) (1978), 979–1005.
[10] C. Corduneanu, Almost periodic functions, Interscience Tracts in Pure and Applied Mathematics, IntersciencePublishers [John Wiley & Sons], New York-London-Sydney, (22)(1968).
[11] Z. Fan, A. Heinecke and Z. Shen, Duality for frames, J. Fourier Anal. Appl., 22(1) (2016), 71–136.
[12] H. G. Feichtinger, W. Kozek and F. Luef, Gabor analysis over finite abelian groups, Appl. Comput. Harmon. Anal., 26(2) (2009), 230–248.
[13] G. B. Folland, A course in abstract harmonic analysis, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, (1995).
[14] J. P. Gabardo and D. Han, Frames associated with measurable spaces, Adv. Comput. Math., 18(3) (2003), 127–147.
[15] K. Gr¨ochenig, Aspects of Gabor analysis on locally compact abelian groups, Gabor analysis and algorithms, Appl. Numer. Harmon. Anal., Birkh¨auser Boston, Boston, MA, (1998), 211–231.
[16] K. Gr¨ochenig and Y. Lyubarskii, Gabor (super) frames with Hermite functions, Math. Ann., 345(2) (2009), 267–286.
[17] A. Gumber and N. K. Shukla, Uncertainty principle corresponding to an orthonormal wavelet system, Appl. Anal., 2017, DOI: 10.1080/00036811.2016.1274025.
[18] A. Gumber and N. K. Shukla, Finite dual **g**-framelet systems associated with an induced group action, Complex Anal. Oper. Theory, 2017, DOI: 10.1007/s11785-017-0729-6.
[19] X. Guo, Constructions of frames by disjoint frames, Numer. Funct. Anal. Optim., 35(5) (2014), 576-587.
[20] X. Guo, Characterizations of disjointness of g-frames and constructions of g-frames in Hilbert Spaces, Complex Anal. Oper. Theory, 8(7) (2014), 1547–1563.
[21] D. Han and D. R. Larson, On the orthogonality of frames and the density and connectivity of wavelet frames, ActaAppl. Math., 107(1-3) (2009), 211–222.
[22] D. Han and D. R. Larson, Frames, bases and group representations, Mem. Amer. Math. Soc., 147(697), (2000).
[23] D. Han, K. Kornelson, D. Larson and E. Weber, Frames for undergraduates, Student Mathematical Library, 40, American Mathematical Society, Providence, RI, (2007).
[24] E. Hern´andez, D. Labate and G. Weiss, A unified characterization of reproducing systems generated by a finite family. II, J. Geom. Anal., 12(4) (2002), 615–662.
[25] E. Hern´andez, D. Labete, G. Weiss and E. Wilson, Oversampling, quasi-affine frames, and wave packets, Appl. Comput. Harmon. Anal., 16(2) (2004), 111–147.
[26] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis. Vol. I, Springer-Verlag, Berlin-G¨ottingen-Heidelberg, (1963).
[27] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis. Vol. II, Springer-Verlag, New York-Berlin, (1970).
[28] M. S. Jakobsen and J. Lemvig, Co-compact Gabor systems on locally compact abelian groups, J. Fourier Anal. Appl., 22(1) (2016), 36–70.
[29] M. S. Jakobsen and J. Lemvig, Reproducing formulas for generalized translation invariant systems on locally compact abelian groups, Trans. Amer. Math. Soc., 368(12) (2016), 8447–8480.
[30] G. Kaiser, A friendly guide to wavelets, Birkh¨auser Boston, Inc., Boston, MA, (1994).
[31] H. O. Kim, R. Y. Kim, J. K. Lim and Z. Shen, A pair of orthogonal frames, J. Approx. Theory, 147(2) (2007), 196–204.
[32] G. Kutyniok and D. Labate, The theory of reproducing systems on locally compact abelian groups, Colloq. Math., 106(2) (2006), 197–220.
[33] R. Larsen, An introduction to the theory of multipliers, Die Grundlehren dermathematischenWissenschaften, Band 175, Springer-Verlag, New York-Heidelberg, 1971.
[34] D. Labate, G. Weiss and E. Wilson, An approach to the study of wave packet systems, Wavelets, frames and operator theory, 215–235, Contemp. Math., 345, Amer. Math. Soc., Providence, RI, (2004).
[35] Y. Z. Li and Q. F. Lian, Super Gabor frames on discrete periodic sets, Adv. Comput. Math., 38(4) (2013), 763–799.
[36] J. Lopez and D. Han, Discrete Gabor frames in _2(Zd), Proc. Amer. Math. Soc., 141(11) (2013), 3839–3851.
[37] A. Nowak, Characterization of Carath´eodory functions, Ann. Math. Sil., (27) (2013), 93–98.
[38] A. Rahimi, A. Najati and Y. N. Dehghan, Continuous frames in Hilbert spaces, Methods Funct. Anal. Topology, 12(2) (2006), 170–182.
[39] A. Ron and Z. Shen, Generalized shift-invariant systems, Constr. Approx., 22(1) (2005), 1–45.
[40] W. Sun, G-frames and g-Riesz bases, J. Math. Anal. Appl., 322(1) (2006), 437–452.
[41] E. Weber, Orthogonal frames of translates, Appl. Comput. Harmon. Anal., 17(1) (2004), 69–90.
[42] A. Gumber, N.K. Shukla, Orthogonality of a pair of frames over locally compact abeliangroups, J. Math. Anal. Appl. (2018), 1-20.