Combined Properties of Finite Sums & Finite Products near Zero

Tanushree Biswas

ABSTRACT: It was proved that whenever is partitioned into finitely many cells, one cell must contain arbitrary length geo-arithmetic progressions. It was also proved that arithmetic and geometric progressions can be nicely intertwined in one cell of partition, whenever is partitioned into finitely many cells. In this article we shall prove that similar types of results also hold near zero in some suitable dense sub semi-group of ((0, ∞), +), using the Stone-Cech compactification BS.

Key words and phrases. Ramsey theory, Central sets near zero, Finite Sums, Image partition near zero, IP set near zero.

Received 21 June, 2019; Accepted 05 July, 2019 © The author(s) 2019. Published with open access at www.questjournals.org

I. INTRODUCTION

One of the famous Ramsey theoretic results is van der Waerden’s Theorem [11], which says that whenever the set of natural numbers is divided into finitely many classes, one of these classes contains arbitrarily long arithmetic progressions. The analogous statement about geometric progressions is easily seen to be equivalent via the homomorphisms: \( \mathbb{N}, + \rightarrow (\mathbb{N}, \cdot) \) such that \( p(x) = 2^x \), and \( q: (\mathbb{N} \setminus \{1\}, \cdot) \rightarrow (\mathbb{N}, +) \), where \( q(x) \) is the length of the prime factorization of \( x \).

It has been shown in [1, Theorem 3.11] that any set which is multiplicatively large, that is a piecewise syndetic IP set in \((\mathbb{N}, \cdot)\) must contain substantial combined additive and multiplicative structure; in particular it must contain arbitrarily large geo-arithmetic progressions, that is, sets of the form \((a + id)i : i \in \{1, 2, \ldots, k\}\) ·

A well-known extension of van der Waerden’s Theorem allows one to get the additive increment of the arithmetic progression in the same cell as the arithmetic progression. Similarly, for any finite partition of \( N \) there exist some cell \( A \) and \( b, r \in \mathbb{N} \) such that \( \{r, b, br, \ldots, br^n\} \subseteq A \). It is proved in [1, Theorem 1.5] that these two facts can be intertwined.

**Theorem 1.1.** Let \( r, k \in \mathbb{N} \) and \( \mathbb{N} = \bigcup_{i=1}^{k} A_i \). Then there exist \( s \in \{1, 2, \ldots, r\} \) and \( a, b, d \in A_s \) such that \( \{b(a + id)i : i \in \{0, 1, \ldots, k\}\} \subseteq A \).

We know that if \( A \subseteq \mathbb{N} \) belongs to every idempotent in \( \beta \mathbb{N} \), then it is called an IP* set. Given a sequence \( (x_n)_{n=1}^{\infty} \) in \( \mathbb{N} \), let \( FP(x_n)_{n=1}^{\infty} \) be the product analogue of Finite Sum. Given a sequence \( (x_n)_{n=1}^{\infty} \) in \( \mathbb{N} \), we say that \( (y_n)_{n=1}^{\infty} \) is a sum subsystem of \( (x_n)_{n=1}^{\infty} \) if there is a sequence \( (H_n)_{n=1}^{\infty} \) of nonempty finite subsets of \( \mathbb{N} \) such that \( \max H_n < \min H_{n+1} \) and \( y_n = \sum_{t \in H_n} x_t \) for each \( n \in \mathbb{N} \).

**Theorem 1.2.** Let \( (x_n)_{n=1}^{\infty} \) be a sequence in \( \mathbb{N} \) and \( \mathcal{A} \) be an IP* set in \( (\mathbb{N}, +) \). Then there exists a sum subsystem \( (y_n)_{n=1}^{\infty} \) of \( (x_n)_{n=1}^{\infty} \) such that \( FS(y_n)_{n=1}^{\infty} \subseteq \mathcal{F}(FP(y_n))_{n=1}^{\infty} \subseteq \mathcal{A} \).

**Proof.** [2, Theorem 2.6] or see [9, Corollary 16.21].

The algebraic structure of the smallest ideal of \( \beta \mathbb{N} \) has played a significant role in Ramsey Theory. It is known that any central subset of \( (\mathbb{N}, +) \) is guaranteed to have substantial additive structure. But Theorem 16.27 of [9] shows that central sets in \( (\mathbb{N}, +) \) need not have any multiplicative structure at all. On the other hand, in [2] we see that sets which belong to every minimal idempotent of \( \mathbb{N} \), called central* sets, must have significant multiplicative structure.

In case of central* sets a similar result has been proved in [4] for a restricted class of sequences called minimal sequences, where a sequence \( (x_n)_{n=1}^{\infty} \) in \( \mathbb{N} \) is said to be a minimal sequence if...
Theorem 1.3. Let \((y_n)_{n=1}^{\infty}\) be a minimal sequence and \(A\) be a central* set in \((\mathbb{N},+)\). Then there exists a sum subsystem \((x_n)_{n=1}^{\infty}\) of \((y_n)_{n=1}^{\infty}\) such that \(\text{FS}((x_n)_{n=1}^{\infty}) \cup \mathbb{F}P ((x_n)_{n=1}^{\infty}) \subseteq A\).

Proof.[2, Theorem 2.4].

A similar result in this direction in the case of dyadic rational numbers has been proved by Bergelson, Hindman and Leader.

Theorem 1.4. There exists a finite partition \(\mathbb{D} \setminus \{0\} = \bigcup_{i=1}^{r} A_i\) such that there do not exist \(i \in \{1, 2, \ldots, r\}\) and a sequence \((x_n)_{n=1}^{\infty}\) with \(\text{FS}((x_n)_{n=1}^{\infty}) \cup \mathbb{F}P ((x_n)_{n=1}^{\infty}) \subseteq A_i\).

Proof.[3, Theorem 5.9].

In [3], the authors also presented the following conjecture and question.

Conjecture 1.5. There exists a finite partition \(\mathbb{Q} \setminus \{0\} = \bigcup_{i=1}^{r} A_i\) such that there do not exist \(i \in \{1, 2, \ldots, r\}\) and a sequence \((x_n)_{n=1}^{\infty}\) with \(\text{FS}((x_n)_{n=1}^{\infty}) \cup \mathbb{F}P ((x_n)_{n=1}^{\infty}) \subseteq A_i\).

Problem 1.6. Does there exists a finite partition \(\mathbb{H} \setminus \{0\} = \bigcup_{i=1}^{r} A_i\) such that there do not exist \(i \in \{1, 2, \ldots, r\}\) and a sequence \((x_n)_{n=1}^{\infty}\) with \(\text{FS}((x_n)_{n=1}^{\infty}) \cup \mathbb{F}P ((x_n)_{n=1}^{\infty}) \subseteq A_i\)?

In the section 2, we shall first work on some combined algebraic properties near 0 in the ring of quaternions, denoted by \(\mathbb{H}\). The ring being non-abelian, is a division ring having an idempotent 0. In section 3, for any suitable dense subsemigroup \(S\) of \((\{0, \infty\}, +)\), our aim is to establish partition regularity in the ring \(\mathbb{H}\) using additive and multiplicative structure of \(\mathbb{H}\).

II. COMBINED ALGEBRAIC AND MULTIPLICATIVE PROPERTIES NEARAN IDEMPOTENT IN RELATION WITH QUATERNION RINGS

In the following discussion our aim is to extend Theorem 1.2 and Theorem 1.3 for dense subsemigroups \((\mathbb{H}, +)\) in the appropriate context.

Definition 2.1. If \(S\) is a dense subsemigroup of \((\mathbb{H}, +)\), we define \(0^*(S) = \{p \in \mathbb{H}S; \ (r > 0)(B_d(r) \subseteq p)\}\).

It is proved in [7], that \(0^*(S)\) is a compact right topological subsemigroup of \((\mathbb{H}S, +)\) which is disjoint from \(K(\beta S)\) and hence gives some new information which are not available from \(K(\beta S)\). Being right topological semigroup \(0^*(S)\) contains minimal idempotents of \(0^*(S)\). A subset \(A\) of \(S\) is said to be IP* set near 0 if it belongs to every idempotent of \(0^*(S)\) and a subset \(C\) of \(S\) is said to be central* set near 0 if it belongs to every minimal idempotent of \(0^*(S)\).

Definition 2.2. Let \(S\) be a dense subsemigroup of \((\mathbb{H}, +)\). A subset \(A\) of \(S\) is said to be an IP set near 0 if there exists a sequence \((x_n)_{n=1}^{\infty}\) with \(\sum_{n=1}^{\infty} x_n\) converging such that \(\text{FS}((x_n)_{n=1}^{\infty}) \subseteq A\). We call a subset \(D\) of \(S\) is an IP* set near 0 if for every subset \(C\) of \(S\) which is IP set near 0, \(C \cap D\) is IP set near 0.

From [10, Theorem 3.2], it follows that for a dense subsemigroup \(S\) of \((\mathbb{H}, +)\) a subset \(A\) of \(S\) is an IP set near 0 if only if there exists some idempotent \(p \in 0^*(S)\) with \(A \subseteq p\). Further it can be easily observed that a subset \(D\) of \(S\) is an IP* set near 0 if and only if it belongs to every idempotent of \(0^*(S)\). Given \(c \in \mathbb{H}\) \(\setminus \{0\}\) and \(p \in \beta \mathbb{H}_d\), the product \(c \cdot p\) and \(p \cdot c\) are defined in \((\beta \mathbb{H}_d, \cdot)\). One has \(A \subseteq c \cap \mathbb{H}\) is a member of \(c \cdot p\) and similarly for \(p \cdot c\).

Lemma 2.3. Let \(S\) be a dense subsemigroup of \((\mathbb{H}, +)\) such that \(S \cap \mathbb{H}\) is a sub-semigroup of \((\mathbb{H} \setminus \{0\}, \cdot)\). If \(A\) is an IP set near 0 in \(S\) then \(A\) is also an IP set near 0 for every \(\mathbb{S} \cap \mathbb{H}_d(1)\) \(\setminus \{0\}\). Further if \(A\) is an IP* set near 0 in \((\mathbb{S}, +)\) then both \(A^*\) and \(A^*\) are IP sets near 0 for every \(\mathbb{S} \cap \mathbb{H}_d(1)\) \(\setminus \{0\}\).

Proof. Since \(A\) is an IP set near 0 then by [7, Theorem 3.1] there exists a sequence \((x_n)_{n=1}^{\infty}\) in \(S\) with the property that \(\sum_{n=1}^{\infty} x_n\) converges and \(\text{FS}((x_n)_{n=1}^{\infty}) \subseteq A\). This implies that \(\sum_{n=1}^{\infty} (s \cdot x_n)\) is also convergent and \(\text{FS}((sx_n)_{n=1}^{\infty}) \subseteq A\). This proves that \(A\) is also IP set near 0. Similarly, we can prove that \(A^{-1}\) is also

*Corresponding Author: Tanushree Biswas
IP set near 0 for every s ∈ S ∩ B_d(1) \ {0}. For the second let A be an IP* set near 0 and s ∈ S ∩ B_d(1) \ {0}. To prove that s^{-1}A is an IP* set near 0 it is sufficient to show that if B is any IP set near 0 then B ∩ s^{-1}A ≠ ∅. Since B is an IP set near 0, sB is also an IP set near 0 by the first part of the proof, so that A ∩ sB ≠ ∅. Choose t ∈ sB ∩ A and k ∈ B such that t = sk. Therefore k ∈ s^{-1}A so that B ∩ s^{-1}A ≠ ∅.

Given A ∈ S and s ∈ S, s^{-1}A = {t ∈ S : st ∈ A} and -s + A = {t ∈ S : s + t ∈ A}. In case of product we must keep in mind the order of elements of product is noncommutative here.

**Definition 2.4.** Let \((x_n)_{n=1}^\infty\) be a sequence in the ring \((\mathbb{H},+\cdot)\), and let k ∈ \mathbb{N}. Then FP(\((x_n)_{n=1}^k\)) is the set of all products of terms of \((x_n)_{n=1}^k\) in any order with no repetitions. Similarly FP(\((x_n)_{n=1}^\infty\)) is the set of all products of terms of \((x_n)_{n=1}^\infty\) in any order with no repetitions.

**Theorem 2.5.** Let S be a dense subsemigroup of \((\mathbb{H},+\cdot)\), such that S ∩ B_d(1) \ {0} is a subsemigroup of \((B_d(1) \ {0},\cdot)\). Also let \((x_n)_{n=1}^\infty\) be a sequence in S such that \(\sum_{n=1}^\infty x_n\) converges to 0 and A be a IP* set near 0 in S. Then there exists a subsemigroup \((y_n)_{n=1}^\infty\) of \((x_n)_{n=1}^\infty\) such that \(\text{FP}(y_n)_{n=1}^\inftyÅ)\), and \(\text{FP}(y_n)_{n=1}^\infty\) is a left ideal of \((0+,\cdot)\).

**Proof.** Since \(\sum_{n=1}^\infty x_n\) converges to 0, from [7, Theorem 3.1] it follows that we can find some idempotent p ∈ \(0^+(S)\) for which \(\text{FS}(\sum_{n=1}^\infty x_n) \subseteq p\). In fact, if \(\mathbb{T} = \cap_{n=1}^\infty \mathbb{S}_p \mathbb{S}(\sum_{n=1}^\infty x_n) \subseteq 0^+(S)\) and \(p \in T\). Again, since A is an IP* set near 0 in S, by the above Lemma 2.3 for every s ∈ S ∩ B_d(1) \ {0}, both \(s^{-1}A, As^{-1} \subseteq p\). Let \(A^* = \{s ∈ A : s + A, Ap\text{ chosen with the following properties: }\)

(a) \(\in\{1, 2, \ldots, m-1\}\); Max H, ≤ Min H; (b) If \(y_1 = \sum_{i ∈ E_1} x_i\) then \(\sum_{i ∈ E_1} x_i\) ∈ \(A^*\) and \(\text{FS}(\sum_{i ∈ E_1} x_i) \subseteq A\).

We observe that \(\{\sum_{i ∈ E_1} x_i : H \in P_f(N), \min H > \max Hm, \text{let } E_1 = \text{FS}(y_i = 1)\text{ and } E_2 = A^*(y_i = 1)\text{. Now consider } D = B ∩ A^* \cap (s + A') \cap \bigcap_{s ∈ E_1} (s^{-1}A') \cap \bigcap_{s ∈ E_2} (A^* s^{-1})\text{.}

Then \(D \in p\). Now choose \(y_{m+1} ∈ D\) and \(H_{m+1} ∈ P_f(N)\) such that \(\min H_{m+1} > \max H_m\). Putting \(y_{m+1} = \sum_{i ∈ E_1} x_i\) shows that the induction can be continued, and this proves the theorem.

**III. AN APPLICATION OF ADDITIVE AND MULTIPLICATIVE STRUCTURE OF BS**

We shall like to produce an alternative proof of the above Theorem 3.1 using additive and multiplicative structure of BS. We need the following notion.

**Theorem 3.1.** Let u, v ∈ \mathbb{N}. Let M be a finite image partition regular matrix over \(\mathbb{N}\) of order \(u \times v\), and let N be an infinite image partition regular matrix near 0 over a dense subsemigroup S of \(((0,\infty),+)\). Then

\[
\begin{pmatrix}
M & O \\
O & N
\end{pmatrix}
\]

is image partition regular near 0 over S.

**Definition 3.2.** Let S be a subsemigroup of \(((0,\infty),+)\) and let A be a matrix, finite or infinite with entries from Q. Then I(A) = {p ∈ \(0^+\) : for every P ∈ p, there exists \(\bar{x}\) with entries from S such that all entries of \(A \bar{x}\) are in P}.

The following lemma can be easily proved as [8, Lemma 2.5].

**Lemma 3.3.** Let A be a matrix, finite or infinite with entries from Q.

(a) The set I(A) is compact and I(A) ≠ \emptyset if and only if A is image partition regular near 0.
(b) If A is finite image partition regular matrix, then I(A) is a sub-semigroup of \((0^+,+)\).

Next, we shall investigate the multiplicative structure of I(A). In the following Lemma 3.4, we shall see that if A is an image partition regular near 0, then I(A) is a left ideal of \((0^+,\cdot)\). It is also a two-sided ideal of \((0^+,\cdot)\), provided A is a finite image partition regular near 0.

**Lemma 3.4.** Let A be a matrix, finite or infinite with entries from Q.

(a) If A is an image partition regular near 0, then I(A) is a left ideal of \((0^+,\cdot)\).
(b) If A is a finite image partition regular near 0, then I(A) is a two-sided ideal of \( (0^+, \cdot) \).

Proof. (a). Let Abe auximage partition regular matrix, where \( u, v \in \mathbb{N} \setminus \{0\} \). Let \( p \in 0^+ \) and \( q \in I(A) \). Also let \( U \in p \cdot q \). Then \( \{ x \in S : x \cup U \} \in q \). Choose \( z \in \{ x \in S : x \cup U \} \). Then \( z^{-1}U \in q \). So there exists \( \tilde{x} \) with entries from S such that \( \tilde{y} = A\tilde{x} \).

\[
\tilde{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{u-1} \end{pmatrix} \quad \text{and} \quad \tilde{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{v-1} \end{pmatrix}; \tilde{y} \text{ and } \tilde{x} \text{ are } u \times 1 \text{ and } v \times 1 \text{ matrices respectively.}
\]

Now \( y_i \in z^{-1}U \) for \( 0 \leq i < u \) implies that \( zy_i \in U \) for \( 0 \leq i < u \). Let \( \tilde{x} = z\tilde{x} \) and \( \tilde{y} = z\tilde{y} \). Then \( \tilde{y} = A\tilde{x} \).

(b). Let A be a \( u \times v \) matrix, where \( u, v \in \mathbb{N} \). Suppose that A be a \( x \times y \) matrix partition regular near 0, where \( u, v \in \mathbb{N} \). Then \( \{ x \in S : x \cup U \} \in q \). Therefore, \( \{ x \in S : x \cup U \} \in q \). Suppose that \( \{ x \in S : x \cup U \} \in q \). So there exists \( \tilde{x} \) with entries from S such that all entries of \( A\tilde{x} \) are in U. Therefore \( p, q \in I(A) \) is a left ideal of \( (0^+, \cdot) \).

Alternative proof of Theorem 2.2.7. Let \( r \in \mathbb{N} \) be given and \( \epsilon > 0 \). Let \( Q = U_{i=1}^r E_i \). Suppose that A be a \( u \times v \) matrix where \( u, v \in \mathbb{N} \). Also let \( A = \begin{pmatrix} M & 0 \\ O & N \end{pmatrix} \). Now by previous lemma 3.4, I(M) is a two sided ideal of \( (0^+, \cdot) \). So \( K(0^+, \cdot) \subseteq I(A) \). Also by lemma 3.4, I(M) is a left ideal of \( (0^+, \cdot) \). Hence, \( K(0^+, \cdot) \cap I(N) \neq \emptyset \). \( \therefore \) choose \( q \in I(M) \cap I(N) \). Now choose \( p \in I(M) \cap I(N) \). Since \( Q = U_{i=1}^r E_i \), there exist \( k \in \{ 1, 2, \ldots, r \} \) such that \( E_k \in p \). Thus, by definition of I(M) and I(N), there exist \( \tilde{x} \in S^u \) and \( \tilde{y} \in S^v \) such that \( M\tilde{x} \in E_k^u \) and \( N\tilde{y} \in E_k^v \).

Take \( \tilde{z} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \). Then \( A\tilde{z} = \begin{pmatrix} M\tilde{x} \\ N\tilde{y} \end{pmatrix} \). Hence \( A\tilde{z} \in E_k^u \cdot N\tilde{y} \in E_k^v \).

REFERENCES


[12]. DR. TANUSHREE BISWAS, ASSISTANT PROFESSOR, ST. XAVIER’S UNIVERSITY, KOLKATA