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Research Paper

Fixed point theorems for (α, β)-admissible Geraghty type contractive mappings in bipolar metric space

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Abstract: *In this paper, we will introduce a new and simple approach of (α, β)- admissible Geraghty type contractive mappings in bipolar metric spaces. Further, we will prove some fixed point theorems for above mentioned contractive mappings in complete bipolar metric spaces. At the end, we shall construct some comparative examples to show the usability of our main results. As an application we discuss about the Ulam-Hyers stability.*

Keywords: *Fixed point, Geraghty type contractive mappings, contravariant mappings, bipolar metric space*

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I. Introduction:

In 1922, Banach [1] introduced Banach contraction principle as the first constructive method to get a fixed point for a self map on a complete metric space. Fixed point theory has a wide range of applications in various field of sciences and Mathematics. Many authors had generalized the Banach contraction principle. In continuation of this, in 1973, Geraghty [5] generalized the Banach contraction principle and gave a new direction to the researchers for getting new fixed point theorems. Some of the authors generalized Geraghty contraction in various spaces (see [3], [12]).

In 2012, Samet *et al*. [13] introduced the concepts of α-contractive and α-admissible mappings and proved various fixed point theorems of α-admissible contractive mappings in complete metric spaces. Recently, in 2015, Chandok [3] introduced the concept of (α, β)- admissible Geraghty type contractive mappings and proved some fixed point theorems of such kind of mappings in complete metric spaces.

To get a new approach for fixed point results in 2016, Mutlu and Gürdal [9] introduced the concept of bipolar metric space. The major difference between the previously defined spaces and bipolar is of distance function. In bipolar metric space, the distance function is from the cartesian product of two different sets to non-negative real numbers.

Since then, many authors have proved several fixed point results in bipolar metric space see [4], [6-8], [10], [12].

Here, we will also prove some new fixed point results for Geraghty type contractions via newly defined (α, β)- admissible.

II. Preliminaries:

We need to recall some basic definitions, introduce some new notations and definitions which is used for the fixed point theorems for (α, β)-admissible Geraghty type contractive mappings in bipolar metric spaces.

Definition 2.1. In 2016, Mutlu and Gürdal [9] introduced the concept of bipolar metric space.

Let X and Y are two non-empty sets and $d : X \times Y \to [0, \infty)$ be a function satisfying the following conditions:

(BP1) $d(x, y) = 0$ if and only if $x = y$, where $(x, y) \in X \times Y$;

(BP2) $d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$;

(BP3) $d(x_1, y_2) \le d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Then d is called bipolar metric and (X, Y, d) is called bipolar metric space.

If $X \cap Y = \emptyset$, then space is called disjoint otherwise joint. The set X is called left pole and Y is called right pole of bipolar metric space (X, Y, d) and any element of left pole (X), right pole (Y) and $X \cap Y$ is called left element, right element and central element respectively.

Definition 2.2. Let (X, Y, d) be a bipolar metric space. Then any sequence $(x_n) \subseteq X$ is called left sequence and is said to be convergent to right element say 'y' if $d(x_n, y) \to 0$ as $n \to \infty$. Similarly, a right sequence $(y_n) \subseteq Y$ is said to be convergent to a left element say 'x' if $d(x, y_n) \to 0$ as $n \to \infty$.

Definition 2.3. Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.
Let $T : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function:
(i) If $T(X_1) \subseteq X_2$ and $T(Y_1) \subseteq Y_2$, then T is called covariant map and is denoted by $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.
(ii) If $T(X_1) \subseteq Y_2$ and $T(Y_1) \subseteq X_2$, then T is called contravariant map and is denoted by $T : (X_1, Y_1, d_1) \nrightarrow (X_2, Y_2, d_2)$.

Definition 2.4. Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.
(i) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left continuous at a point $x_0 \in X_1$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_2(Tx_0, Ty) < \varepsilon$ whenever $d_1(x_0, y) < \delta$.
(ii) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right continuous at a point $y_0 \in Y_1$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_2(Tx, Ty_0) < \varepsilon$ whenever $d_1(x, y_0) < \delta$.
(iii) A map T is called continuous, if it is left continuous at each $x_0 \in X_1$ and right continuous at each $y_0 \in Y_1$.
(iv) A contravariant map $T : (X_1, Y_1, d_1) \nrightarrow (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

Definition 2.5. Let (X, Y, d) be a bipolar metric space.
(i) A sequence $\{(x_n, y_n)\}$ on the set $X \times Y$ is called a bisequence on (X, Y, d).
(ii) If both the sequences (x_n) and (y_n) converge, then bisequence $\{(x_n, y_n)\}$ is said to be convergent. If both the sequences (x_n) and (y_n) converge to same point v and $v \in X \cap Y$, then this bisequence is said to be biconvergent.
(iii) A bisequence $\{(x_n, y_n)\}$ on (X, Y, d) is said to be Cauchy bisequence, if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, y_m) < \epsilon$ for all $n, m \geq N$.
(iv) A bipolar metric space is said to be complete if every Cauchy bisequence is convergent in this space.

Definition 2.6. [10] Let X and Y be two non-empty sets. Let $T : (X, Y) \rightrightarrows (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible(covariant) if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$$

For all $x \in X$ and $y \in Y$.

Definition 2.7. [10] Let X and Y be two non-empty sets. Let $T : (X, Y) \nrightarrow (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible(contravariant) if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Ty, Tx) \geq 1$$

For all $x \in X$ and $y \in Y$.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ which satisfying the following conditions:
(i) ψ is continuous;
(ii) ψ is strictly increasing
(iii) $\psi(0) = 0$.

Let Θ be the family of functions $\theta : [0, \infty) \to [0,1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\theta(t_n) \to 1$ as $t_n \to 0$.

III. Main Results:

In this section, we will introduce new notations for (α,β)-admissible Geraghty type contractive mappings and prove various fixed point theorems for such type of mappings in complete bipolar metric spaces.

Definition 3.1. Let X and Y be two non-empty sets. Let $T : (X, Y) \rightrightarrows (X, Y)$ be a covariant mapping and $\alpha, \beta : X \times Y \to [0, \infty)$. Then T is called (α, β)-admissible(covariant) mapping if $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$
for all $x \in X$ and $y \in Y$.

Definition 3.2. Let X and Y be two non-empty sets. Let $T : (X, Y) \nrightarrow (X, Y)$ be a contravariant mapping and $\alpha, \beta : X \times Y \to [0, \infty)$. Then T is called (α, β)-admissible(contravariant) mapping if $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ implies $\alpha(Ty, Tx) \geq 1$ and $\beta(Ty, Tx) \geq 1$
for all $x \in X$ and $y \in Y$.

Definition 3.3. Let X and Y be two non-empty sets. Consider (X, Y, d) be a bipolar metric space, $T : (X, Y) \nrightarrow (X, Y)$ be a contravariant mapping and $\alpha, \beta : X \times Y \to [0, \infty)$. A mapping T is called (α, β)-Geraghty type contractive mapping if there exist a $\theta \in \Theta$, such that for all $x \in X$ and $y \in Y$ and $\psi \in \Psi$, which satisfying the following condition:

$$\alpha(x,Tx)\,\beta(Ty,y)\,\psi(d(Ty,Tx)) \le \theta(\,\psi(d(x,y)))\,\psi\big(d(x,y)\big). \qquad (3.1)$$

Theorem 3.4. Let (X,Y,d) be a complete bipolar metric space, $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant mapping and $\alpha,\beta : X \times Y \to [0,\infty)$. Suppose that the following condition are satisfied:

(i) T is (α,β)-admissible mapping;
(ii) T is an (α,β)-Geraghty type contractive mapping;
(iii) there exist $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\beta(x_0,Tx_0) \ge 1$;
(iv) T is continuous mapping.

Then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\beta(x_0,Tx_0) \ge 1$. Now we construct the bisequence $\{(x_n,y_n)\}$ as $Tx_n = y_n$ and $Ty_n = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Since T is (α,β)-admissible mapping,

So, $\alpha(x_0,y_0) = \alpha(x_0,Tx_0) \ge 1$,
$\beta(x_0,y_0) = \beta(x_0,Tx_0) \ge 1$,
$\alpha(x_1,y_0) = \alpha(Ty_0,Tx_0) \ge 1$,
$\beta(x_1,y_0) = \beta(Ty_0,Tx_0) \ge 1$,

using mathematical induction, we get

$$\alpha(x_{n+1},y_n) \ge 1 \text{ and } \beta(x_{n+1},y_n) \ge 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \qquad (3.2)$$

Since T is (α,β)-admissible mapping,

So, $\alpha(x_1,y_1) = \alpha(x_1,Tx_1) \ge 1$,
$\beta(x_1,y_1) = \beta(x_1,Tx_1) \ge 1$,

hence by induction, we obtain

$$\alpha(x_n,y_n) \ge 1 \text{ and } \beta(x_n,y_n) \ge 1 \text{ for } n \in \mathbb{N} \cup \{0\}. \qquad (3.3)$$

Putting $x = x_n$ and $y = y_n$ in equation (3.1) and using equation (3.3), we have

$$\begin{aligned}\psi\big(d(x_n,y_n)\big) &= \psi\big(d(Ty_{n-1},Tx_n)\big)\\ &\le \alpha(x_n,Tx_n)\beta(Ty_{n-1},y_{n-1})\psi\big(d(Ty_{n-1},Tx_n)\big)\\ &\le \theta(\,\psi\big(d(x_n,y_{n-1})\big))\psi\big(d(x_n,y_{n-1})\big). \end{aligned} \qquad (3.4)$$

Similarly, putting $x = x_{n+1}$ and $y = y_n$ in equation (3.1) and using equation (3.3), we get $\psi\big(d(x_{n+1},y_n)\big) = \psi\big(d(Ty_n,Tx_n)\big) \le \alpha(x_n,Tx_n)\beta(Ty_n,y_n)\psi\big(d(Ty_n,Tx_n)\big)$

$$\le \theta(\,\psi\big(d(x_n,y_n)\big))\psi\big(d(x_n,y_n)\big). \qquad (3.5)$$

From equation (3.4), we get

$$\begin{aligned}\psi\big(d(x_n,y_n)\big) &\le \theta(\,\psi\big(d(x_n,y_{n-1})\big))\psi\big(d(x_n,y_{n-1})\big)\\ &\le \psi\big(d(x_n,y_{n-1})\big).\end{aligned} \qquad (3.6)$$

Hence, by using the properties of ψ, we conclude that

$$d(x_n,y_n) \le d(x_n,y_{n-1}) \text{ for all } n \in \mathbb{N}.$$

From equation (3.5), we get

$$\begin{aligned}\psi\big(d(x_{n+1},y_n)\big) &\le \theta(\,\psi\big(d(x_n,y_n)\big))\psi\big(d(x_n,y_n)\big)\\ &\le \psi\big(d(x_n,y_n)\big).\end{aligned} \qquad (3.7)$$

Hence, by using the properties of ψ, we conclude that

$$d(x_{n+1},y_n) \le d(x_n,y_n) \text{ for all } n \in \mathbb{N}.$$

From the above, we conclude that the sequences $\{d(x_n,y_{n-1})\}$ and $\{d(x_n,y_n)\}$ are monotonically decreasing and for the non-negative monotonically decreasing sequences $\{d(x_n,y_{n-1})\}$ and $\{d(x_n,y_n)\}$, there exist some $r_1 \ge 0$ and $r_2 \ge 0$, such that

$$d(x_n,y_{n-1}) \to r_1,\ d(x_n,y_n) \to r_2 \text{ as } n \to \infty \qquad (3.8)$$

Further from equation (3.6), it implies that

$$\frac{\psi\big(d(x_n,y_n)\big)}{\psi\big(d(x_{n-1},y_{n-1})\big)} \le \theta(\,\psi\big(d(x_n,y_{n-1})\big)) < 1. \qquad (3.9)$$

As $n \to \infty$ in above inequality, we obtain

$\lim_{n\to\infty} \theta(\,\psi\big(d(x_n,y_{n-1})\big)) = 1$ and $\theta \in \Theta$, $\lim_{n\to\infty} \psi\big(d(x_n,y_{n-1})\big) = 0$, which gives that

$$\lim_{n\to\infty} d(x_n,y_{n-1}) = r_1 = 0. \qquad (3.10)$$

Further from equation (3.7), it implies that

$$\frac{\psi\big(d(x_{n+1},y_n)\big)}{\psi\big(d(x_{n+1},y_n)\big)} \le \theta(\,\psi\big(d(x_n,y_n)\big)) < 1. \qquad (3.11)$$

As $n \to \infty$ in above inequality, we obtain that

$\lim_{n\to\infty} \theta(\,\psi\big(d(x_n,y_n)\big)) = 1$,

and $\theta \in \Theta$.

$\lim_{n\to\infty} \psi(d(x_n, y_n)) = 0,$

which gives that

$\lim_{n\to\infty} d(x_n, y_n) = r_2 = 0.$ (3.12)

Now, we shall prove that $\{(x_n, y_n)\}$ is a Cauchy bisequence. Let us assume that $\{(x_n, y_n)\}$ is not Cauchy bisequence. Then there exist $\delta > 0$ for which we can find subsequence (x_{n_k}, y_{m_k}) with $n_k > m_k > k$ such that

$d(x_{n_k}, y_{m_k}) \geq \delta.$ (3.13)

Further, corresponding to m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ which satisfying equation (3.13), we get

$$d(x_{n_{k-1}}, y_{m_k}) < \delta. \quad (3.14)$$

Using triangle inequality, we obtain

$0 < \delta \leq d(x_{n_k}, y_{m_k}) \leq d(x_{n_k}, y_{n_{k-1}}) + d(x_{n_{k-1}}, y_{n_{k-1}}) + d(x_{n_{k-1}}, y_{m_k}).$ (3.15)

Letting $k \to \infty$ and using equations (3.10), (3.12) and (3.14), we obtain

$$\lim_{k\to\infty} d(x_{n_k}, y_{m_k}) = \delta. \quad (3.16)$$

Again, using triangle inequality, we have

$d(x_{m_k}, y_{n_k-1}) \leq d(x_{m_k}, y_{n_k}) + d(x_{n_k}, y_{n_k}) + d(x_{n_k}, y_{n_{k-1}}).$ (3.17)

Letting $k \to \infty$ and using equations (3.10), (3.12) and (3.13), we get

$$\lim_{k\to\infty} d(x_{m_k}, y_{n_k-1}) = \delta. \quad (3.18)$$

Putting $x = x_{n_k}$ and $y = y_{m_k}$ in equation (3.1), we get

$$\begin{aligned}\psi(d(x_{n_k}, y_{m_k}) &= \psi\left(d(Ty_{n_{k-1}}, Tx_{m_k})\right) \\ &\leq \alpha(x_{m_k}, Tx_{m_k})\beta(Ty_{n_{k-1}}, y_{n_{k-1}})\psi\left(d(Ty_{n_{k-1}}, Tx_{m_k})\right) \\ &\leq \theta(\psi\left(d(x_{m_k}, y_{n_{k-1}})\right))\psi\left(d(x_{m_k}, y_{n_{k-1}})\right). \quad (3.19)\end{aligned}$$

Therefore,

$\psi(d(x_{n_k}, y_{m_k}) \leq \theta(\psi\left(d(x_{m_k}, y_{n_{k-1}})\right))\psi\left(d(x_{m_k}, y_{n_{k-1}})\right).$

On taking limit $k \to \infty$, we obtain

$\psi(\delta) \leq \theta(\psi\left(d(x_{m_k}, y_{n_{k-1}})\right))\psi(\delta),$

that is

$1 \leq \lim_{k\to\infty} \theta(\psi\left(d(x_{m_k}, y_{n_{k-1}})\right))$

$\Rightarrow \lim_{k\to\infty} \theta(\psi\left(d(x_{m_k}, y_{n_{k-1}})\right)) = 1.$

Consequently, we get

$\lim_{k\to\infty} d(x_{m_k}, y_{n_k-1}) = 0,$

which is a contradiction.

Hence, $\{(x_n, y_n)\}$ is a Cauchy bisequence and (X, Y, d) is a complete bipolar metric space. So, $\{(x_n, y_n)\}$ is convergent and in fact biconvergent. So, there exists $u \in X \cap Y$ such that

$(x_n) \to u, (y_n) \to u$ as $n \to \infty$.

As T is a continuous mapping, so

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n$ implies that $T \lim_{n\to\infty} x_n = u$.

By combining both, we get

$Tu = u.$

Hence, T has a fixed point.

Example 3.5. Let $X = [0, +\infty)$ and $Y = (-\infty, 0]$ and let $d : X \times Y \to [0, +\infty)$ be a function such that $d(x, y) = |x - y|$ for all $(x, y) \in X \times Y$.

Then, clearly (X, Y, d) be a complete bipolar metric space.

Define $T : (X, Y) \rightrightarrows (X, Y)$ such that $Tx = -\frac{x}{2}$ is a continuous mapping and $\alpha, \beta : X \times Y \to [0, \infty)$ such that $\alpha(x, y) = \beta(x, y) = 1$ for$(x, y) \in X \times Y$.

Clearly, T is (α, β)-admissible mapping and there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$ and $X \cap Y = \{0\}$ and $T0 = 0$.

Taking $\psi(t) = t$ and $\theta(t) = \frac{1}{2}$.

Left hand side of equation (3.1) becomes

$\alpha(x, Tx)\, \beta(Ty, y)\, \psi(d(Ty, Tx)) = \frac{1}{2}|x - y|,$

Right hand side becomes

$\theta(\psi(d(x,y)))\,\psi(d(x,y)) = \frac{1}{2}|x-y|$.
for all $(x,y) \in X \times Y$.
which implies equation (3.1) holds.
Hence, T is an (α,β)-Geraghty type contractive mapping.
All the conditions of Theorem 3.4 are satisfied.
So, T has a fixed point and $x = 0$ is the fixed point of T.

Definition 3.6. Let X and Y be two non-empty sets. Consider (X,Y,d) be a bipolar metric space, $T : (X,Y) \rightleftarrows (X,Y)$ be a contravariant mapping and $\alpha,\beta : X \times Y \to [0,\infty)$. A mapping T is called (α,β)-Generalized Geraghty type rational contractive mapping if there exist a
$\theta \in \Theta$, such that for all $x \in X$ and $y \in Y$ and $\psi \in \Psi$,
which satisfying the following condition:

$$\alpha(x,Tx)\,\beta(Ty,y)\,\psi(d(Ty,Tx)) \le \theta(\psi(M(x,y)))\,\psi(M(x,y)), \tag{3.20}$$

where $M(x,y) = max\{d(x,y), d(x,Tx), d(Ty,y), \frac{d(x,Tx)d(Ty,y)}{1+d(x,y)}, \frac{d(x,Tx)d(Ty,y)}{1+d(Ty,Tx)}\}$.

Theorem 3.7. Let (X,Y,d) be a complete bipolar metric space, $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant mapping and $\alpha,\beta : X \times Y \to [0,\infty)$. Suppose that the following condition are satisfied:
(i) T is (α,β)-admissible mapping;
(ii) T is an (α,β)-Generalized Geraghty type rational contractive mapping;
(iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\beta(x_0, Tx_0) \ge 1$;
(iv) T is continuous mapping.
Then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\beta(x_0, Tx_0) \ge 1$. Now we construct the bisequence $\{(x_n, y_n)\}$ as $Tx_n = y_n$ and $Ty_n = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.
Since T is (α,β)-admissible mapping,
So, $\alpha(x_0, y_0) = \alpha(x_0, Tx_0) \ge 1$,
$\beta(x_0, y_0) = \beta(x_0, Tx_0) \ge 1$,
$\alpha(x_1, y_0) = \alpha(Ty_0, Tx_0) \ge 1$,
$\beta(x_1, y_0) = \beta(Ty_0, Tx_0) \ge 1$,
using mathematical induction, we get

$$\alpha(x_{n+1}, y_n) \ge 1 \text{ and } \beta(x_{n+1}, y_n) \ge 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.21}$$

Since T is (α,β)-admissible mapping,
So, $\alpha(x_1, y_1) = \alpha(x_1, Tx_1) \ge 1$,
$\beta(x_1, y_1) = \beta(x_1, Tx_1) \ge 1$,
hence by induction, we obtain

$$\alpha(x_n, y_n) \ge 1 \text{ and } \beta(x_n, y_n) \ge 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.22}$$

Putting $x = x_n$ and $y = y_n$ in equation (3.20) and using equation (3.22), we have

$$\psi(d(x_n, y_n)) = \psi(d(Ty_{n-1}, Tx_n)) \le \alpha(x_n, Tx_n)\,\beta(Ty_{n-1}, y_{n-1})\,\psi(d(Ty_{n-1}, Tx_n))$$
$$\le \theta(\psi(M(x_n, y_{n-1})))\psi(M(x_n, y_{n-1})), \tag{3.23}$$

where $M(x_n, y_{n-1}) = max\{d(x_n, y_{n-1}), d(x_n, Tx_n), d(Ty_{n-1}, y_{n-1}), \frac{d(x_n,Tx_n), d(Ty_{n-1},y_{n-1})}{1+d(x_n,y_{n-1})}, \frac{d(x_n,Tx_n), d(Ty_{n-1},y_{n-1})}{1+d(Ty_{n-1},Tx_n)}\}$.

$$M(x_n, y_{n-1}) = max\{d(x_n, y_{n-1}), d(x_n, y_n), d(x_n, y_{n-1}), \frac{d(x_n,y_n), d(x_n,y_{n-1})}{1+d(x_n,y_{n-1})}, \frac{d(x_n,y_n), d(x_n,y_{n-1})}{1+d(x_n,y_n)}\},$$
$$= max\{d(x_n, y_{n-1}), d(x_n, y_n)\}. \tag{3.24}$$

Now, if $M(x_n, y_{n-1}) = d(x_n, y_n)$, then equation (3.23) becomes

$$\psi(d(x_n, y_n)) \le \theta(\psi(d(x_n, y_n)))\psi(d(x_n, y_n))$$
$$< \psi(d(x_n, y_n)).$$

which is a contradiction by using the properties of ψ.
So, $M(x_n, y_{n-1}) = d(x_n, y_{n-1})$ and equation (3.23) becomes

$$\psi(d(x_n, y_n)) \le \theta(\psi(d(x_n, y_{n-1})))\psi(d(x_n, y_{n-1}))$$
$$\le \psi(d(x_n, y_{n-1})). \tag{3.25}$$

By the properties of ψ, we can say that $d(x_n, y_n) \le d(x_n, y_{n-1})$ for all $n \in \mathbb{N}$.
Similarly, putting $x = x_{n+1}$ and $y = y_n$ in equation (3.20) and using equation (3.22), we get $\psi(d(x_{n+1}, y_n)) = \psi(d(Ty_n, Tx_n)) \le \alpha(x_n, Tx_n)\beta(Ty_n, y_n)\psi(d(Ty_n, Tx_n))$

$$\le \theta(\psi(M(x_n, y_n)))\psi(M(x_n, y_n)), \tag{3.26}$$

where $M(x_n, y_n) = max\{d(x_n, y_n), d(x_n, Tx_n), d(Ty_n, y_n),$

$\frac{d(x_n ,Tx_n),d(Ty_n ,y_n)}{1+d(x_n ,y_n)}, \frac{d(x_n ,Tx_n),d(Ty_n ,y_n)}{1+d(Ty_n ,Tx_n)}\}.$

$M(x_n , y_n) = max\{ d(x_n , y_n), d(x_n , y_n), d(x_{n+1} , y_n),$
$\frac{d(x_n ,y_n),d(x_{n+1} ,y_n)}{1+d(x_n ,y_n)}, \frac{d(x_n ,y_n),d(x_{n+1} ,y_n)}{1+d(x_{n+1} ,y_n)}\},$

$$= max\{ d(x_n , y_{n+1}), d(x_n , y_n)\}. \quad (3.27)$$

Now, if $M(x_n , y_n) = d(x_{n+1} , y_n)$, then equation (3.26) becomes

$$\psi(d(x_{n+1}, y_n)) \le \theta(\psi(d(x_{n+1} , y_n)))\psi(d(x_{n+1} , y_n)) < \psi(d(x_{n+1} , y_n)).$$

which is a contradiction by using the properties of ψ.

So, $M(x_n , y_n) = d(x_n , y_n)$ and equation (3.26) becomes

$$\psi(d(x_{n+1} , y_n)) \le \theta(\psi(d(x_n , y_n)))\psi(d(x_n , y_n)) \le \psi(d(x_n , y_n)). \quad (3.28)$$

By the properties of ψ, we can say that $d(x_{n+1} , y_n) \le d(x_n , y_n)$ for all $n \in \mathbb{N}$.

From the above, we conclude that the sequences $\{d(x_n , y_{n-1})\}$ and $\{d(x_n , y_n)\}$ are monotonically decreasing and for the non-negative monotonically decreasing sequences $\{d(x_n , y_{n-1})\}$ and $\{d(x_n , y_n)\}$, there exist some $r_1 \ge 0$ and $r_2 \ge 0$ such that

$$d(x_n , y_{n-1}) \to r_1 , \quad d(x_n , y_n) \to r_2 \text{ as } n \to \infty \quad (3.29)$$

Further from equation (3.25), it implies that

$$\frac{\psi(d(x_n ,y_n))}{\psi(d(x_{n-1} ,y_{n-1}))} \le \theta(\psi(d(x_n , y_{n-1}))) < 1. \quad (3.30)$$

As $n \to \infty$ in above inequality, we obtain

$\lim_{n\to\infty} \theta(\psi(d(x_n , y_{n-1}))) = 1$ and $\theta \in \Theta$, $\lim_{n\to\infty} \psi(d(x_n , y_{n-1}))) = 0$, which gives that

$$\lim_{n\to\infty} d(x_n , y_{n-1}) = r_1 = 0. \quad (3.31)$$

Further from equation (3.28), it implies that

$$\frac{\psi(d(x_{n+1} ,y_n))}{\psi(d(x_{n+1} ,y_n))} \le \theta(\psi(d(x_n , y_n))) < 1. \quad (3.32)$$

As $n \to \infty$ in above inequality, we obtain

$\lim_{n\to\infty} \theta(\psi(d(x_n , y_n))) = 1$ and $\theta \in \Theta$,

So, $\lim_{n\to\infty} \psi(d(x_n , y_n))) = 0$, which gives that

$$\lim_{n\to\infty} d(x_n , y_n) = r_2 = 0. \quad (3.33)$$

Now, we shall prove that $\{(x_n , y_n)\}$ is a Cauchy bisequence. Let us assume that $\{(x_n , y_n)\}$ is not Cauchy bisequence. Then there exist $\delta > 0$ for which we can find subsequence (x_{n_k} , y_{m_k}) with $n_k > m_k > k$ such that

$$d(x_{n_k} , y_{m_k}) \ge \delta. \quad (3.34)$$

Further, corresponding to m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ which satisfying equation (3.34), we get

$$d(x_{n_{k-1}} , y_{m_k}) < \delta. \quad (3.35)$$

Using triangle inequality, we obtain

$$0 < \delta \le d(x_{n_k} , y_{m_k}) \le d(x_{n_k} , y_{n_{k-1}}) + d(x_{n_{k-1}} , y_{n_{k-1}}) + d(x_{n_{k-1}} , y_{m_k}). \quad (3.36)$$

Letting $k \to \infty$ and using equations (3.31), (3.33) and (3.34), we obtain

$$\lim_{k\to\infty} d(x_{n_k} , y_{m_k}) = \delta. \quad (3.37)$$

Again, using triangle inequality, we have

$$d(x_{m_k} , y_{n_{k-1}}) \le d(x_{m_k} , y_{n_k}) + d(x_{n_k} , y_{n_k}) + d(x_{n_k} , y_{n_{k-1}}). \quad (3.38)$$

Letting $k \to \infty$ and using equations (3.31), (3.33) and (3.34), we get

$$\lim_{k\to\infty} d(x_{m_k} , y_{n_{k-1}}) = \delta. \quad (3.39)$$

Putting $x = x_{n_k}$ *and* $y = y_{m_k}$ in equation (3.20), we get

$$\begin{aligned}\psi(d(x_{n_k} , y_{m_k}) &= \psi\left(d(Ty_{n_{k-1}}, Tx_{m_k})\right) \\ &\le \alpha(x_{m_k} , Tx_{m_k})\, \beta(Ty_{n_{k-1}}, y_{n_{k-1}})\, \psi\left(d(Ty_{n_{k-1}}, Tx_{m_k})\right) \\ &\le \theta(\psi\left(M(x_{m_k} , y_{n_{k-1}})\right))\, \psi\left(M(x_{m_k}, y_{n_{k-1}})\right), \end{aligned} \quad (3.40)$$

where $M(x_{m_k} , y_{n_{k-1}}) = max\{d(x_{m_k} , y_{n_{k-1}}), d(x_{m_k} , Tx_{m_k}), d(Ty_{n_{k-1}} , y_{n_{k-1}}),$
$\frac{d(x_{m_k} ,Tx_{m_k})\, d(Ty_{n_{k-1}} ,y_{n_{k-1}})}{1+d(x_{m_k} ,y_{n_{k-1}})}, \frac{d(x_{m_k} ,Tx_{m_k}) d(Ty_{n_{k-1}} ,y_{n_{k-1}})}{1+d(Ty_{n_{k-1}}, Tx_{m_k})}\},$

$= max\{d(x_{m_k} , y_{n_{k-1}}), d(x_{m_k} , y_{m_k}), d(x_{n_k} , y_{n_{k-1}}),$

$$\frac{d(x_{m_k}, y_{m_k})\, d(x_{n_k}, y_{n_{k-1}})}{1+d(x_{m_k}, y_{n_{k-1}})}, \frac{d(x_{m_k}, y_{m_k}) d(x_{n_k}, y_{n_{k-1}})}{1+d(Ty_{n_{k-1}}, Tx_{m_k})}\}.$$

Therefore,

$$\psi(d(x_{n_k}, y_{m_k}) \le \theta(\psi(M(x_{m_k}, y_{n_{k-1}})))\, \psi(M(x_{m_k}, y_{n_{k-1}})). \quad (3.41)$$

On taking limit $k \to \infty$, we obtain

$\psi(\delta) \le \theta(\psi(M(x_{m_k}, y_{n_{k-1}})))\, \psi(\delta),$

that is,

$1 \le \lim_{k\to\infty} \theta(\psi(M(x_{m_k}, y_{n_{k-1}}))),$

which implies that $\lim_{k\to\infty} \theta(\psi(M(x_{m_k}, y_{n_{k-1}}))) = 1.$

Consequently, we obtain

$\lim_{k\to\infty} M(x_{m_k}, y_{n_k-1}) = 0$,

this implies that

$\lim_{k\to\infty} d(x_{m_k}, y_{n_k-1}) = 0.$

which is a contradiction.

Hence, $\{(x_n, y_n)\}$ is a Cauchy bisequence and (X, Y, d) is a complete bipolar metric space. So, $\{(x_n, y_n)\}$ is convergent and in fact biconvergent. So, there exists $u \in X \cap Y$ such that

$(x_n) \to u, (y_n) \to u$ as $n \to \infty$.

As T is a continuous mapping, so

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n$ implies that $T \lim_{n\to\infty} x_n = u,$

By combining both, we get

$Tu = u.$

Hence, T has a fixed point.

Example 3.8. Let $X = \{0,1,2\}$ and $Y = \{2,3\}$ and let $d : X \times Y \to [0, +\infty)$ be a function such that $d(x, y) = |x - y|$ for all $(x, y) \in X \times Y$.

Then, clearly (X, Y, d) is a complete bipolar metric space.

Define $T : (X, Y) \rightleftarrows (X, Y)$ such that $T0 = 2, T1 = 2, T2 = 2$ and $T3 = 1$ is a continuous mapping and $\alpha, \beta : X \times Y \to [0, \infty)$ such that $\alpha(x, y) = \beta(x, y) = 1$ for all $(x, y) \in X \times Y$. Clearly, T is (α, β)-admissible mapping and there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\beta(x_0, Tx_0) \ge 1$ and $X \cap Y = \{2\}$ and $T2 = 2$.

Taking $\psi(t) = t$ and $\theta(t) = \frac{1}{2}$.

(x,y)	$d(Ty,Tx)$	$d(x,y)$	$d(x,Tx)$	$d(Ty,y)$	$\frac{d(x,Tx)d(Ty,y)}{1+d(x,y)}$	$\frac{d(x,Tx)d(Ty,y)}{1+d(Ty,Tx)}$	$M(x,y)$
(0,2)	0	2	2	0	0	0	2
(0,3)	1	3	2	2	1	2	3
(1,2)	0	1	1	0	0	0	1
(1,3)	1	2	1	2	$\frac{2}{3}$	1	2
(2,2)	0	0	0	0	0	0	0
(2,3)	1	1	0	2	0	0	2

From the above data, the condition of (α, β)-Generalized Geraghty rational contractive mapping is satisfied. Hence, all the condition of above Theorem are satisfied.

So, T has a fixed point and 2 is fixed point under T.

Theorem 3.9. Let (X, Y, d) be a complete bipolar metric space, $T : (X, Y) \rightleftarrows (X, Y)$ is a contravariant mapping and $\alpha, \beta : X \times Y \to [0, \infty)$. Suppose that the following condition are satisfied:

(i) T is (α, β)-admissible mapping;

(ii) T is an (α, β)-Geraghty type contractive mapping;

(iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ and $\beta(x_0, Tx_0) \ge 1$;

(iv) T is continuous mapping;

(v) If for all $x, y \in F(T), F(T)$ denotes the set fixed point of T, with $x \ne y$ and $x, y \in X \cap Y$ such that $\alpha(x, Tx) \ge 1, \alpha(y, Ty) \ge 1$ and $\beta(x, Tx) \ge 1, \beta(y, Ty) \ge 1$.

Then, T has a unique fixed point.

Proof: Following the proof of Theorem 3.4., T has fixed point. To prove the uniqueness of fixed point of contravariant mapping T in complete bipolar metric space, let us assume, if possible, u and v are two distinct fixed point of T such that $\alpha(u, Tu) \ge 1, \alpha(v, Tv) \ge 1$ and $\beta(u, Tu) \ge 1, \beta(v, Tv) \ge 1$ and $u, v \in X \cap Y$.

Now applying, equation (3.1), we obtain

$$\psi(d(v,u)) = \psi(d(Tv,Tu)) \leq \alpha(u,Tu)\beta(Tv,v)\psi\big(d(Tv,Tu)\big),$$
$$\leq \theta(\,\psi(d(u,v)))\,\psi\big(d(u,v)\big),$$
$$<\ \psi\big(d(u,v)\big).$$

which is a contradiction. So, $d(v,u) = 0 \Rightarrow u = v$.

Hence, T has a unique fixed point.

Example 3.10. In the Example 3.5, we can easily say that T satisfies all the conditions of Theorem 3.9. So, T has a unique fixed point.

Clearly, '0' is unique fixed point of T.

Theorem 3.11. Let (X,Y,d) be a complete bipolar metric space, $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant mapping and $\alpha, \beta : X \times Y \to [0,\infty)$. Suppose that the following condition are satisfied:

(i) T is (α,β)-admissible mapping;

(ii) T is an (α,β)-Generalized Geraghty type rational contractive mapping;

(iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;

(iv) T is continuous mapping;

(v) If for all $x, y \in F(T)$, $F(T)$ denotes the set fixed point of T, with $x \neq y$ and $x, y \in X \cap Y$ such that $\alpha(x, Tx) \geq 1$, $\alpha(y,Ty) \geq 1$ and $\beta(x,Tx) \geq 1$, $\beta(y,Ty) \geq 1$.

Then, T has a unique fixed point.

Proof: Following the proof of Theorem 3.7., T has fixed point. To prove the uniqueness of fixed point of contravariant mapping T in complete bipolar metric space, let us assume, if possible, u and v are two distinct fixed point of T such that $\alpha(u, Tu) \geq 1$, $\alpha(v,Tv) \geq 1$ and $\beta(u,Tu) \geq 1$, $\beta(v,Tv) \geq 1$ and $u, v \in X \cap Y$.

Now applying, equation (3.20), we have

$$\psi(d(v,u)) = \psi(d(Tv,Tu)) \leq \alpha(u,Tu)\beta(Tv,v)\psi(d(Tv,Tu)),$$
$$\leq \theta(\,\psi(M(u,v)))\,\psi\big(M(u,v)\big).$$

where $M(u,v) = max.\{d(u,v), d(u,Tu), d(Tv,v), \frac{d(u,Tu)d(Tv,v)}{1+d(u,v)}, \frac{d(u,Tu)d(Tv,v)}{1+d(Tv,Tu)}\}$.

Hence,$\psi(d(v,u)) \ \leq \theta(\,\psi(M(u,v)))\,\psi\big(d(u,v)\big) <\ \psi\big(d(u,v)\big)$,

which is a contradiction. So, $d(v,u) = 0 \Rightarrow u = v$.

Hence, T has a unique fixed point.

Example 3.12. In the Example 3.8., we can easily say that T satisfies all the conditions of Theorem 3.11. So, T has a unique fixed point.

Clearly, '2' is unique fixed point of T.

IV. Application to Ulam-Hyers Stability

Let (X,Y,d) be a bipolar metric space and $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant mapping. Let us consider the fixed point equation

$T\xi = \xi$, (4.1)

and

for some $\varepsilon > 0$

$d(\xi, T\xi) < \varepsilon$ for $\xi \in X$ or $d(T\eta, \eta) < \varepsilon$ for $\eta \in Y$. (4.2)

Any point $\xi \in X \cup Y$ which satisfies the above equation (4.2) is called an ε-solution of the mapping T. We say that the fixed point problem (4.1) is Ulam-Hyers stable in a bipolar metric space if there exists a function $\chi : [0,\infty) \to [0,\infty)$ with $\chi(t) > 0$ for all $t > 0$ such that for each $\varepsilon > 0$ and an ε-solution $\xi \in X \cup Y$, there exists a solution η of the fixed point equation (4.1) such that

$d(\xi,\eta) < \chi(\varepsilon)$ or $d(\eta,\xi) < \chi(\varepsilon)$. (4.3)

Theorem 4.1. Let (X,Y,d) be a complete bipolar metric space, $T : (X,Y) \rightleftarrows (X,Y)$ is a contravariant mapping and $\alpha, \beta : X \times Y \to [0,\infty)$ and all the conditions of Theorem 3.4 are holds with $\psi(t) = t, \alpha, \beta = 1$. In addition to this if $[t(I - \theta(t))]^{-1}$: $[0,\infty) \to [0,\infty)$ exists.

Then, fixed point equation (4.1) is Ulam-Hyers stable.

Proof: By the proof of Theorem 3.4, T has a fixed point say ξ. For arbitrary $\varepsilon > 0$ and η be a ε-solution of the mapping T with $\eta \in Y$ that is

$d(T\eta, \eta) < \varepsilon$ for $\eta \in Y$.

Now,

$$d(\xi,\eta) \leq d(\xi,T\xi) + d(T\eta,T\xi) + d(T\eta,\eta),$$
$$\leq d(T\eta,\eta) + \theta\big(d(\xi,\eta)\big)d(\xi,\eta)\ ,$$
$$d(\xi,\eta)\left(1 - \theta\big(d(\xi,\eta)\big)\right) \leq d(T\eta,\eta) < \varepsilon.$$

Therefore, $d(\xi,\eta) < \chi(\varepsilon)$,

Where $\chi(\varepsilon) = [t(I - \theta(t))]^{-1}$.
Similarly, we can prove for a ε-solution of the mapping T with $\xi \in X$.
Hence, the fixed point equation (4.1) is Ulam-Hyers stable.

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