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Research Paper

On Vector Functions With A Parameter

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ABSTRACT: *A set of equations for removing and adding of a parameter were found in a scalar type vector function. By using the Cauchy-Euler differential operator in an exponential form, equations for calculation of the partial derivative with respect to the parameter were developed. Extensions to multiple vector and higher rank functions were made.*
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I. INTRODUCTION

1.1 General

There are two problems of interest in this paper. Sometimes it matters to add a parameter to a vector function of scalar type. The vector argument can be in two dimensions or higher. The parameter should be an independent variable. There seems not to exist an operator for adding a parameter of this kind. Secondly, partial differentiation of a scalar function is usually either very easy or trivial. However, in two or higher dimensions, there are no immediate general formulas available and the resulting equation is not obvious. The total derivative of a three-dimensional function is well known but is of little help.

$$\frac{dF(x,y,z,t)}{dt} = \frac{\partial F}{\partial t} + \nabla F \cdot \frac{d\vec{r}}{dt} \qquad (1)$$

This supposes the $\vec{r}$ to have dependence on the parameter *t* implicitly. But what about the cases where a vector function of scalar form with a parameter and no other implied dependence of it, is to be treated? Textbooks offer no solutions in this matter, see for example [1], [2]. To fill in this gap in the following a set of formulas for addition and removal of a parameter in a vector function and for differentiation with respect to the parameter in a number of cases is presented. In the following section the basic relations for adding a parameter to a vector function is presented. Then this equation is extended to more complex cases. Next is developed the partial derivative equations of the vector functions for various cases. Finally are presented multiple vector cases and higher rank vector functions. All formal proofs are left out to keep everything simple and readable.

II. OPERATOR EXPRESSIONS

2.1 The Vector Functions

In the following vector functions of varying rank of the type below

$$A(\vec{r}f(\beta t)), \vec{F}(xe^{\beta t}, ye^{\beta t}, ze^{\beta t}), \vec{\vec{T}}(xe^{\beta t}, ye^{\beta t}, ze^{\beta t}) \qquad (2)$$

are treated. The cases of having a constant vector offset in the argument with variations is studied as well. The *A*() is a scalar function of a vector. Thus the parameter function is a multiplier of the vector argument in the treatment.

2.2 The Cauchy-Euler Differential Operator Extended

In a recent article [3] it was shown that the Cauchy-Euler differential operator

$$x\frac{\partial}{\partial x} \qquad (3)$$

in exponential form has interesting properties when applied to a general function in one dimension. The assumption for the function is that it has a Taylor's series at the origin. The exponential differential operator is, on the other hand, defined as a Taylor' power series. In the following one may use the short notation

∂x

instead of the full expression $\frac{\partial}{\partial x}$.

$$e^{\beta x \partial x} = \sum_{n=0}^{\infty} \frac{(\beta x \partial x)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(\beta x \partial x)^n}{n!} \qquad (4)$$

The latter presentation shows the one essential point which makes the great difference to other plain differential operators, the unity. That is a conservative term while the derivatives are dynamic terms. A simple proof is given in [3] of the following equation

$$e^{\beta x \partial x} A(x) = A(xe^{\beta}) \qquad (5)$$

β is a parameter not containing any dependence on *x*. This result can be used over the complex plane in two dimensions

$$e^{\beta z \frac{d}{dz}} A(z) = A(ze^{\beta}) \qquad (6)$$

with $z, \beta \in C$. The proof is similar to the real variable case. As a simple application, a phase shift can be added to the argument with the operator

$$e^{i\phi z \frac{d}{dz}} A(z) = A(ze^{i\phi}) \qquad (7)$$

The result (5) can be extended to three dimensions as follows

$$e^{\beta \vec{r} \cdot \nabla} A(\vec{r}) = e^{\beta x \partial x} e^{\beta y \partial y} e^{\beta z \partial z} A(\vec{r}) = e^{\beta x \partial x} e^{\beta y \partial y} e^{\beta z \partial z} a(x, y, z) \qquad (8)$$

It is not surprising that this will have a form analogous to the one-dimensional case, being straightforward to prove from equation (8) by applying each of the operators in sequence

$$e^{\beta \vec{r} \cdot \nabla} A(\vec{r}) = A(\vec{r} e^{\beta}) \qquad (9)$$

This important result will be used repeatedly in the following. The parameter β can be replaced with an arbitrary function $f(t)$

$$e^{ln(f(t)) \vec{r} \cdot \nabla} A(\vec{r}) = A(\vec{r} f(t)) \qquad (10)$$

Equation (9) can be modified slightly with $\beta = ln(\alpha)$ to get

$$e^{ln(\alpha) \vec{r} \cdot \nabla} A(\vec{r}) = A(\vec{r} \alpha) \qquad (11)$$

This relation tells us that an arbitrary vector function can be acted upon with an operator to add a multiplier to the argument vector. The inverse operation will be, as expected

$$e^{-ln(\alpha) \vec{r} \cdot \nabla} A(\vec{r} \alpha) = A(\vec{r}) \qquad (12)$$

One is able to fully control how to turn on and off the parameter. It means that one can implant a parameter to a vector function which did not have any previous dependence of it. Of course, one can always multiply a vector argument by a new parameter to the same effect. However, now it is known what is the external operator to produce the addition or removal. In the following, this is utilized.

2.3 The Partial Derivative with Respect to the Parameter

The first task is to solve the partial derivative

$$\frac{\partial}{\partial t} A(\vec{r}e^{\beta t}) \quad (13)$$

One can now apply equation (9) to lift up the parameter from the inside

$$\frac{\partial}{\partial t} e^{\beta t \vec{r}\cdot\nabla} A(\vec{r}) \quad (14)$$

and then perform the differentiation to get

$$\frac{\partial}{\partial t} A(\vec{r}e^{\beta t}) = (\beta\vec{r}\cdot\nabla) e^{\beta t \vec{r}\cdot\nabla} A(\vec{r}) \quad (15)$$

This can be done since the argument of the exponential operator commutes with itself. Finally the original vector function is restored

$$\frac{\partial}{\partial t} A(\vec{r}e^{\beta t}) = (\beta\vec{r}\cdot\nabla) A(\vec{r}e^{\beta t}) \quad (16)$$

This is the central equation. The essential requirement is that no variable in this equation has any *t* dependence.

2.4 Relation to Taylor's Series

Since

$$(\partial t)^n A(\vec{r}e^{\beta t}) = (\beta\vec{r}\cdot\nabla)^n A(\vec{r}e^{\beta t}) \quad (17)$$

a Taylor's series can be built as follows

$$A(\vec{r}e^{\beta t}) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} (\beta\vec{r}\cdot\nabla)^n A(\vec{r}e^{\beta t_0}) \quad (18)$$

As $t_0 \to 0$ one gets

$$A(\vec{r}e^{\beta t}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\beta\vec{r}\cdot\nabla)^n A(\vec{r}) \quad (19)$$

and the familiar result is recognized, equation (16).

2.5| The Linear Parameter Case

Next the method used above is applied to another case which is not directly supported by equation (16)

$$\frac{\partial}{\partial t}B(\vec{r}t) \tag{20}$$

The parameter t is to become an exponential function so that one can apply the equation above. The variable is changed as follows

$$ln(t) = u \tag{21}$$

$$\frac{\partial u}{\partial t} = e^{-u} \tag{22}$$

$$\frac{\partial}{\partial t}B(\vec{r}t) = \frac{\partial u}{\partial t}\frac{\partial}{\partial u}B(\vec{r}e^{u}) \tag{23}$$

Equation (9) can be applied to obtain

$$\frac{\partial}{\partial t}B(\vec{r}t) = e^{-u}\frac{\partial}{\partial u}e^{u\vec{r}\cdot\nabla}B(\vec{r}) \tag{24}$$

and the differentiation will lead to

$$\frac{\partial}{\partial t}B(\vec{r}t) = e^{-u}(\vec{r}\cdot\nabla)B(\vec{r}e^{u}) \tag{25}$$

Then the original variable is substituted back to restore the vector function

$$\frac{\partial}{\partial t}B(\vec{r}t) = \frac{1}{t}(\vec{r}\cdot\nabla)B(\vec{r}t) \tag{26}$$

2.6 The Log Parameter Case

Continue by varying the built-in parameter, to a logarithm

$$\frac{\partial}{\partial t}B(\vec{r}\cdot ln(\gamma t)) = \frac{1}{t\cdot ln(\gamma t)}(\vec{r}\cdot\nabla)B(\vec{r}\cdot ln(\gamma t)) \tag{27}$$

This result is obtained along similar lines as in the preceding case by using a new variable $ln(\gamma t) = e^{u}$.

2.7 The Trigonometric Functions of the Parameter

The next application is a *sin*(t) function and it is processed analogously as Above

$$\frac{\partial}{\partial t}C(\vec{r}\cdot sin(\alpha t)) = \alpha cot(\alpha t)(\vec{r}\cdot\nabla)C(\vec{r}\cdot sin(\alpha t)) \tag{28}$$

The natural extension of *cos*(t) function follows the same tracks

$$\frac{\partial}{\partial t}C(\vec{r}\cdot cos(\alpha t)) = -\alpha tan(\alpha t)(\vec{r}\cdot\nabla)C(\vec{r}\cdot cos(\alpha t)) \tag{29}$$

2.8 The Case of an Arbitrary Parameter Function

The case of an arbitrary function *f*(*t*) can be solved

$$\frac{\partial}{\partial t}E(\vec{r}\cdot f(t)) = \frac{1}{f(t)(\frac{df^{-1}(z)}{dz})_{z=f(t)}}(\vec{r}\cdot\nabla)E(\vec{r}\cdot f(t)) \qquad (30)$$

This result is obtained along similar lines as in the preceding case. Here is a derivative of the inverse function of the *f*(*t*). Thus success of this equation is dependent on its existence and range of validity.

$$\frac{\partial}{\partial t}F(\vec{r}+\vec{a}e^{\beta t}) \qquad (31)$$

By looking once more to equation (16) it is realized that the independence of the two vectors is the key. Equation (9) can be applied to lift up the parameter from the inside but now one is working with the vector $\vec{a}$ entirely

$$\frac{\partial}{\partial t}e^{\beta t\vec{a}\cdot\nabla_a}F(\vec{r}+\vec{a}) \qquad (32)$$

and then perform the differentiation and

$$\frac{\partial}{\partial t}F(\vec{r}+\vec{a}e^{\beta t}) = (\beta\vec{a}\cdot\nabla_a)e^{\beta t\vec{a}\cdot\nabla_a}F(\vec{r}+\vec{a}) \qquad (33)$$

The original vector function is restored

$$\frac{\partial}{\partial t}F(\vec{r}+\vec{a}e^{\beta t}) = (\beta\vec{a}\cdot\nabla_a)F(\vec{r}+\vec{a}e^{\beta t}) \qquad (34)$$

The ∇_a operates only on vector $\vec{a}$, not on $\vec{r}$, just as ∇ operates only on vector $\vec{r}$ not on $\vec{a}$. Now the capability of creating more equations, like the following cases, is available

$$\frac{\partial}{\partial t}G(\vec{r}+\vec{a}t) = \frac{(\vec{a}\cdot\nabla_a)}{t}G(\vec{r}+\vec{a}t) \qquad (35)$$

$$\frac{\partial}{\partial t}H(\vec{r}+\vec{a}f(t)) = \frac{(\vec{a}\cdot\nabla_a)}{f(t)(\frac{df^{-1}(z)}{dz})_{z=f(t)}}H(\vec{r}+\vec{a}f(t)) \qquad (36)$$

The basic formula can be extended to the two-vector case, having obvious multiple vector extendability

$$e^{\beta\vec{r}\cdot\nabla+\alpha\vec{a}\cdot\nabla_a}A(\vec{r}+\vec{a}) = A(\vec{r}e^{\beta}+\vec{a}e^{\alpha}) \qquad (37)$$

This result immediately leads to

$$\frac{\partial}{\partial t}J(\vec{r}e^{\alpha t}+\vec{a}e^{\beta t}) = (\alpha\vec{r}\cdot\nabla+\beta\vec{a}\cdot\nabla_a)J(\vec{r}e^{\alpha t}+\vec{a}e^{\beta t}) \qquad (38)$$

Likewise, other similar cases for completeness can be found below.

$$\frac{\partial}{\partial t}K(\vec{r}t+\vec{a}t)=\frac{1}{t}(\vec{r}\cdot\nabla+\vec{a}\cdot\nabla_a)K(\vec{r}t+\vec{a}t) \quad (39)$$

$$\frac{\partial}{\partial t}L(\vec{r}ln(\alpha t)+\vec{a}ln(\beta t))=(\frac{\vec{r}\cdot\nabla}{t\cdot ln(\alpha t)}+\frac{\vec{a}\cdot\nabla_a}{t\cdot ln(\beta t)})L(\vec{r}ln(\alpha t)+\vec{a}ln(\beta t)) \quad (40)$$

$$\frac{\partial}{\partial t}M(\vec{r}sin(\alpha t)+\vec{a}sin(\beta t))=[\alpha cot(\alpha t)\vec{r}\cdot\nabla+\beta cot(\beta t)\vec{a}\cdot\nabla_a]M(\vec{r}sin(\alpha t)+\vec{a}sin(\beta t)) \quad (41)$$

$$\frac{\partial}{\partial t}M(\vec{r}cos(\alpha t)+\vec{a}cos(\beta t))=-[\alpha tan(\alpha t)\vec{r}\cdot\nabla+\beta tan(\beta t)\vec{a}\cdot\nabla_a]M(\vec{r}cos(\alpha t)+\vec{a}cos(\beta t)) \quad (42)$$

$$\frac{\partial}{\partial t}N(\vec{r}f(t)+\vec{a}f(t))=\frac{\vec{r}\cdot\nabla+\vec{a}\cdot\nabla_a}{f(t)(\frac{df^{-1}(z)}{dz})_{z=f(t)}}N(\vec{r}f(t)+\vec{a}f(t)) \quad (43)$$

It is tempting to have an option of getting different parameter functions for both vectors above, like equation (37).

$$e^{f(t)\vec{r}\cdot\nabla+g(t)\vec{a}\cdot\nabla_a}A(\vec{r}+\vec{a})=A(\vec{r}e^{f(t)}+\vec{a}e^{g(t)}) \quad (44)$$

This is correct since the operators act on different independent vectors and commute. The functions *f*(*t*)*; g*(*t*) have no role at that point. Thus one has managed to expand the number of options greatly. One is then able to finish this case as follows

$$\frac{\partial}{\partial t}K(\vec{r}e^{f(t)}+\vec{a}e^{g(t)})=(\frac{\partial f(t)}{\partial t}\vec{r}\cdot\nabla+\frac{\partial g(t)}{\partial t}\vec{a}\cdot\nabla_a)K(\vec{r}e^{f(t)}+\vec{a}e^{g(t)}) \quad (45)$$

The result is not similar to what perhaps might be expected since now the two functions are different. On the other hand one can derive the following

$$\frac{\partial}{\partial t}K(\vec{r}h(t)+\vec{a}j(t))=(\frac{h'(t)}{h(t)}\vec{r}\cdot\nabla+\frac{j'(t)}{j(t)}\vec{a}\cdot\nabla_a)K(\vec{r}h(t)+\vec{a}j(t)) \quad (46)$$

These results an be attened back to one dimension, like the following

$$e^{f(t)x\partial x+g(t)a\partial a}A(x+a)=A(xe^{f(t)}+ae^{g(t)}) \quad (47)$$

2.10 Higher Rank Vector Functions

As a natural extension of the methodology above it can be applied to a function in vector form. It is expressed as

$$\vec{F}(x,y,z) \quad (48)$$

Its components in Cartesian coordinates are

$$F_x(x,y,z),F_y(x,y,z),F_z(x,y,z) \quad (49)$$

Therefore, it is not, in general, a function of a vector but of the coordinates only. One can accept true vectors as well as vector-like functions which do not behave correctly in rotations. Their transformation properties are not important in the present treatment. One is able to apply equations (8) and (9)

$$e^{\beta\vec{r}\cdot\nabla}\vec{F}(x,y,z)=\vec{F}(xe^{\beta},ye^{\beta},ze^{\beta}) \tag{50}$$

This will lead to

$$\frac{\partial}{\partial t}\vec{F}(xe^{\beta t},ye^{\beta t},ze^{\beta t})=(\beta\vec{r}\cdot\nabla)\vec{F}(xe^{\beta t},ye^{\beta t},ze^{\beta t}) \tag{51}$$

The next extension would be a dyadic or a tensor of rank two in an analogous way:

$$\vec{\vec{T}}(x,y,z) \tag{52}$$

$$e^{\beta\vec{r}\cdot\nabla}\vec{\vec{T}}(x,y,z)=\vec{\vec{T}}(xe^{\beta},ye^{\beta},ze^{\beta}) \tag{53}$$

and then one obtains

$$\frac{\partial}{\partial t}\vec{\vec{T}}(xe^{\beta t},ye^{\beta t},ze^{\beta t})=(\beta\vec{r}\cdot\nabla)\vec{\vec{T}}(xe^{\beta t},ye^{\beta t},ze^{\beta t}) \tag{54}$$

Similar equations for special parameter functions as in the preceding paragraphs are possible.

2.11 Generalization to Multiple Parameters The slightly generalized vector function

$$\vec{F}(xe^{\alpha t},ye^{\beta t},ze^{\gamma t}) \tag{55}$$

is different from the form treated above. Processing as before, the following is obtained

$$\frac{\partial}{\partial t}\vec{F}(xe^{\alpha t},ye^{\beta t},ze^{\gamma t})=(\alpha x\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}+\gamma z\frac{\partial}{\partial z})\vec{F}(xe^{\alpha t},ye^{\beta t},ze^{\gamma t}) \tag{56}$$

Now one can regard the constants α,β,γ as components of a constant three-vector, M_x, M_y, M_z.

$$\frac{\partial}{\partial t}\vec{F}(xe^{M_xt},ye^{M_yt},ze^{M_zt})=(M_xx\frac{\partial}{\partial x}+M_yy\frac{\partial}{\partial y}+M_zz\frac{\partial}{\partial z})\vec{F}(xe^{M_xt},ye^{M_yt},ze^{M_zt}) \tag{57}$$

By defining a new vector form Cauchy-Euler operator as

$$\vec{\Delta_{CE}}=\hat{e_x}x\frac{\partial}{\partial x}+\hat{e_y}y\frac{\partial}{\partial y}+\hat{e_z}z\frac{\partial}{\partial z} \tag{58}$$

The vectors $\hat{e_x},\hat{e_y},\hat{e_z}$ are the unit vectors of the Cartesian coordinate system. The equation above is equal to

$$\frac{\partial}{\partial t}\vec{F}(xe^{M_xt},ye^{M_yt},ze^{M_zt})=(\vec{M}\cdot\vec{\Delta_{CE}})\vec{F}(xe^{M_xt},ye^{M_yt},ze^{M_zt}) \tag{59}$$

The following can then be alleged

$$e^{t\vec{M}\cdot\vec{\Delta_{CE}}}\vec{F}(x,y,z)=\vec{F}(xe^{M_xt},ye^{M_yt},ze^{M_zt}) \tag{60}$$

which can be proven in the same manner as the similar equations in earlier chapters. The equations (59), (60) are generalizations covering earlier basic results. By defining a new vector form Cauchy-Euler operator

$$\vec{\Delta_{CE}} = \hat{e_x}\frac{\partial}{\partial x} + \hat{e_y}\frac{\partial}{\partial y} + \hat{e_z}\frac{\partial}{\partial z} \qquad (61)$$

The results obtained here can be equally applied to vector functions of any rank.

2.12 Orthogonal Coordinate Systems

The expressions obtained above are valid in other orthogonal coordinate systems, like the spherical polar coordinates. A vector function can be set up in the following form, working as a direct continuation of the preceding paragraph.

$$\vec{G}(re^{M_r t}, \theta e^{M_\theta t}, \phi e^{M_\phi t}) \qquad (62)$$

and one is able to obtain again something very familiar

$$\frac{\partial}{\partial t}\vec{G}(re^{M_r t}, \theta e^{M_\theta t}, \phi e^{M_\phi t}) = (M_r r\frac{\partial}{\partial r} + M_\theta \theta\frac{\partial}{\partial \theta} + M_\phi \phi\frac{\partial}{\partial \phi})\vec{G}(re^{M_r t}, \theta e^{M_\theta t}, \phi e^{M_\phi t})$$

By defining a new vector form Cauchy-Euler opera

$$\vec{\Delta_{CE}} = \hat{e}_r r\frac{\partial}{\partial r} + \hat{e}_\theta \theta\frac{\partial}{\partial \theta} + \hat{e}_\phi \phi\frac{\partial}{\partial \phi} \qquad (64)$$

The vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ are the unit vectors of the coordinate system. They have the common definition as normal to each family of surfaces ($r = const, \theta = const, \phi = const$) and pointing in the direction of increasing coordinate. The equation above is equal to

$$\frac{\partial}{\partial t}\vec{G}(re^{M_r t}, \theta e^{M_\theta t}, \phi e^{M_\phi t}) = (\vec{M} \cdot \vec{\Delta_{CE}})\vec{G}(re^{M_r t}, \theta e^{M_\theta t}, \phi e^{M_\phi t}) \qquad (65)$$

The following is again the immediate conclusion

$$e^{t\vec{M}\cdot\vec{\Delta_{CE}}}\vec{G}(r, \theta, \phi) = \vec{G}(re^{M_r t}, \theta e^{M_\theta t}, \phi e^{M_\phi t}) \qquad (66)$$

II. DISCUSSION

Basic equations for adding and removing a parameter are presented in a vector function of scalar type, equation (9). The parameter is a multiplier to the vector argument. For multiple vector arguments, it turns out that one can affect one of the vectors fed in as a summed argument. Eventually, one can separately affect any with a different parameter function, equation (44).

A set of formulas are developed for calculation of the partial derivative of a vector function, with respect to a parameter being a multiplier of the vector argument. It was required that no variables in the function contain any dependence on the same parameter. Equation (16) is the central formula for creation of new formulas. A set of cases with some typical parameter functions residing in the vector functions were presented. Equation (45) is a generalization to partial derivative of a two-vector case. Extensions to multiple vectors and higher ranks are shown with analogous results. Equations (59) and (60) are the most general expressions in terms of a generalized exponential Cauchy-Euler operator in three dimensions.

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